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A REGULARITY CRITERION FOR THE ANGULAR VELOCITY COMPONENT IN AXISYMMETRIC NAVIER-STOKES EQUATIONS

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Abstract. We study the non-stationary Navier-Stokes equations in the entire three-dimensional space under the assumption that the data are axisymmetric. We extend the regularity criterion for axisymmetric weak solutions given in [10].

1. Introduction

Consider the Navier-Stokes equations in the entire three-dimensional space; i.e., the system of PDE's

$$\begin{gathered}\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{0} \quad \text{in } (0,T)\times\mathbb{R}^3 \\ \operatorname{div}\mathbf{v} = 0 \quad \text{in } (0,T)\times\mathbb{R}^3 \\ \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) \quad \text{in } \mathbb{R}^3\,,\end{gathered} \tag{1.1}$$

where $\mathbf{v} : (0,T)\times\mathbb{R}^3 = \Omega_T \mapsto \mathbb{R}^3$ is the velocity field, $p : \Omega_T \mapsto \mathbb{R}$ is the pressure, $0 < T \le \infty$, $\nu > 0$ is constant viscosity coefficient and $\mathbf{v}_0$ is the initial velocity. To avoid technical difficulties, we take the forcing term on the right-hand side equal to zero. However, it is not difficult to formulate conditions on $\mathbf{f}$ under which the statement of Theorem 1 remains true. We leave this relatively easy exercise to the kind reader.

The question of smoothness and uniqueness of weak solutions to (1.1) is one of the most challenging problems in the theory of PDE's. The solution is known to be unique (in the class of all weak solutions satisfying the energy inequality) if it belongs to the class $L^{t,s}(\Omega_T)$ with $\frac{2}{t}+\frac{3}{s}\le 1$, $t\in[2,+\infty]$, $s\in[3,+\infty]$ (see [3, 12]). Moreover, if the weak solution belongs to $L^{t,s}(\Omega_T)$ with $\frac{2}{t}+\frac{3}{s}\le 1$, $t\in[2,+\infty]$,

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$s \in [3, +\infty]$ and the input data are "smooth enough" then the solution is smooth. (See [13] for $s > 3$, [2] for $s = 3$.)

In the case of the planar flow the weak solution is known to be unique and smooth as the data of the problem allow (see [7], [4]). Thus a natural question, namely what can be said about the axisymmetric flow, appears. The first results in this direction were obtained in the late sixties for $v_\varphi = 0$ (see [5, 15]) and later also in [6].

The case $v_\varphi \neq 0$, including the z-axis, was for the first time considered in the paper [9] where for $v_r \in L^{t,s}(\Omega_T)$ with $\frac{2}{t} + \frac{3}{s} \leq 1$, $t \in [2, +\infty]$, $s \in (3, +\infty]$ the smoothness and thus also the uniqueness in the class of weak solutions satisfying the energy inequality was obtained. In the same paper the authors prove a regularity criterion for the angular velocity component. This criterion was improved in [10]. The author shows the smoothness and the uniqueness in the class of weak solutions satisfying the energy inequality for $v_\varphi \in L^{t,s}(\Omega_T)$ with $t \in (2, +\infty]$, $s \in (4, +\infty]$, $\frac{2}{t} + \frac{3}{s} < 1$. See also [1], where the authors give several other smoothness criteria for the vorticity components. Another approach to this problem, based on the smallness of the swirl, can be found in [16]. Note that, except for the $L^{\infty,3}(\Omega_T)$ case, the criterion for v_r is optimal from the scaling argument (see [11] for discussion of this issue). On the other hand, for v_φ, we would like to have rather equality than strict inequality $\frac{2}{t} + \frac{3}{s} < 1$. Moreover, the restriction $s > 4$ seems to be artificial. In this paper, we will give a partial answer to the latter problem. Note that, unfortunately, we do not get s close to 3 and moreover, the criterion is not optimal from the viewpoint of the scaling. However, our main result reads as follows.

Theorem 1.1. *Let $\mathbf{v}$ be a weak solution to problem* (1.1) *satisfying the energy inequality with $\mathbf{v}_0 \in W^{2,2}(\mathbb{R}^3)$ so that $\nabla \mathbf{v}_0 \in L^1(\mathbb{R}^3)$ and $(v_0)_\varphi r \in L^\infty(\mathbb{R}^3)$. Let $\mathbf{v}_0$ be axisymmetric. Suppose further that the angular component v_φ of $\mathbf{v}$ belongs to $L^{t,s}(\Omega_T)$ for some $t \in \left(\frac{8s}{7s-24}, \infty\right]$, $s \in (\frac{24}{7}, 4]$, $\frac{2}{t} + \frac{3}{s} < \frac{7}{4} - \frac{3}{s}$. Then $(\mathbf{v}, p)$, where p is the corresponding pressure, is the axisymmetric strong solution to problem* (1.1) *which is unique in the class of all weak solutions satisfying the energy inequality.*

Note that the case $s > 4$ is successfully solved in [10]. Theorem 1.1 extends the result from [10] for $s \in (\frac{24}{7}, 4]$.

Under an axisymmetric solution we understand a pair $(\mathbf{v}, p)$ such that in cylindrical coordinates (r, φ, z), $r \in [0, \infty)$, $\varphi \in [0, 2\pi)$ and $z \in \mathbb{R}$, v_r, v_φ and v_z, considered in cylindrical coordinates, are independent of φ, and p, written in cylindrical coordinates, is also independent of φ.

2. Preliminaries

Denote by (v_r, v_φ, v_z) the cylindrical coordinates of the vector field $\mathbf{v}$ and by $(\omega_r, \omega_\varphi, \omega_z)$ the cylindrical coordinates of $\operatorname{curl} \mathbf{v}$, i.e. $\omega_r = -\frac{\partial v_\varphi}{\partial z}$, $\omega_\varphi = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}$ and $\omega_z = \frac{1}{r} \frac{\partial}{\partial r}(r v_\varphi)$ for $\mathbf{v}$ an axisymmetric field. Moreover, let $\mathbf{w} = (w_1, w_2, w_3)$ denote the cartesian coordinates of $\operatorname{curl} \mathbf{v}$.

We will use the standard notation for the Lebesgue spaces $L^p(\mathbb{R}^3)$ endowed with the standard norm $\|\cdot\|_{p,\mathbb{R}^3}$ and the Sobolev spaces $W^{k,p}(\mathbb{R}^3)$ endowed with the standard norm $\|\cdot\|_{k,p,\mathbb{R}^3}$. By $L^{t,s}(\Omega_T)$, $\Omega_T = (0, T) \times \mathbb{R}^3$ we understand the anisotropic Lebesgue space $L^t(0, T; L^s(\mathbb{R}^3))$. If no confusion can arise we skip writing $\mathbb{R}^3$ and Ω_T, respectively.

Vector-valued functions are printed boldfaced. Nonetheless, we do not distinguish between $L^q(\mathbb{R}^3)^3$ and $L^q(\mathbb{R}^3)$.

In order to keep a simple notation, all generic constants will be denoted by C; thus C can have different values from term to term, even in the same formula.

By $D\mathbf{v}$ we mean the gradient of $\mathbf{v}$ expressed in the cartesian coordinates, while ∇v_r denotes the derivatives with respect to r and z only ($\mathbf{v}$ is axisymmetric). Similarly for v_φ and v_z.

We will use the following inequalities. For the proofs of Lemmas 2.1–2.3 see [9].

Lemma 2.1. *Let $\mathbf{v}$ be a sufficiently smooth vector field. Then there exists a constant $C(p) > 0$, independent of $\mathbf{v}$, such that for $1 < p < \infty$*

$$\|D\mathbf{v}\|_p \le C(p)(\|\mathbf{w}\|_p + \|\operatorname{div}\mathbf{v}\|_p)\,. \tag{2.1}$$

Lemma 2.2. *Let $\mathbf{v}$ be a sufficiently smooth divergence-free axisymmetric vector field. Then there exist constants $C_i(p)$, $i = 1, 2$ and C_j, $j = 3, \dots, 7$ such that for $1 < p < \infty$,*

$$\|\nabla v_r\|_p + \Big\|\frac{v_r}{r}\Big\|_p \le C_1(p)\|\omega_\varphi\|_p \tag{2.2}$$

$$\Big\|\frac{\partial}{\partial r}\Big(\frac{v_r}{r}\Big)\Big\|_p \le C_3\|D^2\mathbf{v}\|_p \tag{2.3}$$

$$\|\nabla v_\varphi\|_p + \Big\|\frac{v_\varphi}{r}\Big\|_p \le C_4\|D\mathbf{v}\|_p \tag{2.4}$$

$$\Big\|\frac{\partial}{\partial r}\Big(\frac{v_\varphi}{r}\Big)\Big\|_p \le C_5\|D^2\mathbf{v}\|_p \tag{2.5}$$

$$\begin{aligned} C_2(p)\|D^2\mathbf{v}\|_p &\le \Big\|\frac{\omega_r}{r}\Big\|_p + \Big\|\frac{\omega_\varphi}{r}\Big\|_p + \|\nabla\omega_r\|_p + \|\nabla\omega_\varphi\|_p + \|\nabla\omega_z\|_p \\ &\le C_6\|D^2\mathbf{v}\|_p \end{aligned} \tag{2.6}$$

$$\Big\|\frac{\partial}{\partial r}\Big(\frac{\omega_\varphi}{r}\Big)\Big\|_p \le C_7\|D^3\mathbf{v}\|_p\,. \tag{2.7}$$

Lemma 2.3. *Let $(\mathbf{v}, p)$ be an axisymmetric smooth solution to the Navier-Stokes equations such that $(v_0)_\varphi r \in L^\infty(\mathbb{R}^3)$. Then also $v_\varphi r \in L^\infty(\Omega_T)$.*

Note that Lemma 2.3 indicates that the singularity may appear only on the $z-$axis, outside, the solution is smooth. However, this result follows easily from the well known fact that the one-dimensional Hausdorff measure of the singular set for the suitable weak solution is zero.

Lemma 2.4. *Let $\mathbf{v}$ be a sufficiently smooth axisymmetric vector field. Then to every $\varepsilon \in (0, 1]$ and $1 < p < \infty$ there exists $C(\varepsilon)$, independent of $\mathbf{v}$ such that*

$$\Big\|\frac{|\omega_\varphi|^p}{r^{p+2-\varepsilon}}\Big\|_1 \le C(\varepsilon)\Big(\Big\|\frac{\omega_\varphi}{r}\Big\|_p^p + \Big\|\nabla\Big(\Big|\frac{\omega_\varphi}{r}\Big|^{p/2}\Big)\Big\|_2^2\Big)\,. \tag{2.8}$$

The proof of the above lemma can be found in [10]. We will also use the following regularity criterion proved in [9].

Lemma 2.5. *Let $\mathbf{v}$ be a weak solution to problem (1.1) satisfying the energy inequality with $\mathbf{v}_0 \in W^{2,2}$, axisymmetric and divergence-free. Suppose that $v_r \in L^{t,s}$, $t \in [2, \infty)$, $s \in (3, \infty]$, $\frac{2}{t} + \frac{3}{s} \le 1$. Then $(\mathbf{v}, p)$, where p is the corresponding pressure, is an axisymmetric strong solution to problem (1.1) which is unique in the class of all weak solutions satisfying the energy inequality.*

The proof of Theorem 1.1 is similar to the proof of the regularity criterion in [10]. Thus we will also need the following lemma. Its proof is based on weighted estimates for singular integral operators and the restriction on k is due to the fact, that not all weights belong to the Muckenhoupt class. See [10, 14] for more information.

Lemma 2.6. *Let $1 < a < \infty$ and $0 \le k < \frac{2}{a}$. Then there exists a constant $C(a,k)$ such that*

$$\left\| \frac{v_r}{r^{1+k}} \right\|_a \le C(a,k) \left\| \frac{\omega_\varphi}{r^k} \right\|_a . \tag{2.9}$$

Finally, we will need the following result on the integrability of gradients of $\mathbf{v}$ with some p's less than 2 (see [8] for the proof)

Lemma 2.7. *Let moreover $D\mathbf{v}_0 \in L^1$. Then the weak solution to (1.1) also satisfies $D\mathbf{v} \in L^{\infty,1}$ and $D^2\mathbf{v} \in L^{p,1}$, $1 \le p < 2$.*

3. Proof of Theorem 1.1

The proof is based on the continuation argument. We have the following (more or less standard) result

Lemma 3.1. *Let $\mathbf{v}_0 \in W^{2,2}$. Then there exists $t_0 > 0$ and $(\mathbf{v}, p)$, a weak solution[1] to system (1.1), which is a strong solution on the time interval $(0, t_0)$ such that $\mathbf{v} \in L^2(0,t_0; W^{3,2}) \cap L^\infty(0,t_0; W^{2,2})$ with $\frac{\partial \mathbf{v}}{\partial t}$ and $\nabla p \in L^2(0,t_0;W^{1,2})$. Moreover, if $\mathbf{v}_0$ is axisymmetric then also the strong solution is axisymmetric.*

Now let $\mathbf{v}_0$ be as in Lemma 3.1 (axisymmetric). We define:

$$t^* = \sup\big\{t > 0 : \text{there exists an axisymmetric strong solution to (1.1) on } (0,t)\big\}$$

It follows from Lemma 3.1 that $t^* > 0$. Now let $\mathbf{v}$ be a weak solution to the Navier-Stokes system as in Theorem 1.1. Due to the uniqueness property (thus the energy inequality is required!), it coincides with the strong solution from Lemma 3.1 on any compact subinterval of $[0,t^*)$. There are two possibilities. Either $t^* = \infty$ and we have the global-in-time regular solution, or $t^* < \infty$. We will exclude the latter by showing that v_r satisfies on $(0,t^*)$ the assumptions of Lemma 2.5. To this aim we will essentially use both the information about the better regularity of one velocity component and the fact that the solution is axisymmetric.

Now, let $0 < \bar{t} < t^*$. Then on $(0,\bar{t})$ $(\mathbf{v}, p)$ is in fact a strong solution to the Navier-Stokes system. It is convenient to write the Navier-Stokes system in the cylindrical coordinates for our purpose.

Thus v_r, v_φ, v_z and p satisfy in $(0,\bar{t}) \times \mathbb{R}^3$ the system

$$\begin{gathered}
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \frac{1}{r} v_\varphi^2 + \frac{\partial p}{\partial r} - \nu \Big[\frac{1}{r} \frac{\partial}{\partial r}\big(r \frac{\partial v_r}{\partial r}\big) + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} \Big] = 0 \\
\frac{\partial v_\varphi}{\partial t} + v_r \frac{\partial v_\varphi}{\partial r} + v_z \frac{\partial v_\varphi}{\partial z} + \frac{1}{r} v_\varphi v_r - \nu \Big[\frac{1}{r} \frac{\partial}{\partial r}\big(r \frac{\partial v_\varphi}{\partial r}\big) + \frac{\partial^2 v_\varphi}{\partial z^2} - \frac{v_\varphi}{r^2} \Big] = 0 \\
\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} + \frac{\partial p}{\partial z} - \nu \Big[\frac{1}{r} \frac{\partial}{\partial r}\big(r \frac{\partial v_z}{\partial r}\big) + \frac{\partial^2 v_z}{\partial z^2} \Big] = 0 \\
\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0 .
\end{gathered}$$

[1]It means that $\mathbf{v}$ is the weak solution and p is the corresponding pressure which can be easily computed and, up to an additive constant, is uniquely determined.

Moreover, the vorticity components satisfy, in $(0,\bar t)\times\mathbb{R}^3$,

$$\frac{\partial\omega_r}{\partial t}+v_r\frac{\partial\omega_r}{\partial r}+v_z\frac{\partial\omega_r}{\partial z}-\frac{\partial v_r}{\partial r}\omega_r-\frac{\partial v_r}{\partial z}\omega_z-\nu\Big[\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial\omega_r}{\partial r}\Big)+\frac{\partial^2\omega_r}{\partial z^2}-\frac{\omega_r}{r^2}\Big]=0$$

$$\frac{\partial\omega_\varphi}{\partial t}+v_r\frac{\partial\omega_\varphi}{\partial r}+v_z\frac{\partial\omega_\varphi}{\partial z}-\frac{v_r}{r}\omega_\varphi+\frac{2}{r}v_\varphi\omega_r-\nu\Big[\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial\omega_\varphi}{\partial r}\Big)+\frac{\partial^2\omega_\varphi}{\partial z^2}-\frac{\omega_\varphi}{r^2}\Big]=0$$

$$\frac{\partial\omega_z}{\partial t}+v_r\frac{\partial\omega_z}{\partial r}+v_z\frac{\partial\omega_z}{\partial z}-\frac{\partial v_z}{\partial z}\omega_z-\frac{\partial v_z}{\partial r}\omega_r-\nu\Big[\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial\omega_z}{\partial r}\Big)+\frac{\partial^2\omega_z}{\partial z^2}\Big]=0\,.$$

The main idea of the proof of Theorem 1.1 is very similar to the idea presented in [10]; we will namely combine the estimates of ω_φ in $L^{\infty,p}$ with the estimates of $\frac{\omega_\varphi}{r}$ in $L^{\infty,q}$ with the idea to get an estimate for v_r which is in the range $\frac{2}{t}+\frac{3}{s}\le 1$.

Let us take $p\in(1,2)$ and multiply the equation for ω_φ by $\frac{\omega_\varphi}{|\omega_\varphi|^{2-p}}$ and integrate (with respect to the measure $rdrdz$). We get (in what follows, $\int f$ denotes $\int_{-\infty}^{\infty}\int_0^\infty frdrdz$)

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\omega_\varphi\|_p^p+\nu\int\frac{|\omega_\varphi|^p}{r^2}+\frac{4(p-1)}{p^2}\nu\int|\nabla|\omega_\varphi|^{p/2}|^2\\&=\int\frac{v_r}{r}|\omega_\varphi|^p+\int\frac{2}{r}v_\varphi\frac{\partial v_\varphi}{\partial z}\frac{\omega_\varphi}{|\omega_\varphi|^{2-p}}\,.\end{aligned}\tag{3.1}$$

For details of the integration by parts, see [6]. Note that all terms are finite because $\omega_\varphi(t)\in L^1\cap L^2$. Next, let us multiply the equation for ω_φ by $\psi(r)|\frac{\omega_\varphi}{r}|^{q-2}\frac{\omega_\varphi}{r}\frac{1}{r^{1-\delta}}$, $\delta>0$ and $\psi(r)$ a cut-off function equal to zero near $r=0$. Now we integrate the equality over $\mathbb{R}^3$ then pass first with $\psi(r)$ to the identity function and finally with δ to zero. Note that we cannot take directly $\delta=0$ as some integrals cannot be controlled, cf. [6]. We get

$$\frac{1}{q}\frac{d}{dt}\Big\|\frac{\omega_\varphi}{r}\Big\|_q^q+\frac{4(q-1)}{q^2}\nu\int\Big|\nabla\Big(\Big|\frac{\omega_\varphi}{r}\Big|^{q/2}\Big)\Big|^2\le\Big|\int\frac{2}{r}v_\varphi\frac{\partial v_\varphi}{\partial z}\frac{\omega_\varphi}{|\omega_\varphi|^{2-q}}\frac{1}{r^q}\Big|.\tag{3.2}$$

To prove Theorem 1.1 we sum (3.1) and (3.2) and estimate all terms on the right-hand side with $q=\frac{5}{6}p$. First we will estimate the term $I_1=\int\frac{v_r}{r}|\omega_\varphi|^p$, where we basically follow [10]. Using the Hölder inequality, the interpolations, the Sobolev embedding inequality, Lemma 2.2 and the Young inequality we finally get

$$I_1\le\delta\big\|\nabla\big(\big|\frac{\omega_\varphi}{r}\big|^{q/2}\big)\big\|_2^2+C(\delta)\|\omega_\varphi\|_2^2\big(\|\omega_\varphi\|_p^p+\big\|\frac{\omega_\varphi}{r}\big\|_q^q\big)$$

with arbitrarily small positive δ. The first term can be included into the left-hand side while the second term can be estimated later on, using the Gronwall inequality.

Next we want to estimate the other term on the right-hand side of (3.1), namely I_2. However, the term $I_2=\int\frac{2}{r}v_\varphi\frac{\partial v_\varphi}{\partial z}\frac{\omega_\varphi}{|\omega_\varphi|^{2-p}}$ can be estimated same way as the term on the right-hand side of (3.2), $I_3=|\int\frac{2}{r}v_\varphi\frac{\partial v_\varphi}{\partial z}\frac{\omega_\varphi}{|\omega_\varphi|^{2-q}}\frac{1}{r^q}|$. This is due to the fact that main problems are near the z-axis and I_2 is of lower order than I_3. (Here we also use the fact that $v_\varphi r\in L^\infty(Q_T)$.) Choose $\varepsilon>0$.

$$I_3\le 2\int\big(\frac{|\omega_\varphi|^{q-1}}{r^X}\big)\big(\frac{|v_\varphi|^\alpha}{r^Y}\big)\big|\frac{\partial v_\varphi}{\partial z}\big|\big(\frac{|v_\varphi|^{1-\alpha}}{r^Z}\big),$$

where

$$X = q + 1 - \frac{2}{q} - \varepsilon, \quad Y = \big(\frac{1+\varepsilon}{2} + \frac{1}{q}\big)(2-q), \quad Z = \frac{2}{q} + \varepsilon - Y, \quad \alpha = 2-q.$$

Using the Young inequality we get

$$I_3 \le \delta \int \frac{\omega_\varphi^q}{r^{\frac{qX}{q-1}}} + C(\delta)\Big(\int \frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)+2}} + \int \big|\frac{\partial}{\partial z}\big|\frac{v_\varphi^q}{r^{\frac{q(1+\varepsilon)}{2}}}\big|\big|^2\Big).$$

Since $\frac{qX}{q-1} = q + 2 - \varepsilon\frac{q}{q-1}$, we can use Lemma 2.4 and

$$\begin{aligned} I_3 &\le \delta C(\varepsilon)\Big(\int \big|\nabla\big(\big|\frac{\omega_\varphi}{r}\big|^{q/2}\big)\big|^2 + \int \big|\frac{\omega_\varphi}{r}\big|^q\Big) \\ &\quad + C(\delta)\Big(\int \frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)+2}} + \int \big|\frac{\partial}{\partial z}\big|\frac{v_\varphi^q}{r^{\frac{q(1+\varepsilon)}{2}}}\big|\big|^2\Big). \end{aligned}$$

Taking sufficiently small δ we can include the first term to the left-hand side of (3.2), the second term can be later estimated using the Gronwall inequality. We have to deal with the last two terms on the right-hand side. Summing up the estimates of I_1 and I_3,

$$\begin{aligned} &\frac{d}{dt}\big(\|\omega_\varphi\|_p^p + \|\frac{\omega_\varphi}{r}\|_q^q\big) + \int \Big(\frac{|\omega_\varphi|^p}{r^2} + |\nabla|\omega_\varphi|^{p/2}|^2 + |\nabla(|\frac{\omega_\varphi}{r}|^{q/2})|^2\Big) \\ &\le C\int |\frac{\omega_\varphi}{r}|^q + C\|\omega_\varphi\|_2^2\big(\|\omega_\varphi\|_p^p + \|\frac{\omega_\varphi}{r}\|_q^q\big) \\ &\quad + C\Big(\int \frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)+2}} + \int |\nabla|\frac{v_\varphi^q}{r^{\frac{q(1+\varepsilon)}{2}}}||^2\Big). \end{aligned} \tag{3.3}$$

To estimate the last two terms, we test the equation for v_φ by $|v_\varphi|^{2q-2}v_\varphi/r^{q(1+\varepsilon)}$. We get

$$\begin{aligned} &\frac{2}{q}\frac{d}{dt}\int \frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)}} + \frac{2q-1}{q^2}\nu\int |\nabla|\frac{v_\varphi^q}{r^{\frac{q(1+\varepsilon)}{2}}}||^2 + \frac{(2q)^2 - q^2(1+\varepsilon)^2}{(2q)^2}\nu\int \frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)+2}} \\ &= \big(-1 - \frac{1+\varepsilon}{2}\big)\int \frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)}}\frac{v_r}{r}, \end{aligned}$$

i.e. together with (3.3),

$$\begin{aligned} &\frac{d}{dt}\Big(\|\omega_\varphi\|_p^p + \|\frac{\omega_\varphi}{r}\|_q^q + \int \frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)}}\Big) \\ &+ \int \Big(\frac{|\omega_\varphi|^p}{r^2} + |\nabla|\omega_\varphi|^{p/2}|^2 + |\nabla(|\frac{\omega_\varphi}{r}|^{q/2})|^2 + |\nabla|\frac{v_\varphi^q}{r^{\frac{q(1+\varepsilon)}{2}}}||^2 + \frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)+2}}\Big) \\ &\le C\int |\frac{\omega_\varphi}{r}|^q + C\int \frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)}}\frac{|v_r|}{r} + C\|\omega_\varphi\|_2^2\big(\|\omega_\varphi\|_p^p + \|\frac{\omega_\varphi}{r}\|_q^q\big). \end{aligned} \tag{3.4}$$

Denoting by I_4 the second integral on the right-hand side, we have

$$I_4 = \int \big(\frac{|v_r|}{r^{1+k}}\big)\big(\frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)}}\big)^\alpha\big(\frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)+2}}\big)^\beta|v_\varphi|^\gamma,$$

where $k \in [0,1]$, $\gamma = \frac{5+k}{3}$, $\beta = \frac{5}{12}(1-k) + \varepsilon\frac{5+k}{12}$ and $\alpha = \frac{5k+7}{12} - \frac{5+k}{5p} - \varepsilon\frac{5+k}{12}$. Recall that $q = \frac{5}{6}p$. Let $1 < a < \frac{2}{k}$. Hence

$$I_4 \le \big\|\frac{v_r}{r^{1+k}}\big\|_a\big\|\frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)}}\big\|_1^\alpha\big\|\frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)+2}}\big\|_1^\beta\|v_\varphi\|_{\frac{\gamma a}{a(1-\alpha-\beta)-1}}^\gamma. \tag{3.5}$$

Using Lemma 2.6 we get

$$\|\frac{v_r}{r^{1+k}}\|_a \le C\|\frac{\omega_\varphi}{r^k}\|_a$$

and furthermore

$$\|\frac{\omega_\varphi}{r^k}\|_a \le \|\omega_\varphi\|_a^{1-k}\|\frac{\omega_\varphi}{r}\|_a^k. \tag{3.6}$$

Since $k \in (0,1]$, $1 < a < \frac{2}{k}$, under the assumption that $p < a < \frac{5}{2}p$ (which will be verified later) we have

$$\|\omega_\varphi\|_a \le \|\omega_\varphi\|_p^{\frac{3p-a}{2a}}\|\omega_\varphi\|_{3p}^{\frac{3(a-p)}{2a}}, \tag{3.7}$$

$$\|\frac{\omega_\varphi}{r}\|_a \le \|\frac{\omega_\varphi}{r}\|_q^{\frac{5p-2a}{4a}}\|\frac{\omega_\varphi}{r}\|_{3q}^{\frac{6a-5p}{4a}}, \tag{3.8}$$

Using (3.6), (3.7) and (3.8), we get

$$I_4 \le \|\omega_\varphi\|_p^{\lambda_1}\|\omega_\varphi\|_{3p}^{\lambda_2}\|\frac{\omega_\varphi}{r}\|_q^{\lambda_3}\|\frac{\omega_\varphi}{r}\|_{3q}^{\lambda_4}\|\frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)}}\|_1^{\alpha}\|\frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)+2}}\|_1^{\beta}\|v_\varphi\|^{\gamma}_{\frac{\gamma a}{a(1-\alpha-\beta)-1}}, \tag{3.9}$$

where

$$\lambda_1 = \frac{3p-a}{2a}(1-k), \quad \lambda_2 = \frac{3(a-p)}{2a}(1-k),$$
$$\lambda_3 = \frac{5p-2a}{4a}k, \quad \lambda_4 = \frac{6a-5p}{4a}k.$$

Denote $B = \frac{10ap}{10ap(1-\beta)+15p-15a-3ka}$. Then

$$\begin{aligned} I_4 &\le \delta\Big(\|\omega_\varphi\|_{3p}^p + \|\frac{\omega_\varphi}{r}\|_{3q}^q + \|\frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)+2}}\|_1\Big) \\ &\quad + C(\delta)\|v_\varphi\|^{\gamma B}_{\frac{\gamma a}{a(1-\alpha-\beta)-1}}\|\omega_\varphi\|_p^{\lambda_1 B}\|\frac{\omega_\varphi}{r}\|_q^{\lambda_3 B}\|\frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)}}\|_1^{\alpha B} \end{aligned}$$

and consequently

$$\begin{aligned} I_4 &\le \delta C\Big(\|\nabla(|\omega_\varphi|^{p/2})\|_2^2 + \|\nabla(|\frac{\omega_\varphi}{r}|^{q/2})\|_2^2 + \|\frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)+2}}\|_1\Big) \\ &\quad + C(\delta)\|v_\varphi\|^{\gamma B}_{\frac{\gamma a}{a(1-\alpha-\beta)-1}}\Big(\|\omega_\varphi\|_p^p + \|\frac{\omega_\varphi}{r}\|_q^q + \|\frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)}}\|_1\Big). \end{aligned}$$

Thus if $v_\varphi \in L^{t,s}(\Omega_T)$ with $t = \gamma B$ and $s = \frac{\gamma a}{a(1-\alpha-\beta)-1}$ we get, using the Gronwall inequality

$$\begin{aligned} &\Big(\|\omega_\varphi\|_p^p + \|\frac{\omega_\varphi}{r}\|_q^q + \int \frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)}}\Big)(t) \\ &+ \int_0^t\int\Big(\frac{|\omega_\varphi|^p}{r^2} + |\nabla|\omega_\varphi|^{p/2}|^2 + |\nabla(|\frac{\omega_\varphi}{r}|^{q/2})|^2 + |\nabla|\frac{v_\varphi^q}{r^{\frac{q(1+\varepsilon)}{2}}}||^2 + \frac{|v_\varphi|^{2q}}{r^{q(1+\varepsilon)+2}}\Big) \\ &\le C(\mathbf{v}_0). \end{aligned} \tag{3.10}$$

Using this estimate we will be able to show $v_r \in L^{3,9}(\Omega_T)$. To this aim it is sufficient to verify that $\omega_\varphi \in L^{\infty,\frac{3}{2}}$ and $\nabla|\omega_\varphi|^{\frac{3}{4}} \in L^{2,2}$. Note that we cannot simply use (3.10) with $p = \frac{3}{2}$, because our t and s imply that p is very close to 1.

We will test the equation for ω_φ by $|\omega_\varphi|^{-1/2}\omega_\varphi$. Then with (3.10) at disposal

$$\begin{aligned}&\frac{2}{3}\frac{d}{dt}\|\omega_\varphi\|_{\frac32}^{\frac32} + \nu\int\frac{|\omega_\varphi|^{\frac32}}{r^2} + \frac{8}{9}\nu\int|\nabla(|\omega_\varphi|^{\frac34})|^2\\ &\le \Big|\int\frac{v_r}{r}|\omega_\varphi|^{\frac32}\Big| + \Big|\int\frac{2}{r}v_\varphi\frac{\partial v_\varphi}{\partial z}|\omega_\varphi|^{-1/2}\omega_\varphi\Big|\\ &\equiv I_5 + I_6\end{aligned}$$

To estimate I_5 recall that we know that there exists $\eta_0 > 0$ such that $\frac{\omega_\varphi}{r}$ is bounded in $L^{\infty,1+\eta}$ and in $L^{1+\eta,3(1+\eta)}$ for $0 < \eta \le \eta_0$. Thus

$$I_5 \le C\|\frac{\omega_\varphi}{r}\|_{3(1+\eta)}\|v_r\|_{\frac{3(1+\eta)}{1+2\eta}}\|\omega_\varphi\|_{\frac32}^{\frac12} \le C\|\frac{\omega_\varphi}{r}\|_{3(1+\eta)}\|\omega_\varphi\|_{\frac32}^{\frac32-\frac{2\eta}{1+\eta}}\|v_r\|_2^{\frac{2\eta}{1+\eta}}$$

and we can estimate I_5 using the Gronwall inequality. Finally, to estimate I_6 we will use the fact that there exists $\eta_1 > 0$ such that for $0 < \eta \le \eta_1$, $\frac{|v_\varphi|^{1+\eta}}{r^{\frac{1+\eta}{2}(1+\varepsilon)}} \in L^{\infty,2}$ and its gradient is bounded in $L^{2,2}$. In fact we will use the same information for $|\frac{v_\varphi}{r^{\frac12}}|^{1+\eta}$, but this information is weaker since the main problems are near the axis (recall that due to Lemma 2.3 $v_\varphi r \in L^\infty(\Omega_T)$).

$$\begin{aligned}I_6 &= \Big|\int\frac{2}{r}v_\varphi\frac{\partial v_\varphi}{\partial z}|\omega_\varphi|^{-1/2}\omega_\varphi\Big|\\ &\le C\Big|\int|\omega_\varphi|^{\frac12}(\frac{v_\varphi}{\sqrt r})^{1-\eta}\frac{\partial}{\partial z}(\frac{v_\varphi}{\sqrt r})^{1+\eta}\Big|\\ &\le C\|\omega_\varphi\|_{\frac32}^{\frac12}\|(\frac{v_\varphi}{\sqrt r})^{6(1-\eta)}\|_1^{\frac16}\|\nabla|\frac{v_\varphi}{\sqrt r}|^{1+\eta}\|_2\\ &= C\|\omega_\varphi\|_{\frac32}^{\frac12}\|(\frac{v_\varphi}{\sqrt r})^{1+\eta}\|_r^{\frac{r}{6}}\|\nabla|\frac{v_\varphi}{\sqrt r}|^{1+\eta}\|_2,\end{aligned}$$

where $r = 6\frac{1-\eta}{1+\eta} < 6$. Thus I_6 can be again estimated by means of the Gronwall inequality.

Since

$$\begin{gathered}s = \frac{\gamma a}{a(1-\alpha-\beta)-1} = \frac{10ap\gamma}{6a\gamma-10p},\\ t = \gamma B = \frac{10ap\gamma}{10ap(1-\beta)+15p-15a-3ka},\\ \gamma = \frac{5+k}{3},\end{gathered}$$

we compute

$$\frac{2}{t} + \frac{6}{s} = \frac{7+5k}{2(5+k)} + \frac{9}{5p} - \frac{9}{a(5+k)} - \frac{\varepsilon}{2}.$$

Now, using that $a < \frac{2}{k}$, $\varepsilon > 0$, we get

$$\frac{2}{t} + \frac{6}{s} < \frac{7-4k}{2(5+k)} + \frac{9}{5p}.$$

Recall that $\alpha = \frac{5k+7}{12} - \frac{5+k}{5p} - \varepsilon\frac{5+k}{12}$ needs to be greater than zero. This implies $p > \frac{12}{5}\frac{5+k}{5k+7}$. Using this we get

$$\frac{2}{t} + \frac{6}{s} < \frac{7-4k}{2(5+k)} + \frac{9}{5p} < \frac{7}{4}. \tag{3.11}$$

Note that taking a sufficiently close to $\frac{2}{k}$ and p close to $\frac{12}{5}\frac{5+k}{5k+7}$, we get $\frac{2}{t}+\frac{6}{s}$ arbitrarily close to $\frac{7}{4}$. Moreover we need $\frac{1}{t}\geq 0$. Thus

$$\frac{1}{t}=\frac{7+5k}{4(5+k)}+\frac{9}{2a(5+k)}-\frac{9}{10p}-\frac{\varepsilon}{4}\geq 0$$

and consequently $p>\frac{18}{35}\frac{5+k}{1+2k}$. The lowest s we get taking $\frac{1}{t}=0$ and α almost equal to zero. Therefore we take

$$p=\frac{12}{5}\frac{5+k}{5k+7}=\frac{18}{35}\frac{5+k}{1+2k},$$

which implies $k=\frac{7}{13}$, $p=\frac{48}{35}$ and $s=\frac{24}{7}+\varepsilon$ for arbitrarily small ε. Note that since $k\in(\frac{7}{13},1)$ and $p\in(\frac{6}{5},\frac{48}{35})$, taking a close to $\frac{2}{k}$, we indeed have $p<a<\frac{5}{2}p$. The theorem is proved.

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