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POSITIVE SOLUTIONS OF FOUR-POINT BOUNDARY-VALUE PROBLEMS FOR HIGHER-ORDER WITH p-LAPLACIAN OPERATOR

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Abstract. In this paper, we study the existence of positive solutions for nonlinear four-point singular boundary-value problems for higher-order equation with the p-Laplacian operator. Using the fixed-point index theory, we find conditions for the existence of one solution, and of multiple solutions.

1. Introduction

In this paper, we study the quasi-linear equation, with p-Laplacian operator,

$$(\phi_p(u^{(n-1)}))' + g(t)f(u(t), u'(t), \dots, u^{(n-2)}(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

subject to the boundary conditions

$$\begin{gathered} u^{(i)}(0) = 0 \quad 0 \le i \le n-3, \\ u^{(n-2)}(0) - B_0(u^{(n-1)}(\xi)) = 0 \quad n \ge 3, \\ u^{(n-2)}(1) + B_1(u^{(n-1)}(\eta)) = 0 \quad n \ge 3, \end{gathered} \tag{1.2}$$

where $\phi_p(s)$ is the p-Laplacian operator; i.e., $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_q = \phi_p^{-1}$, $\frac{1}{p} + \frac{1}{q} = 1$. $\xi, \eta \in (0, 1)$ is prescribed and $\xi < \eta$, $g : (0, 1) \to [0, \infty)$, B_0, B_1 are both nondecreasing continuous odd functions defined on $(-\infty, +\infty)$.

In recent years, the existence of positive solutions for nonlinear boundary-value problems with p-Laplacian operator received wide attention. Recently, for the existence of positive solutions of multi-points boundary-value problems for second-order ordinary differential equation, some authors have obtained the existence results [3, 1, 2, 4, 8]. However, the multi-points boundary-value problems treated in the above mentioned references do not discuss the problems with singularities and the higher-order p-Laplacian operator. For the singular case of multi-point boundary-value problems for higher-order p-Laplacian operator, with the author's acknowledge, no one has studied the existence of positive solutions in this case. Therefore this paper mainly studies the existence of positive solutions for nonlinear singular boundary-value problem (1.1), (1.2).

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In this paper, by constructing an integral equation which is equivalent to the problem (1.1), (1.2), we research the existence of positive solutions when g and f satisfy some suitable conditions.

For the rest of this paper, we make the following assumptions:

(H1) $f \in C([0,+\infty)^{n-1},[0,+\infty))$;
(H2) $g:(0,1) \to [0,+\infty)$ and $0 < \int_0^1 g(t)dt < \infty$;
(H3) B_0, B_1 are both increasing, continuous, odd functions defined on $(-\infty,+\infty)$ and at least one of them satisfies the condition that there exists one $b>0$ such that

$$0 < B_i(v) \le bv, \quad \forall v \ge 0,\ i=0,\ \text{or } i=1.$$

It is easy to check that condition (H2) implies

$$0 < \int_0^1 \phi_q\big(\int_0^s g(s_1)ds_1\big)ds < +\infty.$$

This paper is organized as follows. In section 2, we present some preliminaries and lemmas that will be used to prove our main results. In section 3, we discuss the existence of single solution of the systems (1.1). In section 4, we study the existence of at least two solutions of the systems (1.1). In section 5, we give two examples as an application.

2. Preliminaries and Lemmas

Let

$$B = \big\{u \in C^{n-2}[0,1] : u^{(i)}(0)=0,\ 0 \le i \le n-3\big\}.$$

Then B is a Banach space with the norm $\|u\| = \max_{t\in[0,1]} |u^{(n-2)}(t)|$. And let

$$K = \big\{u \in B : u^{(n-2)}(t) \ge 0,\ u^{(n-2)}(t) \text{ is concave function, } t \in [0,1]\big\}.$$

Obviously, K is a cone in B and $0 \le u^{(i)}(t) \le \|u\|$ on $[0,1]$. Set $K_r = \{u \in K : \|u\| \le r\}$. We can easily get the following Lemmas.

Lemma 2.1. *Suppose condition (H_2) holds. Then there exists a constant $\theta \in (0,1/2)$ that satisfies*

$$0 < \int_\theta^{1-\theta} g(t)dt < \infty.$$

Furthermore, the function

$$A(t) = \int_\theta^t \phi_q\Big(\int_s^t g(s_1)ds_1\Big)ds + \int_t^{1-\theta} \phi_q\Big(\int_t^s g(s_1)ds_1\Big)ds, \quad t \in [\theta, 1-\theta]$$

is positive continuous function on $[\theta,1-\theta]$, therefore $A(t)$ has minimum on $[\theta,1-\theta]$. Hence we suppose that there exists $L>0$ such that $A \ge L$, $t \in [\theta,1-\theta]$.

Lemma 2.2. *Let $u \in K$ and $\theta \in (0,1/2)$ in Lemma 2.1. Then*

$$u^{(n-2)}(t) \ge \theta\|u\|, \quad t \in [\theta,1-\theta].$$

The proof of the above lemma is similar to the proof of in [9, Lemma 2.2], so we omit it.

Lemma 2.3. *Suppose that conditions (H1)–(H3) hold. Then $u(t) \in K \cap C^{n-1}(0,1)$ is a solution of boundary-value problem (1.1), (1.2) if and only if $u(t) \in B$ is a solution of the integral equation*

$$u(t) = \int_0^t \int_0^{s_1} \dots \int_0^{s_{n-3}} w(s_{n-2}) ds_{n-2} ds_{n-3} \dots ds_1,$$

where

$$w(t) = \begin{cases} B_0 \circ \phi_q\big(\int_\xi^\delta g(s) f(u(s), u'(s), \dots, u^{(n-2)}(s)) ds\big) \\ + \int_0^t \phi_q\big(\int_s^\delta g(r) f(u(r), u'(r), \dots, u^{(n-2)}(r)) dr\big) ds & 0 \le t \le \delta, \\ B_1 \circ \phi_q\big(\int_\delta^\eta g(s) f(u(s), u'(s), \dots, u^{(n-2)}(s)) ds\big) \\ + \int_t^1 \phi_q\big(\int_\delta^s g(r) f(u(r), u'(r), \dots, u^{(n-2)}(r)) dr\big)) ds & \delta \le t \le 1. \end{cases} \tag{2.1}$$

Here δ is unique solution of the equation $g_1(t) = g_2(t)$, where

$$\begin{aligned} g_1(t) &= B_0 \circ \phi_q\big(\int_\delta^\eta g(s) f(u(s), u'(s), \dots, u^{(n-2)}(s)) ds\big) \\ &\quad + \int_t^1 \phi_q\big(\int_\delta^s g(r) f(u(r), u'(r), \dots, u^{(n-2)}(r)) dr\big) ds, \\ g_2(t) &= B_1 \circ \phi_q(\int_\delta^\eta g(s) f(u(s), u'(s), \dots, u^{(n-2)}(s)) ds) \\ &\quad + \int_t^1 \phi_q(\int_\delta^s g(r) f(u(r), u'(r), \dots, u^{(n-2)}(r)) dr) ds. \end{aligned}$$

The equation $g_1(t) = g_2(t)$ has unique solution in $(0,1)$ because $g_1(t)$ is strictly increasing on $[0,1)$, and $g_1(0) = 0$, while $g_2(t)$ is strictly decreasing on $(0,1]$, and $g_2(1) = 0$.

Proof. Necessity. By the equation of the boundary condition and (H3), we have $u^{(n-1)}(\xi) \ge 0$, $u^{(n-1)}(\eta) \le 0$, then there exist a constant $\delta \in [\xi, \eta] \subset (0,1)$ such that $u^{(n-1)}(\delta) = 0$. Firstly, by integrating the equation of the problems (1.1) on (δ, t), we have

$$\phi_p(u^{(n-1)}(t)) = \phi_p(u^{(n-1)}(\delta)) - \int_\delta^t g(s) f\big(u(s), u'(s), \dots, u^{(n-2)}(s)\big) ds,$$

then

$$u^{(n-1)}(t) = -\phi_q\Big(\int_\delta^t g(s) f\big(u(s), u'(s), \dots, u^{(n-2)}(s)\big) ds\Big), \tag{2.2}$$

thus

$$u^{(n-2)}(t) = u^{(n-2)}(\delta) - \int_\delta^t \phi_q\Big(\int_\delta^s g(r) f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big) dr\Big) ds. \tag{2.3}$$

By $u^{(n-1)}(\delta) = 0$ and condition (1.2), letting $t = \eta$ on (2.2), we have

$$u^{(n-2)}(1) = -B_1\big(u^{(n-1)}(\eta)\big) = B_1 \circ \phi_q\Big(\int_\delta^\eta g(s) f\big(u(s), u'(s), \dots, u^{(n-2)}(s)\big) ds\Big).$$

Then by (2.3), we have

$$\begin{aligned} u^{(n-2)}(\delta) &= B_1 \circ \phi_q\Big(\int_\delta^\eta g(s) f\big(u(s), u'(s), \dots, u^{(n-2)}(s)\big) ds\Big)) \\ &\quad + \int_\delta^1 \phi_q\Big(\int_\delta^s g(r) f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big) dr\Big) ds. \end{aligned} \tag{2.4}$$

Then

$$\begin{aligned} u^{(n-2)}(t) &= B_1 \circ \phi_q\Big(\int_\delta^\eta g(s) f\big(u(s), u'(s), \dots, u^{(n-2)}(s)\big) ds\Big) \\ &\quad + \int_t^1 \phi_q\Big(\int_\delta^s g(r) f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big) dr\Big) ds. \end{aligned} \tag{2.5}$$

Integrating (2.5) for $n-2$ times on $(0,t)$, we have

$$\begin{aligned} u(t) &= \int_0^t \int_0^{s_1} \dots \int_0^{s_{n-3}} B_1 \circ \phi_q\Big(\int_\delta^\eta g(s) f\big(u(s), u'(s), \\ &\quad \dots, u^{(n-2)}(s)\big) ds\Big) ds_{s_{n-2}} \dots ds_2\, ds_1 \\ &\quad + \int_0^t \int_0^{s_1} \dots \int_0^{s_{n-3}} (\int_{s_{n-2}}^1 \phi_q\Big(\int_\delta^s g(r) f\big(u(r), u'(r), \\ &\quad \dots, u^{(n-2)}(r)\big) dr\Big) ds) ds_{s_{n-2}} \dots ds_2\, ds_1. \end{aligned}$$

Similarly, for $t \in (0,\delta)$, integrating problems (1.1) on $(0,\delta)$, we have

$$\begin{aligned} u(t) &= \int_0^t \int_0^{s_1} \dots \int_0^{s_{n-3}} B_0 \circ \phi_q\Big(\int_\xi^\delta g(s) f\big(u(s), u'(s), \\ &\quad \dots, u^{(n-2)}(s)\big) ds\Big) ds_{s_{n-2}} \dots ds_2\, ds_1 \\ &\quad + \int_0^t \int_0^{s_1} \dots \int_0^{s_{n-3}} (\int_0^{s_{n-2}} \phi_q\Big(\int_s^\delta g(r) f\big(u(r), u'(r), \\ &\quad \dots, u^{(n-2)}(r)\big) dr\Big) ds) ds_{s_{n-2}} \dots ds_2\, ds_1. \end{aligned}$$

Therefore, for any $t \in [0,1]$, $u(t)$ can be expressed as

$$u(t) = \int_0^t \int_0^{s_1} \dots \int_0^{s_{n-3}} w(s_{n-2}) ds_{n-2} ds_{n-3} \dots ds_1,$$

where $w(t)$ is expressed as (2.1).

Sufficiency. Suppose that $u(t) = \int_0^t \int_0^{s_1} \dots \int_0^{s_{n-3}} w(s_{n-2}) ds_{n-2} ds_{n-3} \dots ds_1$. Then by (2.1), we have

$$u^{(n-1)}(t) = \begin{cases} \phi_q\Big(\int_t^\delta g(s) f\big(u(s), u'(s), \dots, u^{(n-2)}(s)\big) ds\Big) ds \geq 0, & 0 \leq t < \delta, \\ -\phi_q\Big(\int_\delta^t g(s) f\big(u(s), u'(s), \dots, u^{(n-2)}(s)\big) ds\Big) ds \leq 0, & \delta < t \leq 1, \end{cases} \tag{2.6}$$

So that $(\phi_p(u^{(n-1)}))' + g(t) f(u(t), u'(t), \dots, u^{(n-2)}(t)) = 0$, $0 < t < 1$, $t \neq \delta$. These imply that (1.1) holds. Furthermore, by letting $t = 0$ and $t = 1$ on (2.1) and (2.6), we obtain the boundary-value equations of (1.2). The proof is complete. $\square$

Now, we define a mapping $T : K \to C^{n-1}[0,1]$ given by

$$(Tu)(t) = \int_0^t \int_0^{s_1} \dots \int_0^{s_{n-3}} w(s_{n-2}) ds_{n-2} ds_{n-3} \dots ds_1,$$

where $w(t)$ is given by (2.1).

Lemma 2.4. *Suppose that conditions (H1), (H2) hold. Then the solution $u(t) \in K$ of (1.1), (1.2) satisfies*

$$u(t) \le u'(t) \le \dots \le u^{(n-3)}(t), \quad t \in [0,1],$$

and for $\theta \in (0, 1/2)$ in Lemma 2.1, we have

$$u^{(n-3)}(t) \le \frac{1}{\theta} u^{(n-2)}(t), \quad t \in [\theta, 1-\theta].$$

Proof. If $u(t)$ is the solution of problem (1.1), (1.2), then $u^{(n-2)}(t)$ is concave function, and $u^{(i)}(t) \ge 0$, $i = 0, 1, \dots, n-2$, $t \in [0,1]$, Thus we have

$$u^{(i)}(t) = \int_0^t u^{(i+1)}(s) ds \le t u^{(i+1)}(t) \le u^{(i+1)}(t), \quad i = 0, 1, \dots, n-4,$$

i.e., $u(t) \le u'(t) \le \dots \le u^{(n-3)}(t)$, $t \in [0,1]$. Next, by Lemma 2.2, for $t \in [\theta, 1-\theta]$, we have $u^{(n-2)}(t) \ge \theta \| u^{(n-2)} \|$. Then from

$$u^{(n-3)}(t) = \int_0^t u^{(n-2)}(s) ds \le \| u^{(n-2)} \|,$$

we have

$$u^{(n-3)}(t) \le \frac{1}{\theta} u^{(n-2)}(t), \quad t \in [\theta, 1-\theta].$$

The proof is complete. ☐

Lemma 2.5. *The operator $T : K \to K$ is completely continuous.*

Proof. Because

$$\begin{aligned} &(Tu)^{(n-1)}(t) \\ &= w'(t) = \begin{cases} \phi_q\Big(\int_t^\delta g(s) f\big(u(s), u'(s), \dots, u^{(n-2)}(s)\big) ds \Big) \ge 0 & 0 \le t \le \delta, \\ -\phi_q\Big(\int_\delta^t g(s) f\big(u(s), u'(s), \dots, u^{(n-2)}(s)\big) ds \Big) \le 0 & \delta \le t \le 1, \end{cases} \end{aligned}$$

is continuous, decreasing on $[0,1]$ and satisfies $(Tu)^{(n-1)}(\delta) = 0$. Then, $Tu \in K$ for each $u \in K$ and $(Tu)^{(n-2)}(\delta) = \max_{t \in [0,1]} (Tu)^{(n-2)}(t)$. This shows that $TK \subset K$. Furthermore, it is easy to check by Arzela-ascoli Theorem that $T : K \to K$ is completely continuous. ☐

Obviously, we can obtain the following results,

$$w(0) - B_0 w'(\xi) = 0, \quad w(1) + B_1 w'(\eta) = 0.$$

Our main tool of this paper is the following fixed point index theorem.

Theorem 2.6 ([5, 6])**.** *Suppose E is a real Banach space and $K \subset E$ is a cone. Let $\Omega_r = \{u \in K : \|u\| \le r\}$, and the operator $T : \Omega_r \to K$ be completely continuous and satisfy $Tx \ne x$ for all $x \in \partial\Omega_r$. Then*

(i) *If $\|Tx\| \le \|x\|$ for all $x \in \partial\Omega_r$, then $i(T, \Omega_r, K) = 1$;*
(ii) *If $\|Tx\| \ge \|x\|$ for all $x \in \partial\Omega_r$, then $i(T, \Omega_r, K) = 0$.*

For convenience, we set

$$\theta^* = \frac{2}{L}, \quad \theta_* = \frac{1}{(b+1)\phi_q\big(\int_0^1 g(r)dr\big)}.$$

where L is the constant in Lemma 2.1. By Lemma 2.4, we can also set

$$\begin{aligned}
f^0 &= \lim_{u_{n-1}\to 0} \max_{0\le u_1\le\dots\le u_{n-2}\le u_{n-1}/\theta} \frac{f(u_1,u_2,\dots,u_{n-1})}{(u_{n-1})^{p-1}},\\
f_\infty &= \lim_{u_{n-1}\to \infty} \min_{0\le u_1\le\dots\le u_{n-2}\le u_{n-1}/\theta} \frac{f(u_1,u_2,\dots,u_{n-1})}{(u_{n-1})^{p-1}},\\
f_0 &= \lim_{u_{n-1}\to 0} \min_{0\le u_1\le\dots\le u_{n-2}\le u_{n-1}/\theta} \frac{f(u_1,u_2,\dots,u_{n-1})}{(u_{n-1})^{p-1}},\\
f^\infty &= \lim_{u_{n-1}\to \infty} \max_{0\le u_1\le\dots\le u_{n-2}\le u_{n-1}/\theta} \frac{f(u_1,u_2,\dots,u_{n-1})}{(u_{n-1})^{p-1}}.
\end{aligned}$$

3. Existence of Positive Solutions

In this section, we present our main results.

Theorem 3.1. *Suppose that condition (H1)–(H3) hold. Assume that f also satisfies*

- (A1) *$f(u_1,u_2,\dots,u_{n-1}) \ge (mr)^{p-1}$ for $\theta r \le u_{n-1} \le r$, $0 \le u_1 \le \dots \le u_{n-2} \le u_{n-1}/\theta$;*
- (A2) *$f(u_1,u_2,\dots,u_{n-1}) \le (MR)^{p-1}$ for $0 \le u_{n-1} \le R$, $0 \le u_1 \le \dots \le u_{n-2} \le u_{n-1}/\theta$, where $m \in (\theta^*,\infty), M \in (0,\theta_*)$.*

Then the boundary-value problem (1.1), (1.2) *has a solution u such that $\|u\|$ lies between r and R.*

Theorem 3.2. *Suppose that condition (H1)–(H3) hold. Assume that f also satisfies*

- (A3) *$f^0 = \varphi \in [0,(\theta_*/4)^{p-1})$;*
- (A4) *$f_\infty = \lambda \in (2\theta^*/\theta)^{p-1},\infty)$.*

Then the boundary-value problem (1.1), (1.2) *has a solution u which is bounded in the norm $\|\cdot\|$.*

Theorem 3.3. *Suppose that condition (H1)–(H3) hold. Assume that f also satisfies*

- (A5) *$f^\infty = \lambda \in [0,(\theta_*/4)^{p-1})$;*
- (A6) *$f_0 = \varphi \in ((2\theta^*/\theta)^{p-1},\infty)$.*

Then the boundary-value problem (1.1), (1.2) *has a solution u which is bounded in the norm $\|\cdot\|$.*

Proof of Theorem 3.1. Without loss of generality, we suppose that $r < R$ and $0 < B_0(v) \le bv$ for all $v \ge 0$. For any $u \in K$, by Lemma 2.2, we have

$$u^{(n-2)}(t) \ge \theta\|u\|, \quad t \in [\theta, 1-\theta]. \tag{3.1}$$

We define the following two open subset of E:

$$\Omega_1 = \{u \in K : \|u\| < r\}, \quad \Omega_2 = \{u \in K : \|u\| < R\}.$$

For each $u \in \partial\Omega_1$, by (3.1) we have

$$r = \|u\| \ge u^{(n-2)}(t) \ge \theta\|u\| = \theta r, \quad t \in [\theta, 1-\theta].$$

For $t \in [\theta, 1-\theta]$ and $u \in \partial\Omega_1$, we shall discuss it from three perspectives.
(i) If $\delta \in [\theta, 1-\theta]$, thus for $u \in \partial\Omega_1$, by (A1) and Lemma 2.3, we have

$$\begin{aligned}
2\|Tu\| &= 2(Tu)^{(n-2)}(\delta)\\
&\geq \int_0^\delta \phi_q\Big(\int_s^\delta g(r)f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big)ds\\
&\quad + \int_\delta^1 \phi_q\Big(\int_\delta^s g(r)f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big)ds\\
&\geq \int_\theta^\delta \phi_q\Big(\int_s^\delta g(r)f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big)ds\\
&\quad + \int_\delta^{1-\theta} \phi_q\Big(\int_\delta^s g(r)f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big)ds\\
&\geq mrA(\delta) \geq mrL\\
&> 2r = 2\|u\|.
\end{aligned}$$

(ii) If $\delta \in (1-\theta, 1]$, thus for $u \in \partial\Omega_1$, by (A1) and Lemma 2.3, we have

$$\begin{aligned}
\|Tu\| &= (Tu)^{(n-2)}(\delta)\\
&\geq B_0 \circ \phi_q\Big(\int_\xi^\delta g(r)f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big)\\
&\quad + \int_0^\delta \phi_q(\int_s^\delta g(r)f(u(r), u'(r), \dots, u^{(n-2)}(r))dr)ds\\
&\geq \int_\theta^{1-\theta} \phi_q\Big(\int_s^{1-\theta} g(r)f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big)ds\\
&\geq mrA(1-\theta) \geq mrL\\
&> 2r > r = \|u\|.
\end{aligned}$$

(iii) If $\delta \in (0, \theta)$, thus for $u \in \partial\Omega_1$, by (A1) and Lemma 2.3, we have

$$\begin{aligned}
\|Tu\| &= (Tu)^{(n-2)}(\delta)\\
&\geq B_1 \circ \phi_q\Big(\int_\delta^\eta g(r)f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big)\\
&\quad + \int_\delta^1 \phi_q\Big(\int_\delta^s g(r)f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big)ds\\
&\geq \int_\theta^{1-\theta} \phi_q\Big(\int_\theta^s g(r)f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big)ds\\
&\geq mrA(\theta) \geq mrL\\
&> 2r > r = \|u\|.
\end{aligned}$$

Therefore, under all condition, we have $\|Tu\| > \|u\|$ for all $u \in \partial\Omega_1$. Then by Theorem 2.6,

$$i(T, \Omega_1, K) = 0. \tag{3.2}$$

On the other hand, for $u \in \partial\Omega_2$, we have $u^{(n-2)}(t) \le \|u\| = R$, by (A2),

$$\begin{aligned}
\|Tu\| &= (Tu)^{(n-2)}(\delta) \\
&\le B_0 \circ \phi_q\Big(\int_0^1 g(r) f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big) \\
&\quad + \int_0^1 \phi_q\Big(\int_s^\delta g(r) f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big)ds \\
&\le bMR\phi_q\big(\int_0^1 g(r)dr\big) + MR\phi_q\big(\int_0^1 g(r)dr\big) \\
&= (b+1)MR\phi_q\big(\int_0^1 g(r)dr\big) \\
&\le R = \|u\|.
\end{aligned}$$

Thus $\|Tu\| < \|u\|$ for all $u \in \partial\Omega_2$. Then by Theorem 3.1, we have

$$i(T, \Omega_2, K) = 1. \tag{3.3}$$

Therefore, by (3.2), (3.3), $r < R$ we have

$$i(T, \Omega_2 \setminus \overline{\Omega}_1, K) = 1.$$

Then operator T has a fixed point $u \in (\Omega_2 \setminus \overline{\Omega}_1)$ and $r \le \|u\| \le R$. This completes the proof $\square$

Proof of Theorem 3.2. First, from $f^0 = \varphi \in [0, (\theta_*/4)^{p-1})$, for $\epsilon = (\theta_*/4)^{p-1} - \varphi$, there exists an appropriately small positive number ρ, such that $0 \le u_{n-1} \le \rho$. Since $u_{n-1} \ne 0$, we have

$$f(u_1, u_2, \dots, u_{n-1}) \le (\varphi + \epsilon)(u_{n-1})^{p-1} \le (\theta_*/4)^{p-1}\rho^{p-1} = (\theta_*\rho/4)^{p-1}. \tag{3.4}$$

Then let $R = \rho$, $M = \frac{\theta_*}{4} \in (0, \theta_*)$, thus by (3.4)

$$\begin{gathered}
f(u_1, u_2, \dots, u_{n-1}) \le (MR)^{p-1}, \\
0 \le u_{n-1} \le R, \quad 0 \le u_1 \le \dots \le u_{n-2} \le u_{n-1}/\theta.
\end{gathered}$$

So condition (A2) holds.

Next, by condition (A4), $f_\infty = \lambda \in ((2\theta^*/\theta)^{p-1}, \infty)$, then for $\epsilon = \lambda - (2\theta^*/\theta)^{p-1}$, there exists an adequately big positive number $r \ne R$, such that $u_{n-1} \ge \theta r$, $0 \le u_1 \le \dots \le u_{n-2} \le u_{n-1}/\theta$, we have

$$f(u_1, u_2, \dots, u_{n-1}) \ge (\lambda - \epsilon)(u_{n-1})^{p-1} \ge (2\theta^*/\theta)^{p-1}(\theta r)^{p-1} = (2\theta^* r)^{p-1}, \tag{3.5}$$

Let $m = 2\theta^* > \theta^*$, thus by (3.5), condition (A1) holds. Therefore by Theorem 3.1 we know that the results of Theorem 3.2 hold. The proof is complete. $\square$

Proof of Theorem 3.3. First, by condition (A6), $f_0 = \varphi \in ((2\theta^*/\theta)^{p-1}, \infty)$, then for $\epsilon = \varphi - (2\theta^*/\theta)^{p-1}$, there exists an appropriately small positive number r, such that $0 \le u_{n-1} \le r$, $u_{n-1} \ne 0$, we have

$$f(u_1, u_2, \dots, u_{n-1}) \ge (\varphi - \epsilon)(u_{n-1})^{p-1} = (2\theta^*/\theta)^{p-1}(u_{n-1})^{p-1},$$

thus when $\theta r \le u_{n-1} \le r$, we have

$$f(u_1, u_2, \dots, u_{n-1}) \ge (2\theta^*/\theta)^{p-1}(\theta r)^{p-1} = (2\theta^* r)^{p-1}. \tag{3.6}$$

Let $m = 2\theta^* > \theta^*$, so by (3.6), condition (A1) holds.

Next, by condition (A5): $f^\infty = \lambda \in [0, (\theta_*/4)^{p-1})$, then for $\epsilon = (\theta_*/4)^{p-1} - \lambda$, there exists an suitably big positive number $\rho \neq r$, such that $u_{n-1} \geq \rho$, we have

$$f(u_1, u_2, \dots, u_{n-1}) \leq (\lambda + \epsilon)(u_{n-1})^{p-1} \leq (\theta_*/4)^{p-1}(u_{n-1})^{p-1}. \tag{3.7}$$

If f is unbounded, by the continuation of f on $[0, \infty)^{n-1}$, then exists constant $R \geq \rho$, $R \neq r$, and a point $(u_{01}, u_{02}, \dots, u_{0(n-1)}) \in [0, \infty)^{n-1}$ such that

$$\rho \leq u_{0(n-1)} \leq R$$

and

$$f(u_1, u_2, \dots, u_{n-1}) \leq f(u_{01}, u_{02}, \dots, u_{0(n-1)}), \quad 0 \leq u_{n-1} \leq R.$$

Thus, by $\rho \leq u_{0(n-1)} \leq R$, we know

$$\begin{aligned} f(u_1, u_2, \dots, u_{n-1}) &\leq f(u_{01}, u_{02}, \dots, u_{0(n-1)}) \\ &\leq (\theta_*/4)^{p-1}(u_{0(n-1)})^{p-1} \\ &\leq (\theta_* R/4)^{p-1}. \end{aligned}$$

Choose $M = \frac{\theta_*}{4} \in (0, \theta_*)$. Then, we have

$$\begin{gathered} f(u_1, u_2, \dots, u_{n-1}) \leq (MR)^{p-1}, \\ 0 \leq u_{n-1} \leq R, \quad 0 \leq u_1 \leq \dots \leq u_{n-2} \leq u_{n-1}/\theta. \end{gathered}$$

If f is bounded, we suppose $f(u_1, u_2, \dots, u_{n-1}) \leq \overline{M}^{p-1}$, $u_{n-1} \in [0, \infty)$, $\overline{M}^{p-1} \in R_+$, there exists an adequately big positive number $R > 4\overline{M}/\theta_*$, then choose $M = \theta_*/4 \in (0, \theta_*)$, for $0 \leq u_1 \leq \dots \leq u_{n-2} \leq u_{n-1}/\theta$, we have

$$f(u_1, u_2, \dots, u_{n-1}) \leq \overline{M}^{p-1} \leq (\theta_* R/4)^{p-1} = (MR)^{p-1}, \quad 0 \leq u_{n-1} \leq R$$

Therefore, condition (A2) holds. Therefore, by Theorem 3.1, we know that the results of Theorem 3.3 holds. The proof is complete. □

4. Existence of Many Positive Solutions

Next, we discuss the existence of many positive solutions.

Theorem 4.1. *Suppose that conditions (H1)–(H3) and (A2) hold. Assume that f also satisfies*

(A7) $f_0 = +\infty$;
(A8) $f_\infty = +\infty$.

Then the boundary-value problem (1.1), (1.2) *has at least two solutions u_1, u_2 such that*

$$0 < \|u_1\| < R < \|u_2\|.$$

Proof. First, by condition (A7), for any $M > \frac{2}{\theta L}$, there exists a constant $\rho_* \in (0, R)$ such that

$$\begin{gathered} f(u_1, u_2, \dots, u_{n-1}) \geq (Mu_{n-1})^{p-1}, \\ 0 < u_{n-1} \leq \rho_*, \quad 0 \leq u_1 \leq \dots \leq u_{n-2} \leq u_{n-1}/\theta. \end{gathered} \tag{4.1}$$

Set $\Omega_{\rho_*} = \{u \in K : \|u\| < \rho_*\}$, for any $u \in \partial\Omega_{\rho_*}$. By (4.1) and Lemma 2.2, similar to the proof of Theorem 3.1, we have from the three perspectives,

$$\|Tu\| \geq \|u\|, \ \forall \ u \in \partial\Omega_{\rho_*}.$$

Then by Theorem 2.6, we have

$$i(T, \Omega_{\rho_*}, K) = 0. \tag{4.2}$$

Next, by condition (A8), for any $\overline{M} > \frac{2}{\theta L}$, there exists a constant $\rho_0 > 0$ such that

$$\begin{gathered} f(u_1, u_2, \dots, u_{n-1}) \geq (\overline{M} u_{n-1})^{p-1}, \\ u_{n-1} > \rho_0, \quad 0 \leq u_1 \leq \dots \leq u_{n-2} \leq u_{n-1}/\theta. \end{gathered} \tag{4.3}$$

We choose a constant $\rho^* > \max\{R, \frac{\rho_0}{\theta}\}$, obviously $\rho_* < R < \rho^*$. Set $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$. For any $u \in \partial\Omega_{\rho^*}$, by Lemma 2.2, we have

$$u^{(n-2)}(t) \geq \theta\|u\| = \theta\rho^* > \rho_0, \quad t \in [\theta, 1-\theta].$$

Then by (4.3) and also similar to the proof of Theorem 3.1, we have from the three perspectives,

$$\|Tu\| \geq \|u\| \quad \forall u \in \partial\Omega_{\rho^*}.$$

Then by Theorem 2.6, we have

$$i(T, \Omega_{\rho^*}, K) = 0. \tag{4.4}$$

Finally, set $\Omega_R = \{u \in K : \|u\| < R\}$, For each $u \in \partial\Omega_R$, by (A2), Lemma 2.2 and also similar to the proof of Theorem 3.1, we can also have

$$\|Tu\| \leq \|u\| \quad \forall u \in \partial\Omega_R.$$

Then by Theorem 2.6, we have

$$i(T, \Omega_R, K) = 1. \tag{4.5}$$

Therefore, by (4.2), (4.4), (4.5), $\rho_* < R < \rho^*$ we have

$$i(T, \Omega_R \setminus \overline{\Omega}_{\rho_*}, K) = 1, \quad i(T, \Omega_{\rho^*} \setminus \overline{\Omega}_R, K) = -1.$$

Then T have fixed point $u_1 \in \Omega_R \setminus \overline{\Omega}_{\rho_*}$, and fixed point $u_2 \in \Omega_{\rho^*} \setminus \overline{\Omega}_R$. Obviously, $u_1,\ u_2$ are all positive solutions of problem (1.1),(1.2) and $0 < \|u_1\| < R < \|u_2\|$. The proof is complete. □

Theorem 4.2. *Suppose that conditions (H1)–(H3)and (A1) hold. Assume that f also satisfies*

- (A9) $f^0 = 0$;
- (A10) $f^\infty = 0$.

Then the boundary-value problem (1.1), (1.2) *has at least two solutions u_1, u_2 such that* $0 < \|u_1\| < r < \|u_2\|$.

Proof. First, from $f^0 = 0$, for $\eta_1 \in (0, \theta_*)$, there exists a constant $\rho_* \in (0, r)$ such that

$$\begin{gathered} f(u_1, u_2, \dots, u_{n-1}) \leq (\eta_1 u_{n-1})^{p-1}, \\ 0 < u_{n-1} \leq \rho_*, \quad 0 \leq u_1 \leq \dots \leq u_{n-2} \leq u_{n-1}/\theta. \end{gathered} \tag{4.6}$$

Set $\Omega_{\rho_*} = \{u \in K : \|u\| < \rho_*\}$, for each $u \in \partial\Omega_{\rho_*}$, by (4.6), we have

$$\begin{aligned}
\|Tu\| &= (Tu)^{(n-2)}(\delta) \\
&\le B_0 \circ \phi_q\Big(\int_0^1 g(r) f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big) \\
&\quad + \int_0^1 \phi_q\Big(\int_s^\delta g(r) f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big)ds \\
&\le B_0 \circ \phi_q\Big(\int_0^1 g(r) f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big) \\
&\quad + \phi_q\Big(\int_0^1 g(r) f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big) \\
&\le (b+1)\eta_1 \rho_* \phi_q\Big(\int_0^1 g(r)dr\Big) \\
&\le \rho_* = \|u\|.
\end{aligned}$$

i.e., $\|Tu\| \le \|u\|$ for all $u \in \partial\Omega_{\rho_*}$. Then by Theorem 2.6, we have

$$i(T, \Omega_{\rho_*}, K) = 1. \tag{4.7}$$

Next, let $f^*(x) = \max_{0 \le u_{n-1} \le x, 0 \le u_1 \le \dots \le u_{n-2} \le u_{n-1}/\theta} f(u_1, u_2, \dots, u_{n-1})$, note that $f^*(x)$ is monotone increasing with respect to $x \ge 0$. Then from $f^\infty = 0$, it is easy to see that

$$\lim_{x\to\infty} \frac{f^*(x)}{x^{p-1}} = 0.$$

Therefore, for any $\eta_2 \in (0, \theta_*)$, there exists a constant $\rho^* > r$ such that

$$f^*(x) \le (\eta_2 x)^{p-1}, \quad x \ge \rho^*. \tag{4.8}$$

Set $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$, for each $u \in \partial\Omega_{\rho^*}$, by (4.8), we have

$$\begin{aligned}
\|Tu\| &= (Tu)^{(n-2)}(\delta) \\
&\le B_0 \circ \phi_q\Big(\int_0^1 g(r) f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big) \\
&\quad + \int_0^1 \phi_q\Big(\int_s^\delta g(r) f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big)ds \\
&\le B_0 \circ \phi_q\Big(\int_0^1 g(r) f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big) \\
&\quad + \phi_q\Big(\int_0^1 g(r) f\big(u(r), u'(r), \dots, u^{(n-2)}(r)\big)dr\Big) \\
&\le (b+1)\phi_q\Big(\int_0^1 g(r) f^*(\rho^*)dr\Big) \\
&\le (b+1)\eta_2 \rho^* \phi_q\Big(\int_0^1 g(r)dr\Big) \\
&\le \rho^* = \|u\|.
\end{aligned}$$

i.e., $\|Tu\| \le \|u\|$ for all $u \in \partial\Omega_{\rho^*}$. Then by Theorem 2.6, we have

$$i(T, \Omega_{\rho^*}, K) = 1. \tag{4.9}$$

Finally, set $\Omega_r = \{u \in K : \|u\| < r\}$, For any $u \in \partial\Omega_r$, by (A_1), Lemma 2.2 and also similar to the previous proof of Theorem 3.1, we can also have $\|Tu\| \geq \|u\|$ for all $u \in \partial\Omega_r$. Then by Theorem 2.6, we have

$$i(T, \Omega_r, K) = 0. \tag{4.10}$$

Therefore, by (4.7), (4.9), (4.10), $\rho_* < r < \rho^*$, we have

$$i(T, \Omega_r \setminus \overline{\Omega}_{\rho_*}, K) = -1, \quad i(T, \Omega_{\rho^*} \setminus \overline{\Omega}_r, K) = 1.$$

Then T has a fixed points $u_1 \in \Omega_r \setminus \overline{\Omega}_{\rho_*}$, and $u_2 \in \Omega_{\rho_*} \setminus \overline{\Omega}_r$. Obviously, u_1, u_2 are all positive solutions of problem (1.1),(1.2) and $0 < \|u_1\| < r < \|u_2\|$. The proof is complete. □

Similar to Theorem 3.1, we obtain the following Theorems.

Theorem 4.3. *Suppose that conditions (H1)–(H3), (A2), (A4), and (A6)hold. Then the boundary-value problem* (1.1), (1.2) *has at least two solutions* u_1, u_2 *such that* $0 < \|u_1\| < R < \|u_2\|$.

Theorem 4.4. *Suppose that conditions (H1)–(H3), (A1), (A3) and (A5) hold. Then the boundary-value problem* (1.1), (1.2) *has at least two solutions* u_1, u_2 *such that* $0 < \|u_1\| < r < \|u_2\|$.

5. Applications

Example 5.1. Consider the following third-order singular boundary-value problem (SBVP), with p-Laplacian,

$$\begin{gathered}
(\phi_p(u''))' + \frac{\sqrt{3}}{36} t^{-\frac{1}{2}} (u')^{1/2} [\frac{1}{5} + \frac{\frac{94}{5} e^{2u'}}{120u + 7e^{u'} + e^{2u'}}] = 0 \quad 0 < t < 1, \\
u(0) = 0, \\
u'(0) - u''(1/4) = 0, \quad u'(1) + 3u''(1/2)) = 0,
\end{gathered} \tag{5.1}$$

where

$$p = \frac{3}{2}, \quad \xi = \frac{1}{4}, \quad \eta = \frac{1}{2}, \quad b = 2, \quad \theta = \frac{1}{4},$$
$$g(t) = \frac{\sqrt{3}}{36} t^{-\frac{1}{2}}, \quad f(u_1,\ u_2) = (u_2)^{1/2} \big[\frac{1}{5} + \frac{94 e^{2u_2}/5}{120u_1 + 7e^{u_2} + e^{2u_2}}\big].$$

Then obviously,

$$q = 3, \quad f^0 = \varphi = \lim_{u_2 \to 0^+} \max_{0 \leq u_1 \leq 4u_2} \frac{f(u_1,\ u_2)}{u_2^{p-1}} = \frac{51}{20},$$
$$f_\infty = l = \lim_{u_2 \to \infty} \min_{0 \leq u_1 \leq 4u_2} \frac{f(u_1,\ u_2)}{u_2^{p-1}} = l = \frac{95}{5}, \quad \int_0^1 g(t)dt = \frac{\sqrt{3}}{18},$$
$$B_0(v) = v < 2v = bv, \quad B_1(v) = 3v, \quad \forall v \geq 0,$$

so conditions (H1)–(H3) hold. Next,

$$\theta_* = \frac{1}{(b+1)\phi_q\big(\int_0^1 g(r)dr\big)} = 36,$$

then $(\theta_*/4)^{p-1} = 3 > \frac{51}{20}$, i.e. $\varphi \in [0, (\theta_*/4)^{p-1})$, so conditions (A3) holds. For $\theta = 1/4$, by calculating, it is easy see that

$$L = \min_{t\in[\theta,1-\theta]} A(t) = \frac{1}{16}(\frac{7}{36} + \frac{\sqrt{3}}{3}).$$

Because

$$(2\theta^*/\theta)^{p-1} = 96 \times (\frac{1}{7+12\sqrt{3}})^{1/2} < \frac{95}{5},$$

we have

$$l \in ((2\theta^*/\theta)^{p-1}, \infty),$$

so conditions (A4) holds. Then by Theorem 3.2, (5.1) has at least a positive solution.

Example 5.2. Consider the following third-order singular boundary-value problem, with p-Laplacian,

$$\begin{gathered}(\phi_p(u''))' + \frac{1}{64\pi^4} t^{-\frac{1}{2}}(1-t)[u + (u')^2 + (u')^4] = 0 \quad 0 < t < 1,\\ u(0) = 0,\\ u'(0) - u''(1/4) = 0, \quad u'(1) + 5u''(1/3)) = 0,\end{gathered} \tag{5.2}$$

where

$$\begin{gathered}p = 4, \quad \xi = \frac{1}{4}, \quad \eta = \frac{1}{3}, \quad \theta = \frac{1}{4},\\ g(t) = \frac{1}{64\pi^4} t^{-\frac{1}{2}}(1-t), \quad f(u_1, u_2) = u_1 + u_2^2 + u_2^4.\end{gathered}$$

Then obviously,

$$\begin{gathered}q = \frac{4}{3}, \quad \int_0^1 g(t)dt = \frac{1}{64\pi^3}, \quad f_\infty = +\infty, \quad f_0 = +\infty,\\ B_0(v) = v < 2v = bv, \quad B_1(v) = 5v \quad \forall v \geq 0,\end{gathered}$$

conditions (H1)–(H3), (A7), (A8) hold. Next,

$$\phi_q\big(\int_0^1 g(t)dt\big) = \frac{1}{4\pi}, \quad \theta_* = \frac{4\pi}{3},$$

we choose $R = 3$, $M = 2$ and for $\theta = \frac{1}{4}$, because of the monotone increasing of $f(u_1, u_2)$ on $[0, \infty) \times [0, \infty)$, then

$$f(u_1,\ u_2) \leq f(12,\ 3) = 12 + 90 = 102, \quad 0 \leq u_2 \leq 3, \quad 0 \leq u_1 \leq 4u_2.$$

Therefore, by

$$M \in (0, \theta_*), \quad (MR)^{p-1} = (6)^3 = 216,$$

we know that

$$f(u_1, u_2) \leq (MR)^{p-1}, \quad 0 \leq u_2 \leq 3,\ 0 \leq u_1 \leq 4u_2,$$

so conditions (A2) holds. Then by Theorem 4.1, (5.2) has at least two positive solutions v_1, v_2 and $0 < \|v_1\| < 3 < \|v_2\|$.

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