Electronic Journal of Differential Equations, Vol. 2007(2007), No. 109, pp. 1–25.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

INITIAL-BOUNDARY VALUE PROBLEMS FOR NONLINEAR PSEUDOPARABOLIC EQUATIONS IN A CRITICAL CASE

ELENA I. KAIKINA

Abstract. We study nonlinear pseudoparabolic equations, on the half-line in a critical case,

$$\begin{gathered}\partial_t(u-u_{xx})-\alpha u_{xx}=\lambda|u|u, \quad x\in\mathbb{R}^+,\ t>0,\\ u(0,x)=u_0(x), \quad x\in\mathbb{R}^+,\\ u(t,0)=0,\end{gathered}$$

where $\alpha>0$, $\lambda\in\mathbb{R}$. The aim of this paper is to prove the existence of global solutions to the initial-boundary value problem and to find the main term of the asymptotic representation of solutions.

1. Introduction

We study the following nonlinear pseudoparabolic equation on the half line, with Dirichlet boundary condition,

$$\begin{gathered}\partial_t(u-u_{xx})-\alpha u_{xx}=\lambda|u|u, \quad x\in\mathbb{R}^+,\ t>0,\\ u(0,x)=u_0(x), \quad x\in\mathbb{R}^+,\\ u(t,0)=0,\end{gathered}\tag{1.1}$$

where $\alpha>0$ and $\lambda\in\mathbb{R}$.

The Cauchy problem for nonlinear pseudoparabolic type equations was studied in many papers (for example, see [11, 12, 13, 14, 15, 26, 30, 31]). The large time asymptotics of solutions to the Cauchy problem was obtained in papers [1]-[4], [18, 22, 23, 27, 28, 33].

In this paper we study the initial boundary-value problem (1.1) in a critical case, when the nonlinear term of equation (1.1) has the same time decay rate as the linear terms. Recently much attention was drown to the study of the global existence and large time asymptotic behavior of solutions to the Cauchy problems for nonlinear equations in the critical cases (see papers [6]-[10, 20, 21, 24, 25] and literature cited therein). A general theory of nonlinear nonlocal equations on a half-line was developed in the book [16], however the case of nonanalytic symbols $K(p)$ in the right-hand complex plane was not studied previously. In the present paper we fill this gap, considering as example the pseudoparabolic type equation

2000 *Mathematics Subject Classification.* 35Q35.
Key words and phrases. Dissipative nonlinear evolution equation; Sobolev equation; large time asymptotic behavior.
©2007 Texas State University - San Marcos.
Submitted March 22, 2007. Published August 7, 2007.

(1.1) with a symbol $K(p) = \alpha p^2/(1-p^2)$. We construct the Green operator for problem (1.1) posing some necessary condition at the singularity point $p = 1$ of the symbol $K(p)$. Another difficulty which we overcome in the present paper is in evaluating the contribution of the boundary data into the large time asymptotic behavior of solutions. In this paper we will prove that in the case of the initial-boundary value problem due to boundary data the solution obtains an additional decay comparing with the case of Cauchy problem. As a result the main term of the asymptotic expansion does not depend on the mean value of the solution instead it is determined by the first moment of the solution. Thus we have to estimate the evolution of the first moments to obtain an optimal time decay estimate of the solution. Also we are interested in the case of large initial data. Using the energy type a-priori estimates for the first moment of the solution we are able to remove the smallness condition for the initial data. The asymptotic behavior of solutions is founded by the standard way developed in the book [19].

Below $\hat{\phi}$ is the Laplace transform of ϕ defined by

$$\hat{\phi}(\xi) = \int_0^{+\infty} e^{-x\xi}\phi(x)dx,$$

Here

$$\mathcal{L}^{-1}\hat{\phi}(\xi) = (2\pi i)^{-1}\int_{-i\infty}^{i\infty} e^{x\xi}\hat{\phi}(\xi)d\xi$$

is the inverse Laplace transform of ϕ. By $\mathbf{C}(\mathbf{I};\mathbf{B})$ we denote the space of continuous functions from a time interval $\mathbf{I}$ to the Banach space $\mathbf{B}$. The usual Lebesgue space is denote by $\mathbf{L}^p(\mathbb{R}^+)$, $1 \le p \le \infty$, the weighted Lebesgue space $\mathbf{L}^{p,a}(\mathbb{R}^+)$ is defined by

$$\mathbf{L}^{p,a}(\mathbb{R}^+) = \left\{\phi \in \mathbf{L}^p(\mathbb{R}^+); \|\phi\|_{\mathbf{L}^{p,a}} = \|\langle x\rangle^a \phi\|_{\mathbf{L}^p} < \infty\right\},$$

where $\langle x\rangle = \sqrt{1+|x|^2}$, $a \ge 0$. Weighted Sobolev spaces we define as follows

$$\mathbf{W}_p^{k,a}(\mathbb{R}^+) = \Big\{\phi \in \mathbf{L}^p(\mathbf{R}^+); \|\phi\|_{\mathbf{W}_p^{k,a}} = \sum_{j=0}^{k}\|\partial^j\phi\|_{\mathbf{L}^{p,a}} < \infty\Big\},$$

where $k \ge 0$, $a \ge 0$, $1 \le p \le \infty$.Also we denote by $\mathbf{H}^{k,a}(\mathbb{R}^+) = \mathbf{W}_2^{k,a}(\mathbb{R}^+)$. Define

$$|\lambda| t\int_0^{+\infty} x(G_0(t,x))^2 dx = \eta,$$

where the heat kernel

$$G_0(t,x) = (4\pi\alpha t)^{-1/2}\frac{x}{\alpha t}e^{-\frac{x^2}{4\alpha t}}.$$

Denote

$$g(t) = 1 + |\theta|\eta\log(1+t), \theta = \int_0^{+\infty} x u_0(x)dx.$$

Now we state the results of this paper.

Theorem 1.1. *Assume that* $\lambda\theta \le -C\varepsilon < 0$. *Let the initial data* $u_0 \in \mathbf{L}^\infty(\mathbb{R}^+) \cap \mathbf{L}^{1,1+a}(\mathbb{R}^+)$, $a \in (0,1]$ *are small* $\|u_0\|_{\mathbf{L}^\infty} + \|u_0\|_{\mathbf{L}^{1,1+a}} \le \varepsilon$. *Then the initial-boundary value problem* (1.1) *has a unique global solution*

$$u \in \mathbf{C}([0,\infty); \mathbf{L}^\infty(\mathbb{R}^+) \cap \mathbf{L}^{1,1+a}(\mathbb{R}^+))$$

satisfying the time decay estimate

$$\|u(t)-\theta G_0(t)g^{-1}(t)\|_{\mathbf{L}^\infty} \le C\langle t\rangle^{-1}g^{-2}(t). \tag{1.2}$$

Using the method of papers [17, 18] we can remove the smallness condition on the initial data $u_0(x)$.

Theorem 1.2. *Assume the initial data* $u_0 \in \mathbf{W}^{2,0}_\infty(\mathbb{R}^+)\cap \mathbf{W}^{2,0}_1(\mathbf{R}^+)\cap \mathbf{L}^{1,1+a}(\mathbb{R}^+)$, $0<a\le 1$, *are such that* $\lambda\theta<0$. *Then the initial-boundary value problem* (1.1) *has a unique global solution*

$$u(t,x)\in \mathbf{C}([0,\infty);\mathbf{L}^\infty(\mathbb{R}^+)\cap \mathbf{L}^{1,1+a}(\mathbb{R}^+))$$

satisfying the time decay estimate

$$\|u(t)-\theta G_0(t)g^{-1}(t)\|_{\mathbf{L}^\infty} \le C\langle t\rangle^{-1}g^{-2}(t)\log\log g(t). \tag{1.3}$$

We start with a section where we obtain preliminary estimates for the linearized initial boundary-value problem corresponding to (1.1). Also we prove a local existence theorem. In the next two sections we prove Theorems 1.1 and 1.2 respectively.

2. Preliminaries

2.1. **Green operator.** Consider the linear initial boundary-value problem on half-line

$$\begin{gathered}\partial_t(u-u_{xx})-\alpha u_{xx}=f(t,x),\quad x\in\mathbb{R}^+,\ t>0,\\ u(0,x)=u_0(x),\quad x\in\mathbb{R}^+\\ u(t,0)=0.\end{gathered}\tag{2.1}$$

Taking Laplace transform with respect to space variable we get

$$\begin{gathered}\partial_t\Big(\widehat{u}-p^2\Big(\widehat{u}-\sum_{j=1}^{2}\frac{\partial_x^{j-1}u(t,0)}{p^j}\Big)\Big)-\alpha p^2\Big(\widehat{u}-\sum_{j=1}^{2}\frac{\partial_x^{j-1}u(t,0)}{p^j}\Big)=\widehat{f}(t,p),\\ \widehat{u}(0,p)=\widehat{u}_0(p),\\ u(t,0)=0.\end{gathered}\tag{2.2}$$

We rewrite

$$(1-p^2)\partial_t\widehat{u}-\alpha p^2\widehat{u}=f_1(t,p), \tag{2.3}$$

where

$$f_1(t,p)=\widehat{f}(t,p)-p^2\sum_{j=1}^{2}\frac{\partial_x^{j-1}u_t(0,t)}{p^j}-\alpha p^2\sum_{j=1}^{2}\frac{\partial_x^{j-1}u(t,0)}{p^j}.$$

Integrating (2.3) with respect to time, we obtain

$$\widehat{u}(t,p)=e^{-K(p)t}\widehat{u}_0(p)+\int_0^t e^{-K(p)(t-\tau)}\frac{1}{1-p^2}f_1(\tau,p)d\tau, \tag{2.4}$$

where

$$K(p)=-\frac{\alpha p^2}{1-p^2}.$$

Note that symbol $K(p)$ is not analytic in right-half complex plane. Therefore we have for solution $u(t,x)$

$$u(t,x)=\frac{1}{2\pi i}\int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty}e^{px}\widehat{u}(t,p)dp.$$

To satisfy zero condition for $x \to +\infty$ we need to have

$$\operatorname{res}(\widehat{u}(t,p),1) = 0. \tag{2.5}$$

We have by definition

$$\begin{aligned} \operatorname{res}(\widehat{u}(t,p),1) &= \lim_{\rho\to 0}\int_{C_{\rho,1}} \widehat{u}(t,p)dp \\ &= \lim_{\rho\to 0}\int_{C_{\rho,1},p\in D^+} \widehat{u}(t,p)dp + \lim_{\rho\to 0}\int_{C_{\rho,1},p\in D^-} \widehat{u}(t,p)dp, \end{aligned}$$

where

$$C_{\rho,1} = \left\{p\in\mathbb{C} : p = 1+\rho e^{i\phi}, \rho>0, \phi\in[0,2\pi)\right\},$$

and by D^+, D^- we denote domains where $\operatorname{Re} K(p) > 0$ and $\operatorname{Re} K(p) < 0$. Since for $p \in C_{\rho,1}$

$$K(p) = -\frac{\alpha(1+\rho e^{i\phi})^2}{\rho e^{i\phi}(2+\rho e^{i\phi})},$$

it is easy to see that

$$\lim_{\rho\to 0}\int_{C_{\rho,1},p\in D^+} \widehat{u}(t,p)dp = 0.$$

We rewrite formula (2.4) in a domain, where $\operatorname{Re} K(p) < 0$ in the form

$$\begin{aligned} \widehat{u}(t,p) = e^{-K(p)t}(\widehat{u}_0(p) + \int_0^{+\infty} e^{-K(p)(t-\tau)}\frac{1}{1-p^2} f_1(\tau,p)d\tau) \\ - \int_t^{+\infty} e^{-K(p)(t-\tau)}\frac{1}{1-p^2} f_1(\tau,p)d\tau. \end{aligned}$$

We have

$$\lim_{\rho\to 0}\int_{C_{\rho,1},p\in D^-} dp\int_t^{+\infty} e^{-K(p)(t-\tau)}\frac{1}{1-p^2} f_1(\tau,p)d\tau = 0.$$

To satisfy (2.5) we have to put the following condition for all $p\in D^-, \operatorname{Re} p > 0$

$$\widehat{u}_0(p) + \int_0^{+\infty} e^{K(p)\tau}\frac{1}{1-p^2} f_1(\tau,p)d\tau = 0. \tag{2.6}$$

Note that function f_1 includes two unknown boundary data $u(t,0)$ and $u_x(0,t)$. Also we have two roots $\phi_j(\xi)$ of equation $\xi = -K(p)$. By direct calculation we obtain

$$\phi_1 = \sqrt{\frac{\xi}{\xi+\alpha}} \quad \text{and} \quad \phi_2 = -\sqrt{\frac{\xi}{\xi+\alpha}}.$$

Since we are interested in $\operatorname{Re}\xi > 0$ and $\operatorname{Re} p > 0$ making the change of variable $\xi = -K(p)$ we rewrite condition (2.6) as one equation with two unknown boundary data $u(t,0)$ and $u_x(t,0)$

$$\widehat{u}_0(\phi_1) + \frac{1}{1-\phi_1^2}\int_0^{+\infty} e^{-\xi\tau} f_1(\tau,\xi)d\tau = 0, \tag{2.7}$$

where

$$f_1(\tau,\xi) = \widehat{f}(\tau,\phi_1) - \phi_1^2\sum_{j=1}^{2}\frac{\partial_x^{j-1}u_\tau(\tau,0)}{\phi_1^j} - \alpha\phi_1^2\sum_{j=1}^{2}\frac{\partial_x^{j-1}u(\tau,0)}{\phi_1^j}.$$

So we have to put only one boundary conditions in the problem (2.1) and rest boundary data we will find from equation (2.7). Putting $u(t,0)=0$ in equation (2.7) we get

$$\mathcal{L}(\widehat{u}_{tx}(t,0)+\alpha u_x(t,0)) = (1-\phi_1^2)\widehat{u}_0(\phi_1)+\widehat{\widehat{f}}(\xi,\phi_1).$$

So that

$$\begin{aligned}
\widehat{u}_{tx}(t,0)+\alpha u_x(t,0) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi t}((1-\phi_1^2)\widehat{u}_0(\phi_1)+\widehat{\widehat{f}}(\xi,\phi_1))d\xi \\
&= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi t}(1-\frac{\xi}{\xi+\alpha})(\widehat{u}_0(\phi_1)+\widehat{\widehat{f}}(\xi,\phi_1))d\xi \\
&= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi t}(\frac{\alpha\widehat{u}_0(\phi_1)}{\xi+\alpha}+\widehat{\widehat{f}}(\xi,\phi_1))d\xi.
\end{aligned}$$

Substituting this representation into (2.4) we obtain

$$\widehat{u}(t,p) = I_1(t,p)+I_2(t,p), \tag{2.8}$$

where

$$\begin{aligned}
I_1(t,p) &= e^{-K(p)t}\widehat{u}_0(p) - \frac{\alpha}{1-p^2}\frac{1}{2\pi i}\int_0^t d\tau e^{-K(p)(t-\tau)}\int_{-i\infty}^{i\infty} e^{\xi\tau}\frac{\widehat{u}_0(\phi_1)}{\xi+\alpha}d\xi \\
&= J_1+J_2
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
I_2(t,p) = \frac{1}{1-p^2}\Big(&\int_0^t e^{-K(p)(t-\tau)}f(\tau,p)d\tau \\
&- \frac{1}{2\pi i}\int_0^t d\tau e^{-K(p)(t-\tau)}\int_{-i\infty}^{i\infty} e^{\xi\tau}\widehat{\widehat{f}}(\xi,\phi_1)d\xi\Big).
\end{aligned} \tag{2.10}$$

Now we consider I_1 in the representation (2.8). Changing the order of integration,

$$\begin{aligned}
J_2 &= \frac{\alpha}{1-p^2}\frac{1}{2\pi i}\int_0^t d\tau e^{-K(p)(t-\tau)}\int_{-i\infty}^{i\infty} e^{\xi\tau}\frac{\widehat{u}_0(\phi_1)}{\xi+\alpha}d\xi \\
&= e^{-K(p)t}\frac{\alpha}{1-p^2}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\widehat{u}_0(\phi_1)}{\xi+\alpha}\frac{e^{(\xi+K(p))t}-1}{K(p)+\xi}d\xi.
\end{aligned}$$

Since ϕ_1 is analytic in right-half complex plane and $\operatorname{Re}\phi_1>0$, $\operatorname{Re}K(p)>0$ for all $\operatorname{Re}p=0$, $\operatorname{Re}\xi>0$ via Cauchy Theorem we obtain

$$\int_{-i\infty}^{i\infty}\frac{\widehat{u}_0(\phi_1)}{\xi+\alpha}\frac{1}{K(p)+\xi}d\xi = 0.$$

Therefore,

$$J_2 = \frac{\alpha}{1-p^2}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\widehat{u}_0(\phi_1)}{\xi+\alpha}\frac{e^{\xi t}}{K(p)+\xi}d\xi.$$

Taking inverse Laplace transform with respect to space variable we obtain

$$\begin{aligned}\mathcal{L}_x^{-1}(J_2) &= \frac{\alpha}{(2\pi i)^2}\int_{-i\infty}^{i\infty} dpe^{px}\frac{1}{1-p^2}\int_{-i\infty}^{i\infty}\frac{\widehat{u}_0(\phi_1)}{\xi+\alpha}\frac{e^{\xi t}}{K(p)+\xi}d\xi \\ &= \frac{\alpha}{(2\pi i)^2}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\frac{\widehat{u}_0(\phi_1)}{\xi+\alpha}\int_{-i\infty}^{i\infty} dpe^{px}\frac{1}{(K(p)+\xi)(1-p^2)} \\ &= \frac{\alpha}{(2\pi i)^2}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\frac{\widehat{u}_0(\phi_1)}{\xi+\alpha}\int_{-i\infty}^{i\infty} dpe^{px}\frac{1}{-\alpha p^2+\xi(1-p^2)}.\end{aligned}$$

Since

$$\begin{aligned}\int_{-i\infty}^{i\infty} dpe^{px}\frac{1}{-(\alpha+\xi)p^2+\xi} &= -\frac{1}{\alpha+\xi}\int_{-i\infty}^{i\infty} dpe^{px}\frac{1}{(p-\phi_1)(p-\phi_2)} \\ &= -2\pi i e^{\phi_2 x}\frac{1}{\alpha+\xi}\frac{1}{\phi_2-\phi_1},\end{aligned}$$

using $\phi_1+\phi_2=0$, we obtain

$$\mathcal{L}_x^{-1}(J_2) = -\frac{\alpha}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^{\xi t+\phi_2 x}\frac{1}{2(\xi+\alpha)^2\phi_2}\widehat{u}_0(-\phi_2).$$

We have

$$\phi_2' = \frac{1}{2\phi_2}\Big(\frac{\xi}{\xi+\alpha}\Big)' = \frac{\alpha}{2\phi_2}\frac{1}{(\xi+\alpha)^2}.$$

Making the change of the variables $p=\phi_2$ we obtain

$$\begin{aligned}\mathcal{L}_x^{-1}(J_2) &= -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^{\xi t+\phi_2 x}\phi_2'\widehat{u}_0(-\phi_2) \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dpe^{-K(p)t+px}\widehat{u}_0(-p) \\ &= e^{-\alpha t}\mathcal{L}_x^{-1}\big\{e^{\frac{\alpha t}{1+|p|^2}}\widehat{u}_0(-p)\big\}.\end{aligned}$$

Substituting this representation into (2.9) we get

$$\mathcal{L}_x^{-1}(I_1) = \mathcal{G}(t)u_0, \tag{2.11}$$

where the Green operator $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t)\phi = e^{-\alpha t}\mathcal{L}_x^{-1}\big\{e^{\frac{\alpha t}{1+|p|^2}}(\hat{\phi}(p)-\hat{\phi}(-p))\big\}. \tag{2.12}$$

Now we consider I_2 in the formula (2.10). In the same way we get

$$\mathcal{L}_x^{-1}(I_2) = \int_0^t d\tau\int_0^{+\infty} dyf(\tau,y)\mathcal{L}_x^{-1}\big\{e^{-K(p)(t-\tau)}\frac{1}{1-p^2}(e^{-py}-e^{py})\big\}. \tag{2.13}$$

We have

$$\begin{aligned}&\mathcal{L}_x^{-1}\big\{e^{-K(p)(t-\tau)}\frac{1}{1-p^2}(e^{-py}-e^{py})\big\} \\ &= \int_{-\infty}^{+\infty} dz\mathcal{L}_{x-z}^{-1}\Big\{e^{-K(p)(t-\tau)}\Big\}\mathcal{L}_z^{-1}\big\{\frac{1}{1-p^2}(e^{-py}-e^{py})\big\}.\end{aligned}$$

Taking into account

$$\mathcal{L}_z^{-1}\big\{\frac{1}{1-p^2}(e^{-py}-e^{py})\big\} = \frac{1}{2}(e^{-|z-y|}-e^{-|z+y|}),$$

we get

$$\begin{aligned}
&\mathcal{L}_x^{-1}\big\{e^{-K(p)(t-\tau)}\frac{1}{1-p^2}(e^{-py}-e^{py})\big\}\\
&=\int_{-\infty}^{+\infty} dz\mathcal{L}_{x-z}^{-1}\left\{e^{-K(p)(t-\tau)}\right\}\frac{1}{2}(e^{-|z-y|}-e^{-|z+y|})\\
&=\int_0^{+\infty}(\mathcal{L}_{x-z}^{-1}\big\{e^{-K(p)(t-\tau)}\big\}-\mathcal{L}_{x-z}^{-1}\big\{e^{-K(p)(t-\tau)}\big\})\frac{1}{2}(e^{-|z-y|}-e^{-|z+y|})dz.
\end{aligned}$$

Substituting into (2.13) we obtain

$$\mathcal{L}_x^{-1}(I_2)=\int_0^t \mathcal{G}(t-\tau)\mathcal{B}f(\tau)d\tau, \tag{2.14}$$

where operator $\mathcal{G}$ was defined into (2.12) and operator $\mathcal{B}$ is defined as

$$\mathcal{B}\phi=\frac{1}{2}\int_0^{+\infty}(e^{-|x-y|}-e^{-|x+y|})\phi(y)dy.$$

Also by direct calculation we can obtain another representation of operator $\mathcal{B}$

$$\mathcal{B}=(1-\partial_x^2)^{-1}.$$

Indeed, we put $\mathcal{B}w=u$. Taking Laplace transform with respect to space variable and using $u(t,0)=0$ we get

$$\widehat{u}(t,\xi)=\frac{1}{1-\xi^2}(\widehat{w}(t,\xi)-u_x(0,t)). \tag{2.15}$$

To satisfy $\lim_{x\to\infty}u(t,x)=0$ we need to put the following condition

$$res_{\xi=1}e^{\xi x}(\widehat{w}(t,\xi)-u_x(0,t))=0.$$

Therefore we obtain

$$u_x(0,t)=\widehat{w}(t,1). \tag{2.16}$$

Substituting (2.16) into (2.15) we find

$$\mathcal{B}w=\int_0^{+\infty}B(x,y)w(y)dy,$$

where

$$B(x,y)=\frac{1}{2}(e^{-|x-y|}-e^{-|x+y|}).$$

Note that $B(x,y)\geq 0$ for any $x\geq 0$, $y\geq 0$. From (2.8),(2.11) and (2.14) we obtain integral formula for solution of (2.1)

$$u(t,x)=\mathcal{G}(t)u_0+\int_0^t\mathcal{G}(t-\tau)\mathcal{B}f(\tau)d\tau. \tag{2.17}$$

We can easily see that

$$\|\mathcal{B}\phi\|_{\mathbf{L}^r}\leq C\|\phi\|_{\mathbf{L}^r} \tag{2.18}$$

for all $1\leq r\leq\infty$ and

$$\|\mathcal{B}\phi\|_{\mathbf{L}^{1,b}}\leq C\|\phi\|_{\mathbf{L}^{1,b}} \tag{2.19}$$

for any $b\geq 0$.

We first collect some preliminary estimates of the Green operator $\mathcal{G}(t)$ in the norms $\|\phi\|_{\mathbf{L}^p}$ and $\|\phi\|_{\mathbf{L}^{1,1+w}}$, where $w\in(0,1)$, $1\leq r\leq\infty$.

2.2. **Preliminary lemmas.** We introduce the operator $\mathcal{G}_0(t)$ given by

$$\mathcal{G}_0(t)\phi = \int_0^{+\infty} G_1(t,x,y)\phi(y)dy,$$

where the kernel is

$$G_1(t,x,y) = (4\pi\alpha t)^{-1/2}\Big(e^{-\frac{(x-y)^2}{4\alpha t}} - e^{-\frac{(x+y)^2}{4\alpha t}}\Big).$$

We first prepare some preliminary estimates of the operator $\mathcal{G}_0(t)$ in the Lebesgue norms $\|\phi\|_{\mathbf{L}^q}$ and $\|\phi\|_{\mathbf{L}^{1,a}} = \|\langle\cdot\rangle^a\phi\|_{\mathbf{L}^1}$, where $a \geq 0$, $1 \leq q \leq \infty$. Denote

$$G_0(t,x) = \partial_y G_1(t,x,y)|_{y=0} = (4\pi\alpha t)^{-1/2}\frac{x}{\alpha t}e^{-\frac{x^2}{4\alpha t}}.$$

Lemma 2.1. *Let $\phi \in \mathbf{L}^r(R^+)$. Then*

$$\|\mathcal{G}_0(t)\phi\|_{\mathbf{L}^q} \leq C\langle t\rangle^{\frac{1}{2}(\frac{1}{q}-\frac{1}{r})}\|\phi\|_{\mathbf{L}^r},$$

for all $t > 0$, $1 \leq q \leq \infty$, $1 \leq r \leq \infty$. Furthermore we assume that $\phi \in \mathbf{L}^{1,1+a}$, then the estimate

$$\|(\cdot)^b(\mathcal{G}_0(t)\phi - \vartheta G_0(t))\|_{\mathbf{L}^q} \leq Ct^{-1+\frac{1}{2q}+\frac{b-a}{2}}\|\phi\|_{\mathbf{L}^{1,1+a}}$$

is valid for all $t > 0$, where $1 \leq q \leq \infty$, $b \in [0, 1+a]$ and

$$\vartheta = \int_0^{+\infty} x\phi(x)dx.$$

Proof. Since

$$|G_1(t,x,y)| \leq Ct^{-1/2}e^{-\frac{C}{t}|x-y|^2}$$

for all $x, y \in \mathbb{R}^+$, by the Young inequality we have for $p = \frac{qr+r-q}{qr}$

$$\begin{aligned}\|\mathcal{G}_0(t)\phi\|_{\mathbf{L}^q} &\leq Ct^{-1/2}\|\int_0^{+\infty} e^{-\frac{C}{t}|x-y|^2}\phi(y)dy\|_{\mathbf{L}^q}\\ &\leq Ct^{-1/2}\|e^{-\frac{C}{t}|x|^2}\|_{\mathbf{L}^p}\|\phi\|_{\mathbf{L}^r} \leq C\langle t\rangle^{\frac{1}{2}(\frac{1}{q}-\frac{1}{r})}\|\phi\|_{\mathbf{L}^r}\end{aligned}$$

for all $t > 0$, where $1 \leq q \leq \infty$, Hence the first estimate of the lemma follows. For the second estimate we write

$$x^b(\mathcal{G}_0(t)\phi - \vartheta G_0(t,x)) = \int_0^{+\infty} x^b(G_1(t,x,y) - G_0(t,x)y)\phi(y)dy$$

for any $b \in [0, 1+a]$. Applying Taylor expansion, we obtain

$$|G_1(t,x,y) - G_0(t,x)y| \leq Ct^{-1-\frac{a}{2}}y^{1+a}(e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2})$$

for all $x, y \in \mathbb{R}^+$. Hence in the domain $y \leq \frac{x}{2}$

$$\begin{aligned}x^b|G_1(t,x,y) - G_0(t,x)y| &\leq Ct^{-1-\frac{a}{2}}y^{a+1}x^b e^{-\frac{C}{t}|x|^2}\\ &\leq Ct^{-1+\frac{b-a}{2}}y^{a+1}e^{-\frac{C}{t}|x|^2}.\end{aligned}$$

By the Lagrange finite differences Theorem we have

$$|G_1(t,x,y)| \leq Ct^{-\frac{1+\nu}{2}}y^\nu e^{-\frac{C}{t}|x-y|^2}$$

for all $x, y \in \mathbb{R}^+$, where $\nu \in [0,1]$. Taking $\nu = 1 + a - b$, in the case $b \in [1, a+1]$ we get for $y \geq \frac{x}{2}$

$$\begin{aligned} x^b|G_1(t,x,y) - G_0(t,x)y| &\leq x^b(|G_1(t,x,y)| + |G_0(t,x)y|) \\ &\leq Ct^{-1+\frac{b-a}{2}} x^b y^{a+1-b} e^{-\frac{C}{t}|x-y|^2} + Ct^{-\frac{3}{2}} x^{b+1} y e^{-\frac{C}{t}|x|^2} \\ &\leq Ct^{-1+\frac{b-a}{2}} y^{a+1}(e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2}). \end{aligned}$$

In the case $b \in [0,1]$, we write

$$\begin{aligned} &x^b|G_1(t,x,y) - G_0(t,x)y| \\ &\leq x^b(|G_1(t,x,y)| + |G_0(t,x)y|)^b |G_1(t,x,y) - G_0(t,x)y|^{1-b} \\ &\leq Ct^{-b}|y|^{(1+a)b} t^{-(1+\frac{a}{2})(1-b)} |y|^{(a+1)(1-b)} \big(e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2}\big) \\ &\leq Ct^{-1+\frac{b-a}{2}} y^{1+a}(e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2}), \end{aligned}$$

for all $x, y \in \mathbb{R}^+$, $y \geq \frac{x}{2}$. Thus we obtain the estimate

$$x^b|G_1(t,x,y) - G_0(t,x)y| \leq Ct^{-1+\frac{b-a}{2}} |y|^{a+1} \Big(e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2}\Big)$$

for all $x, y \in \mathbb{R}^+$, and for any $b \in [0, 1+a]$. Applying the above estimate with Young inequality we find

$$\begin{aligned} &\|(\cdot)^b(\mathcal{G}_0(t)\phi - \vartheta G_0(t))\|_{\mathbf{L}^p} \\ &= \|\int_0^{+\infty} x^b(G_1(t,x,y) - G_0(t,x)y)\phi(y)dy\|_{\mathbf{L}^q_x} \\ &\leq Ct^{-1+\frac{b-a}{2}} \|\int_0^{+\infty} (e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2}) y^{1+a}|\phi(y)|dy\|_{\mathbf{L}^q_x} \\ &\leq Ct^{-1+\frac{1}{2q}+\frac{b-a}{2}} \|\phi\|_{\mathbf{L}^{1,1+a}}. \end{aligned}$$

Thus the second estimate follows and the lemma is proved. □

Denote by $\mathcal{G}(t)$ the expression

$$\mathcal{G}(t)\phi = e^{-\alpha t}\mathcal{L}_x^{-1}\big\{e^{\frac{\alpha t}{1+|p|^2}}(\hat{\phi}(p) - \hat{\phi}(-p))\big\}.$$

Lemma 2.2. *Suppose that the function $\phi \in \mathbf{L}^\infty(\mathbb{R}^+) \cap \mathbf{L}^{1,1+a}(\mathbb{R}^+)$, where $a \in (0,1)$. Then the estimates*

$$\begin{gathered} \|\mathcal{G}(t)\phi\|_{\mathbf{L}^r} \leq e^{-at}\|\phi\|_{\mathbf{L}^r} + C\langle t\rangle^{-\frac{1}{2}(\frac{1}{r_1}-\frac{1}{r})}\|\phi\|_{\mathbf{L}^{r_1}}, \\ \|\mathcal{G}(t)\phi - \vartheta G_0(t)\|_{\mathbf{L}^\infty} \leq Ct^{-1-\frac{a}{2}}\|\phi\|_{\mathbf{L}^{1,1+a}} + e^{-\alpha t}\|\phi\|_{\mathbf{L}^\infty}, \\ \|(\cdot)^b(\mathcal{G}(t)\phi - \vartheta G_0)\|_{\mathbf{L}^1} \leq Ct^{-\frac{1}{2}+\frac{b-a}{2}}\|\phi\|_{\mathbf{L}^{1,1+a}} + e^{-\alpha t}\|(\cdot)^b\phi\|_{\mathbf{L}^1} \end{gathered}$$

are valid for all $t > 0$, where $1 \leq r \leq \infty$.

Proof. Note that the Green operator $\mathcal{G}(t)$ can be represented as

$$\mathcal{G}(t)\phi = \mathcal{G}_0(t)\phi + e^{-\alpha t}\phi(x) + \mathcal{R}(t)\phi, \tag{2.20}$$

where the remainder is

$$\mathcal{R}(t)\phi = \int_0^{+\infty} (R(t,x-y) - R(t,x+y))\phi(y)dy$$

with kernel

$$R(t,x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px}\widehat{R}(t,p)dp,$$

where $\widehat{R}(t,p) = e^{\frac{\alpha tp^2}{1-p^2}} - e^{\alpha tp^2} - e^{-\alpha t}$. From Lemma 2.1 the operator $\mathcal{G}_0(t)$ satisfies the estimates of the lemma . Also it is easily to see that the term $e^{-\alpha t}\phi(x)$ satisfies the estimate of the lemma.

Now we estimate the remainder $\mathcal{R}(t)$. We represent

$$\widehat{R}(t,p) = e^{\frac{\alpha tp^2}{1-p^2}}\big(1 - e^{-\alpha t\frac{p^4}{1-p^2}}\big) - e^{-\alpha t}$$

for all $|p| \le 1$, and

$$\widehat{R}(t,p) = -e^{\alpha tp^2} + e^{-\alpha t}\big(e^{\frac{\alpha t}{1-p^2}} - 1\big)$$

for all $|p| \ge 1$. Then we see that

$$|\partial_p^j \widehat{R}(t,p)| \le C\langle t\rangle^{\frac{j}{2}-1} e^{\frac{\alpha}{2}tp^2} + C\langle t\rangle^2 e^{-\alpha t}(1-p^2)^{-3}$$

for all $\operatorname{Re} p = 0$, $t > 0$, $0 \le j \le 4$. Therefore,

$$\begin{aligned}|R(t,x)| &\le C\langle x\langle t\rangle^{-1/2}\rangle^{-4}\langle t\rangle^{-\frac{1}{2}-1} + C\langle x\rangle^{-4}\langle t\rangle^2 e^{-\alpha t}\\ &\le C\langle x\langle t\rangle^{-\frac{1}{2}}\rangle^{-4}\langle t\rangle^{-\frac{1}{2}-1}\end{aligned}$$

for all $x \in \mathbb{R}$, $t > 0$. Applying this estimate by the Young inequality we find

$$\|\mathcal{R}(t)\phi\|_{\mathbf{L}^r} \le C\langle t\rangle^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1}\|\phi\|_{\mathbf{L}^q}$$

for all $1 \le q \le r \le \infty$ and

$$\|\mathcal{R}(t)\phi\|_{\mathbf{L}^{1,w}} \le C\langle t\rangle^{-1}(\langle t\rangle^{\frac{w}{2}}\|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^{1,w}})$$

for all $t > 0$. Now by representation (2.20) the estimates of the lemma follow. Lemma 2.2 is proved. $\square$

In the next lemma we estimate the Green operator in our basic norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0}(\langle t\rangle\|\phi(t)\|_{\mathbf{L}^\infty} + \langle t\rangle^{-a/2}\|\phi(t)\|_{\mathbf{L}^{1,1+a}}),$$

where $a \in (0,1)$. Note that the $\mathbf{L}^1$-norm is estimated by the norm in $\mathbf{X}$,

$$\begin{aligned}\langle t\rangle^{\frac{1}{2}}\|\phi(t)\|_{\mathbf{L}^1} &= \int_0^{\langle t\rangle}|\phi(t,x)|dx + \int_{\langle t\rangle}^{+\infty}|1+x|^{-1-\alpha}|x|^{1+\alpha}|\phi(t,x)|dx\\ &\le C\langle t\rangle\|\phi(t)\|_{\mathbf{L}^\infty} + C\langle t\rangle^{-\frac{a}{2}}\|\phi(t)\|_{\mathbf{L}^{1,a}} \le C\|\phi\|_{\mathbf{X}}.\end{aligned}$$

We define

$$g(t) = 1 + \kappa\log\langle t\rangle$$

with $\kappa > 0$.

Lemma 2.3. *Let the function $f(t,x)$ have a zero first moment $\int_0^{+\infty} xf(t,x)dx = 0$. Then the following inequality*

$$\|g(t)\int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)\mathcal{B}f(\tau)d\tau\|_{\mathbf{X}} \le C\|\langle t\rangle f\|_{\mathbf{X}}$$

is valid, provided that the right-hand side is finite.

Proof. Since $g^{-1}(t) < C$ in view of Lemma 2.2 we get

$$\begin{aligned} &\| \int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)f(\tau)d\tau\|_{\mathbf{L}^\infty} + \| \int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)f(\tau)d\tau\|_{\mathbf{L}^{1,1+a}} \\ &\le C\|\langle t\rangle f\|_{\mathbf{X}} \end{aligned}$$

for all $0 \le t \le 4$. We now consider $t > 4$. From definition of function $g(t)$ we have

$$Cg^{-1}(t) \ge \langle t\rangle^{-a/4},$$
$$\sup_{\tau\in[\sqrt{t},t]} g^{-1}(\tau) \le C(1 + \frac{\kappa}{2}\log\langle t\rangle) \le Cg^{-1}(t).$$

Therefore by Lemma 2.2 and (2.18)-(2.19) we obtain

$$\begin{aligned} &\| \int_0^t \mathcal{G}(t-\tau)\mathcal{B}f(\tau)d\tau\|_{\mathbf{L}^\infty} \\ &\le C\int_0^{\sqrt{t}} (t-\tau)^{-1-\frac{a}{2}}(\|\mathcal{B}f(\tau)\|_{\mathbf{L}^\infty} + \|\mathcal{B}f(\tau)\|_{\mathbf{L}^{1,1+a}})d\tau \\ &\quad + Cg^{-1}(t)\int_{\sqrt{t}}^{t/2} (t-\tau)^{-1-\frac{a}{2}}(\|\mathcal{B}f(\tau)\|_{\mathbf{L}^\infty} + \|\mathcal{B}f(\tau)\|_{\mathbf{L}^{1,1+a}})d\tau \\ &\quad + Cg^{-1}(t)\int_{\frac{t}{2}}^t \|\mathcal{B}f(\tau)\|_{\mathbf{L}^\infty}d\tau, \end{aligned}$$

hence using the definition of the norm in $\mathbf{X}$, we get

$$\begin{aligned} &\| \int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)\mathcal{B}f(\tau)d\tau\|_{\mathbf{L}^\infty} \\ &\le C\|\langle t\rangle f\|_{\mathbf{X}}\int_0^{\sqrt{t}} (t-\tau)^{-1-\frac{a}{2}}\langle\tau\rangle^{\frac{a}{2}-1}d\tau \\ &\quad + C\|\langle t\rangle f\|_{\mathbf{X}}g^{-1}(t)\int_{\sqrt{t}}^{t/2} (t-\tau)^{-1-\frac{a}{2}}\langle\tau\rangle^{\frac{a}{2}-1}d\tau + C\|\langle t\rangle f\|_{\mathbf{X}}g^{-1}(t)\int_{\frac{t}{2}}^t \langle\tau\rangle^{-2}d\tau \\ &\le C(t^{-1-\frac{a}{4}} + t^{-1}g^{-1}(t))\|\langle t\rangle f\|_{\mathbf{X}} \le Ct^{-1}g^{-1}(t)\|\langle t\rangle f\|_{\mathbf{X}} \end{aligned}$$

and similarly

$$\begin{aligned} &\| \int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)f(\tau)d\tau\|_{\mathbf{L}^{1,1+a}} \\ &\le C\int_0^t g^{-1}(\tau)\|f(\tau)\|_{\mathbf{L}^{1,1+a}}d\tau \\ &\le C\|\langle t\rangle f\|_{\mathbf{X}}\int_0^{\sqrt{t}} \tau^{\frac{a}{2}-1}d\tau + C\|\langle t\rangle f\|_{\mathbf{X}}g^{-1}(t)\int_{\sqrt{t}}^t \tau^{\frac{a}{2}-1}d\tau \\ &\le C\varepsilon(t^{\frac{a}{4}} + g^{-1}t^{a/2})\|\langle t\rangle f\|_{\mathbf{X}} \le Ct^{\frac{a}{2}}g^{-1}\|\langle t\rangle f\|_{\mathbf{X}} \end{aligned}$$

for all $t > 4$. Hence the result of the lemma follows. Lemma 2.3 is proved. □

We now prove the local existence of weak solutions to the initial boundary-value problem (1.1).

Proposition 2.4. *Let $u_0 \in \mathbf{L}^{1,1+a}(\mathbb{R}^+) \cap \mathbf{L}^\infty(\mathbb{R}^+)$, $a \ge 0$. Then for some $T > 0$, there exists a unique solution $u \in \mathbf{C}([0,T];\mathbf{L}^{1,1+a}(\mathbb{R}^+) \cap \mathbf{L}^\infty(\mathbb{R}^+))$ to the problem* (1.1).

Proof. We apply the contraction mapping principle. We choose a functional space
$$\mathbf{Z}=\big\{\phi\in\mathbf{C}([0,T];\mathbf{L}^{1,1+a}(\mathbb{R}^+)\cap\mathbf{L}^{\infty}(\mathbf{R}^+)):\quad ||u||_{\mathbf{Z}}<\infty\big\},$$
with the norm
$$\|u\|_{\mathbf{Z}}=\sup_{t\in[0,T]}(\|u(t)\|_{\mathbf{L}^{1,1+a}}+\|u(t)\|_{\mathbf{L}^{\infty}}).$$
For $v\in\mathbf{Z}$ we define the mapping $\mathcal{M}(v)$ by
$$\mathcal{M}(v)=\mathcal{G}(t)u_0+\lambda\int_0^t d\tau\mathcal{G}(t-\tau)\mathcal{B}|v|^{\sigma}v(\tau). \tag{2.21}$$
Suppose that the norm $\|v\|_{\mathbf{Z}}<\delta$. We first prove that
$$\|\mathcal{M}(v)\|_{\mathbf{Z}}<\delta.$$
Applying the Lemma2.2 we have
$$\begin{aligned}\|\mathcal{M}(v)\|_{\mathbf{L}^{\infty}}&\le\|\mathcal{G}(t)u_0\|_{\mathbf{L}^{\infty}}+|\lambda|\int_0^t\|\mathcal{G}(t-\tau)\mathcal{B}|v|v(\tau)\|_{\mathbf{L}^{\infty}}d\tau\\&\le C\|u_0\|_{\mathbf{L}^{\infty}}+C\int_0^t\|\mathcal{B}|v|v(\tau)\|_{\mathbf{L}^{\infty}}d\tau.\end{aligned}\tag{2.22}$$
Similarly we obtain the estimate
$$\begin{aligned}\|\mathcal{M}(v)\|_{\mathbf{L}^{1,1+a}}&\le\|\mathcal{G}(t)u_0-\vartheta_1G_0\|_{\mathbf{L}^{1,1+a}}+\|G_0\|_{\mathbf{L}^{1,1+a}}\|u_0\|_{\mathbf{L}^1}+e^{-\alpha t}\|u_0\|_{\mathbf{L}^{1,1+a}}\\&\quad+|\lambda|\int_0^t\|\mathcal{G}(t-\tau)\mathcal{B}|v|v(\tau)-\vartheta_2G_0\|_{\mathbf{L}^{1,1+a}}d\tau\\&\quad+|\lambda|\int_0^t\|F(t-\tau)\|_{\mathbf{L}^{1,1+a}}\|\mathcal{B}|v|v(\tau)\|_{\mathbf{L}^1}d\tau\\&\quad+|\lambda|\int_0^t e^{-\alpha(t-\tau)}\|\mathcal{B}|v|v(\tau)\|_{\mathbf{L}^{1,1+a}}d\tau\\&\le C\|u_0\|_{\mathbf{L}^{1,1+a}}+C\int_0^t\|\mathcal{B}|v|v(\tau)\|_{\mathbf{L}^{1,1+a}}d\tau.\end{aligned}\tag{2.23}$$
Using the estimate
$$\|\mathcal{B}|v|v(\tau)\|_{\mathbf{Z}}\le\|v(\tau)\|_{\mathbf{L}^{\infty}}\|v(\tau)\|_{\mathbf{Z}}\le\delta^2,$$
From (2.22) and (2.23), we obtain
$$\|\mathcal{M}(v)\|_{\mathbf{Z}}\le C\|u_0\|_{\mathbf{Z}}+CT\delta^2\le\delta$$
if we choose $\delta\ge 2C\|u_0\|_{\mathbf{Z}}$ and $T>0$ sufficiently small. Therefore, the mapping $\mathcal{M}$ transforms a ball of a radius $\delta>0$ into itself in the space $\mathbf{Z}$. We now estimate the difference
$$\|\mathcal{M}(v_1)-\mathcal{M}(v_2)\|_{\mathbf{Z}}\le\frac{1}{2}\|v_1-v_2\|_{\mathbf{Z}}.$$
We have
$$\|\mathcal{M}(v_1)-\mathcal{M}(v_2)\|_{\mathbf{Z}}\le C\int_0^T||\mathcal{B}(|v_1|v_1(\tau)-|v_2|v_2(\tau))||_{\mathbf{Z}}\langle t-\tau\rangle^{a/2}d\tau.$$
Since
$$||\mathcal{B}(|v_1|v_1(\tau)-|v_2|v_2(\tau))||_{\mathbf{Z}}\le C\delta||v_1-v_2||_{\mathbf{Z}},$$

it follows that

$$\|\mathcal{M}(v_1) - \mathcal{M}(v_2)\|_{\mathbf{Z}} \le CT\delta\|v_1 - v_2\|_{\mathbf{Z}} \le \frac{1}{2}\|v_1 - v_2\|_{\mathbf{Z}}$$

if we choose $T > 0$ sufficiently small. Thus $\mathcal{M}$ is a contraction mapping, therefore there exists a unique solution $u(t,x) \in \mathbf{C}([0,T]; \mathbf{L}^{1,1+a}(\mathbb{R}^+) \cap \mathbf{L}^\infty(\mathbb{R}^+))$ to the problem (1.1). Proposition 2.4 is proved. □

3. Proof of Theorem 1.1 (Small data)

As in reference [20] by making a change of the dependent variable $u(t,x) = v(t,x)e^{-\varphi(t)}$, for the new function $v(t,x)$ we get the equation

$$\partial_t(v - v_{xx}) - \alpha v_{xx} - \lambda e^{-\varphi}|v|v - \varphi' v = 0.$$

We assume that $\varphi(t)$ is such that $\varphi(0) = 1$ and

$$\int_0^{+\infty} x(\lambda e^{-\varphi}|v|v + \varphi' v)dx = 0.$$

Since by construction

$$\int_0^{+\infty} u(x)dx = 0$$

the first moment of new function $v(t,x)$ satisfies a conservation law:

$$\frac{d}{dt}\int_0^{+\infty} xv(t,x)dx = 0,$$

hence $\int_{\mathbf{0}}^{+\infty} xv(t,x)dx = \int_{\mathbf{0}}^{+\infty} xu_0(x)dx$ for all $t > 0$. Thus we consider the initial-boundary value problem for the new dependent variables $(v(t,x), \varphi(t))$,

$$\begin{gathered} \partial_t(v - \Delta v) - \alpha\Delta v = \lambda e^{-\varphi}\big(|v| - \frac{1}{\theta}\int_0^{+\infty} x|v|v dx\big)v, \\ \partial_t\varphi(t) = -\frac{\lambda}{\theta}e^{-\varphi}\int_0^{+\infty} x|v|v dx, \\ v(0,x) = u_0(x), \quad v(t,0) = 0, \quad \varphi(0) = 0. \end{gathered} \tag{3.1}$$

We denote $h(t) = e^{\varphi(t)}$ and write (3.1) as

$$\begin{gathered} \partial_t(v - v_{xx}) - \alpha v_{xx} = f(v,h) \\ v(0,x) = u_0(x), \quad v(t,0) = 0 \\ \partial_t h = -\frac{\lambda}{\theta}\int_0^{+\infty} x|v|v dx, \quad h(0) = 1, \end{gathered} \tag{3.2}$$

where

$$f(v,h) = \lambda h^{-1}\Big(|v| - \frac{1}{\theta}\int_0^{+\infty} x|v|v dx\Big)v.$$

We note that the first moment of the nonlinearity is

$$\int_0^{+\infty} xf(v,h)(t,x)dx = 0$$

for all $t > 0$. We now prove the existence of the solution $(v(t,x), h(t))$ for the problem (3.2) by the successive approximations $(v_m(t,x), h_m(t))$, $m = 1, 2, \dots$, defined as follows

$$\begin{gathered} \partial_t(v_m - \partial_x^2 v_m) - \alpha\partial_x^2 v_m = f(v_{m-1}, h_{m-1}), \\ \partial_t h_m = -\frac{\lambda}{\theta}\int_0^{+\infty} |v_{m-1}|v_{m-1}dx, \\ v_m(0,x) = u_0(x), \quad v_m(t,0) = 0, \quad h_m(0) = 1, \end{gathered} \tag{3.3}$$

for all $m \geq 2$, where $v_1 = \mathcal{G}(t)u_0$, $h_1 = g(t)$, $g(t) = 1 + |\theta|\eta\log\langle t\rangle$.

We now prove by induction the following estimates

$$\begin{gathered} \|v_m\|_{\mathbf{X}} \leq C\varepsilon, \\ \|v_m(t) - \mathcal{G}(t)u_0\|_{\mathbf{L}^{1,1}} \leq C\varepsilon^2 g^{-1}(t), \\ |h_m(t) - g(t)| \leq C\varepsilon(1 + \log g(t)) \end{gathered} \tag{3.4}$$

for all $m \geq 1$, the norm $\|\cdot\|_{\mathbf{X}}$ is defined as above by

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0}(\langle t\rangle\|\phi(t)\|_{\mathbf{L}^\infty} + \langle t\rangle^{-a/2}\|\phi(t)\|_{\mathbf{L}^{1,1+a}}).$$

By virtue of Lemma 2.2 we have

$$\begin{gathered} \|\mathcal{G}(t)u_0\|_{\mathbf{L}^\infty} \leq C\varepsilon\langle t\rangle^{-1}, \\ \|\mathcal{G}(t)u_0\|_{\mathbf{L}^{1,1}} \leq C\varepsilon, \\ \||\cdot|^a(\mathcal{G}(t)u_0 - \theta G_0(t,x))\|_{\mathbf{L}^{1,1}} \leq C\varepsilon, \\ \||\cdot|^a G_0(t,x)\|_{\mathbf{L}^{1,1}} \leq Ct^{a/2}. \end{gathered}$$

Therefore, estimates (3.4) are valid for $m = 1$. We assume that estimates (3.4) are true with m replaced by $m-1$. The integral equations associated with (3.3) are written as

$$\begin{gathered} v_m(t) = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-\tau)\mathcal{B}f(v_{m-1}(\tau), h_{m-1}(\tau))d\tau, \\ h_m(t) = 1 - \frac{\lambda}{\theta}\int_0^t d\tau\int_0^{+\infty} |v_{m-1}|v_{m-1}dx. \end{gathered}$$

We have

$$f(v,h) = \lambda h^{-1}\Big(|v| - \frac{1}{\theta}\int_0^{+\infty} x|v|vdx\Big)v.$$

$$\begin{aligned} \|f(v_{m-1}(t), h_{m-1}(t))\|_{\mathbf{L}^\infty} &\leq Ch_{m-1}^{-1}(t)\|v_{m-1}(t)\|_{\mathbf{L}^\infty}^{1+1}\Big(1 + \frac{1}{|\theta|}\|v_{m-1}(t)\|_{\mathbf{L}^{1,1}}\Big) \\ &\leq C\varepsilon^2\langle t\rangle^{-2}g^{-1}(t) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &\|f(v_{m-1}(t), h_{m-1}(t))\|_{\mathbf{L}^{1,1+a}} \\ &\leq Ch_{m-1}^{-1}(t)\|v_{m-1}(t)\|_{\mathbf{L}^\infty}\|v_{m-1}(t)\|_{\mathbf{L}^{1,1+a}}\Big(1 + \frac{1}{|\theta|}\|v_{m-1}(t)\|_{\mathbf{L}^{1,1}}\Big) \\ &\leq C\varepsilon^2\langle t\rangle^{-1+\frac{a}{2}}g^{-1}(t) \end{aligned} \tag{3.6}$$

for all $t > 0$, provided that $(v_{m-1}(t), h_{m-1}(t))$ satisfies (3.4). This yields the estimate

$$\|\langle t\rangle g(t) f(v_{m-1}(t), h_{m-1}(t))\|_{\mathbf{X}} \le C\varepsilon^2.$$

Since $f(v_{m-1}(\tau), h_{m-1}(\tau))$ have the zero first moment we get via Lemma 2.3

$$\|g(t)\int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)\mathcal{B}f(v_{m-1}(\tau), h_{m-1}(\tau))d\tau\|_{\mathbf{X}} \le C\varepsilon^2$$

hence it follows that

$$\|v_m\|_{\mathbf{X}} \le C\varepsilon, \|v_m(t) - \mathcal{G}(t)u_0\|_{\mathbf{L}^{1,1}} \le C\varepsilon^2 g^{-1}(t). \tag{3.7}$$

To prove the third estimate in (3.4) we need the following lemma, where we evaluate the large time behavior of the first moment of the nonlinearity in equation (1.1) in the critical case. As above we take $\theta = \int_0^{+\infty} x u_0(x)dx$.

Lemma 3.1. *Assume that $u_0 \in \mathbf{L}^\infty(\mathbb{R}^+) \cap \mathbf{L}^{1,1+a}(\mathbb{R}^+)$, that $\|u_0\|_{\mathbf{L}^\infty} + \|u_0\|_{\mathbf{L}^{1,1+a}} = \varepsilon$ is sufficiently small, and that $\theta\lambda \le -C\varepsilon < 0$. Let a function $v(t,x)$ satisfy the estimates*

$$\langle t\rangle \|v\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^{1,1}} \le C\varepsilon,$$
$$\|v(t) - \mathcal{G}(t)u_0\|_{\mathbf{L}^{1,1}} \le C\varepsilon^2 g^{-1}(t)$$

for all $t > 0$. Then the inequality

$$\big|1 - \frac{\lambda}{\theta}\int_0^t d\tau \int_0^{+\infty} x|v|v(\tau,x)dx - g(t)\big| \le C\varepsilon(1 + \log g(t)) \tag{3.8}$$

is valid for all $t > 0$.

Proof. In view of the condition $\|v\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^{1,1}} \le C\varepsilon$ we get

$$|\frac{\lambda}{\theta}\int_0^t d\tau \int_0^{+\infty} x|v|v(\tau,x)dx| \le C\varepsilon t,$$

hence estimate (3.8) is true for all $0 < t < 1$.

We now consider the case $t \ge 1$. By the last estimate of Lemma 2.2 we get

$$\|x(\mathcal{G}(t)u_0 - \theta G_0(t,x))\|_{\mathbf{L}^1} \le C\varepsilon t^{-a/2}.$$

Hence we find

$$\begin{aligned}
&\|x(|v|v - |\theta|\theta(G_0(t,x))^{1+1})\|_{\mathbf{L}^1} \\
&\le C(\|x(v(t) - \mathcal{G}(t)u_0)\|_{\mathbf{L}^1} + \|x(\mathcal{G}(t)u_0 - \theta G_0(t,x))\|_{\mathbf{L}^1}) \\
&\quad \times (\|v\|_{\mathbf{L}^\infty} + \|\mathcal{G}(t)u_0\|_{\mathbf{L}^\infty} + |\theta|\|G_0(t)\|_{\mathbf{L}^\infty}) \\
&\le C\varepsilon^2 t^{-1}(\varepsilon g^{-1}(t) + t^{-\frac{a}{2}})
\end{aligned}$$

for all $t \ge 1$. Since

$$t\int_0^{+\infty} x(G_0(t,x))^2 dx = \frac{\eta}{|\lambda|},$$

it follows that

$$\begin{aligned}
|\int_0^{+\infty} x|v|v(t,x)dx - |\theta|\theta t^{-1}\frac{\eta}{|\lambda|}| &\le C\|x(|v|v - |\theta|\theta(G_0(t,x))^2)\|_{\mathbf{L}^1} \\
&\le C\varepsilon^2 t^{-1}(\varepsilon g^{-1}(t) + t^{-\frac{a}{2}})
\end{aligned}$$

for all $t \geq 1$. Therefore,

$$\begin{aligned}
&|\frac{|\lambda|}{\theta}\int_1^t d\tau \int_0^{+\infty} x|v|v(\tau,x)dx - |\theta|\eta \log t| \\
&\leq \int_1^t \frac{C\varepsilon^2 d\tau}{\tau(1+|\theta|\eta\log(1+\tau))} + C\varepsilon \int_1^t \tau^{-1-\frac{a}{2}} d\tau \\
&\leq C\varepsilon(1+\log g(t))
\end{aligned}$$

for all $t \geq 1$. Thus we obtain (3.8) and complete the proof. □

By virtue of (3.7) and applying Lemma 3.1 we find that

$$|h_m(t) - g(t)| \leq C\varepsilon(1+\log g(t))$$

for all $t > 0$. Thus by induction we see that estimates (3.4) are valid for all $m \geq 1$. In the same way by induction we can prove that

$$\begin{gathered}
\|v_m - v_{m-1}\|_{\mathbf{X}} \leq \frac{1}{4}\|v_{m-1} - v_{m-2}\|_{\mathbf{X}}, \\
\sup_{t>0} g^{-1}(t)|h_m(t) - h_{m-1}(t)| \\
\leq \frac{1}{4}\|v_{m-1} - v_{m-2}\|_{\mathbf{X}} + \frac{1}{4}\sup_{t>0} g^{-1}(t)|h_{m-1}(t) - h_{m-2}(t)|
\end{gathered}$$

for all $m > 2$. Therefore taking the limit $m \to \infty$, we obtain a unique solution $\lim_{m\to\infty} v_m(t,x) = v(t,x) \in \mathbf{X}$, $\lim_{m\to\infty} h_m(t) = h(t) = e^{\varphi(t)} \in \mathbf{C}(0,\infty)$ satisfying the equalities

$$\begin{gathered}
v(t) = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-\tau)\mathcal{B}f(v(\tau),h(\tau))d\tau, \\
h(t) = 1 - \frac{\lambda}{\theta}\int_0^t d\tau \int_0^{+\infty} x|v|v dx,
\end{gathered} \tag{3.9}$$

and the estimates

$$\begin{gathered}
\|v(t) - \mathcal{G}(t)u_0\|_{\mathbf{L}^{1,1}} \leq C\varepsilon^2 g^{-1}(t), \\
|h_m(t) - g(t)| \leq C\varepsilon(1+\log g(t)).
\end{gathered} \tag{3.10}$$

Applying (3.5) and (3.6) to (3.9), we have

$$\|v(t) - \mathcal{G}(t)u_0\|_{\mathbf{L}^\infty} \leq C\varepsilon^2 \langle t\rangle^{-1} g^{-1}(t). \tag{3.11}$$

Then via formulas $u(t,x) = e^{-\varphi(t)}v(t,x) = h^{-1}(t)v(t,x)$ we find the estimates

$$\begin{aligned}
&\|u(t) - \theta G_0(t,x)e^{-\varphi(t)}\|_{\mathbf{L}^\infty} \\
&\leq \|u(t) - (\mathcal{G}(t)u_0)e^{-\varphi(t)}\|_{\mathbf{L}^\infty} + \|(\mathcal{G}(t)u_0 - \theta G_0(t,x))e^{-\varphi(t)}\|_{\mathbf{L}^\infty} \\
&\leq C\varepsilon^2\langle t\rangle^{-1} g^{-1}(t),
\end{aligned} \tag{3.12}$$

where we used the estimate

$$\|(\mathcal{G}(t)u_0 - \theta G_0(t,x))e^{-\varphi(t)}\|_{\mathbf{L}^\infty} \leq Ct^{-1-\frac{a}{2}}\|u_0\|_{\mathbf{L}^{1,1+a}}$$

and (3.11). By (3.10), we have

$$\|\theta G_0(t)h^{-1}(t) - \theta G_0(t,x)g^{-1}(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon t^{-1} g^{-2}(t)|h(t) - g(t)|,$$

hence via (3.12) it follows that

$$\|u(t) - \theta G_0(t,x)g^{-1}(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon^2\langle t\rangle^{-1} g^{-2}(t). \tag{3.13}$$

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2 (Large data)

Before proving Theorem 1.2 we need a lemma, where we compare the solutions of the following two problems

$$\begin{aligned} u_t - u_{xx} + |u|u = f, \quad x \in \mathbb{R}^+,\ t > 0, \\ u(0,x) = u_0(x), \quad u(t,0) = 0, \quad x \in \mathbb{R}^+,\ t > 0 \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} v_t - v_{xx} + \epsilon v^2 = |f|, \quad x \in \mathbb{R}^+,\ t > 0, \\ v(0,x) = |u_0(x)|, \quad v(t,0) = 0, \quad x \in \mathbb{R}^+,\ t > 0 \end{aligned} \tag{4.2}$$

Lemma 4.1. *Suppose that* $u_0 \in \mathbf{L}^\infty(\mathbb{R}^+) \cap \mathbf{C}^0(\mathbb{R}^+)$*, and* $0 \le \epsilon \le 1$*. Then* $|u(t,x)| \le v(t,x)$ *for all* $t \ge 0$*,* $x \in \mathbb{R}^+$*.*

Proof. Define $r = v - u$. Then we obtain

$$\begin{aligned} r_t - r_{xx} + \epsilon v^2 - |u|u = |f| - f, \quad x \in \mathbb{R}^+,\ t > 0, \\ r(0,x) = |u_0(x)| - u_0(x), \quad r(t,0) = 0. \end{aligned} \tag{4.3}$$

We need to prove that $r \ge 0$ for all $t \ge 0$, $x \in \mathbb{R}^+$. Define $R(t) \equiv \inf_{x \in \mathbb{R}^+} r(t,x)$. On the contrary, suppose that there exists a time $T > 0$ such that $R(T) < 0$. By the continuity we can find an interval $[T_1, T]$ such that $R(t) \le 0$ for all $t \in [T_1, T]$ and $R(T_1) = 0$. By [5, Theorem 2.1] there exists a point $\zeta(t) \in \mathbb{R}^+$ such that $R(t) = r(t, \zeta(t))$, moreover $R'(t) = \frac{d}{dt} r(t, \zeta(t))$ almost everywhere on $t \in [T_1, T]$. We have

$$|u|u - \epsilon v^2 = (v - R)^2 - \epsilon v^2 \ge 0$$

for all $t \in [T_1, T]$. For the Laplacian ∂_x^2 at the point of maximum $\zeta(t)$ we have

$$-\partial_x^2 r(t, \zeta(t)) \le 0.$$

Therefore by equation (4.3) we get $R'(t) \ge 0$ for all $t \in [T_1, T]$. Integration with respect to time yields $R(t) \ge 0$. This gives a contradiction, hence $u(t,x) \le |v(t,x)|$ for all $x \in \mathbb{R}^+$ and $t > T_1$. In the same manner we prove that $v + u \ge 0$ for all $x \in \mathbb{R}^n$ and $t > T_1$. Lemma 4.1 is proved. □

Lemma 4.2. *Let* $u_0 \in \mathbf{W}^2_\infty(\mathbb{R}^+) \cap \mathbf{W}^2_1(\mathbb{R}^+)$*. Then we have the following estimate for solution of initial-boundary value problem* (1.1)*:*

$$\|u\|_{\mathbf{L}^2} \le C\langle t \rangle^{-3/4}, \quad \|xu\|_{\mathbf{L}^2} \le C\langle t \rangle^{-1/4}$$

Proof. Multiplying equation (1.1) by $2u$ and integrating with respect to $x \in \mathbb{R}^+$ we get

$$\frac{d}{dt}(\|u(t)\|^2_{\mathbf{L}^2} + \|u_x(t)\|^2_{\mathbf{L}^2}) + 2\alpha \|u_x(t)\|^2_{\mathbf{L}^2} = 2\lambda \|u(t)\|^{\sigma+2}_{\mathbf{L}^{\sigma+2}},$$

hence integrating we see that

$$\begin{aligned} &\|u(t)\|^2_{\mathbf{L}^2} + \|u_x(t)\|^2_{\mathbf{L}^2} + 2\alpha \int_0^t \|u_x(\tau)\|^2_{\mathbf{L}^2} d\tau - 2\lambda \int_0^t \|u(\tau)\|^{\sigma+2}_{\mathbf{L}^{\sigma+2}} d\tau \\ &\le \|u_0\|^2_{\mathbf{L}^2} + \|u_{0x}\|^2_{\mathbf{L}^2} \\ &= \|u_0\|^2_{\mathbf{H}^1} \end{aligned}$$

for all $t \geq 0$. In particular we have

$$\|u\|_{\infty,2} \equiv \sup_{t\geq 0} \|u(t)\|_{\mathbf{L}^2} \leq \|u_0\|_{\mathbf{H}^1}, \tag{4.4}$$

$$\|u\|_{3,3} \equiv \| \|u(t,x)\|_{\mathbf{L}^3_x} \|_{\mathbf{L}^3_t(0,\infty)} \leq C\|u_0\|_{\mathbf{H}^1} \tag{4.5}$$

From equation (1.1) we have

$$u_{xx}(t,0) = 0.$$

Differentiating (1.1), multiplying by $2u_x$ and integrating with respect to $x \in \mathbb{R}^+$ we get

$$\frac{d}{dt}(\|u_x(t)\|^2_{\mathbf{L}^2} + \|u_{xx}(t)\|^2_{\mathbf{L}^2}) + 2\alpha\|u_{xx}(t)\|^2_{\mathbf{L}^2} = 2\lambda \int_0^{+\infty} |u|u_x^2 dx,$$

hence integrating we see that

$$\begin{aligned} &\|u(t)\|^2_{\mathbf{L}^2} + \|u_x(t)\|^2_{\mathbf{L}^2} + 2\alpha \int_0^t \|u_x(\tau)\|^2_{\mathbf{L}^2} d\tau - 2\lambda \int_0^{+\infty} |u|u_x^2 dx \\ &\leq \|u_{0x}\|^2_{\mathbf{L}^2} + \|u_{0xx}\|^2_{\mathbf{L}^2} = \|u_0\|^2_{\mathbf{H}^2} \end{aligned}$$

for all $t \geq 0$. In particular we obtain that the solution $u(t,x) \in \mathbf{C}([0,+\infty), \mathbf{C}^1(\mathbb{R}^+))$. Now we prove the estimates

$$\|u(t)\|_{\mathbf{L}^2} \leq C\langle t\rangle^{-3/4} \tag{4.6}$$

for all $t > 0$. Denote $\Theta(x) = 1$ for all $x > 0$ and $\Theta(x) = -1$ for all $x < 0$; $\Theta(0) = 0$. We multiply equation (1.1) by $x\Theta(u(t,x))$ and integrate with respect to x over $\mathbb{R}^+$ to get

$$\begin{aligned} &\partial_t\Big(\int_0^{+\infty} xu(t,x)\Theta(u(t,x))dx - \int_0^{+\infty} xu_{xx}(t,x)\Theta(u(t,x))dx\Big) \\ &- \alpha \int_0^{+\infty} xu_{xx}(t,x)\Theta(u(t,x))dx \\ &= \lambda \int_0^{+\infty} x|u|^2 dx. \end{aligned} \tag{4.7}$$

Since $u(t,x) \in \mathbf{C}([0,+\infty), \mathbf{C}^1(\mathbb{R}^+))$ we get

$$\int_0^{+\infty} xu_{xx}(t,x)\Theta(u(t,x))dx = 0. \tag{4.8}$$

Also we have

$$\begin{gathered} \int_0^{+\infty} \partial_t xu(t,x)\Theta(u(t,x))dx = \frac{d}{dt}\|xu(t)\|_{\mathbf{L}^1}, \\ \lambda \int_0^{+\infty} x|u|^2 dx \leq 0. \end{gathered} \tag{4.9}$$

Therefore by (4.7), (4.9) and (4.8) we find

$$\frac{d}{dt}\|xu\|_{\mathbf{L}^1} \leq 0. \tag{4.10}$$

Integration of inequality (4.10) yields

$$\|xu\|_{\mathbf{L}^1} \leq C. \tag{4.11}$$

Now we can prove the estimate (4.6) for all $t > 0$. Indeed, using (4.11) in particular, we find

$$\sup_{\xi\in\mathbb{R}} |\widehat{u}_\xi(t,\xi)| \le C\|xu(t)\|_{\mathbf{L}^1} \le C. \tag{4.12}$$

We now multiply equation (1.1) by $2u$, then integrating with respect to $x \in \mathbb{R}^+$ we get

$$\frac{d}{dt}\Big(\|u(t)\|^2_{\mathbf{L}^2} + \|u_x(t)\|^2_{\mathbf{L}^2}\Big) = -2\alpha\|u_x(t)\|^2_{\mathbf{L}^2} + \lambda\|u(t)\|^3_{\mathbf{L}^3}. \tag{4.13}$$

By construction we have

$$\widehat{u}(t,0) = \int_0^{+\infty} u(t,x)dx = 0$$

By the Plansherel theorem using the Fourier splitting method due to [29], we have

$$\begin{aligned}
\|u_x(t)\|^2_{\mathbf{L}^2} &= \|\xi\widehat{u}(t)\|^2_{\mathbf{L}^2}\\
&\ge \int_{|\xi|\ge\delta} |\widehat{u}(t,\xi)|^2|\xi|^2 d\xi \ge \delta^2\int_{|\xi|\ge\delta} |\widehat{u}(t,\xi)|^2 d\xi\\
&= \delta^2\int_{-\infty}^{+\infty} |\widehat{u}(t,\xi)|^2 d\xi - \delta^2\int_{|\xi|<\delta} |\widehat{u}(t,\xi) - \widehat{u}(t,0)|^2 d\xi\\
&= \delta^2\int_0^{+\infty} |\widehat{u}(t,\xi)|^2 d\xi - \delta^2\int_{|\xi|<\delta} |\int_0^\xi \widehat{u}_{\xi_1}(t,\xi_1)d\xi_1|^2 d\xi\\
&\ge \delta^2\|u(t)\|^2_{\mathbf{L}^2} - 2\delta^5\sup_{|\xi|\le\delta}|\widehat{u}_\xi(t,\xi)|^2,
\end{aligned}$$

where $\delta > 0$. Thus from (4.13) we have the inequality

$$\frac{d}{dt}\|u(t)\|^2_{\mathbf{H}^1} \le -\alpha\delta^2\|u(t)\|^2_{\mathbf{H}^1} + 4\alpha\delta^5\sup_{|\xi|\le\delta}|\widehat{u}_\xi(t,\xi)|^2. \tag{4.14}$$

We choose $\alpha\delta^2 = 2(1+t)^{-1}$ and change $\|u(t)\|^2_{\mathbf{H}^1} = (1+t)^{-2}W(t)$. Then via (4.13) we get from (4.14)

$$\frac{d}{dt}W(t) \le C(1+t)^{-1/2}. \tag{4.15}$$

Integration of (4.15) with respect to time yields

$$W(t) \le \|u_0\|^2_{\mathbf{H}^1} + C\big((1+t)^{-\frac{1}{2}+1} - 1\big).$$

Therefore we obtain a time decay estimate in the $\mathbf{L}^2$-norm,

$$\|u(t)\|_{\mathbf{L}^2} \le C(1+t)^{-3/4} \tag{4.16}$$

for all $t > 0$. We now multiply equation (1.1) by x^2u, then integrating with respect to $x \in \mathbb{R}^+$ we get

$$\begin{aligned}
&\frac{d}{dt}\big(\|xu(t)\|^2_{\mathbf{L}^2} + \|xu_x(t)\|^2_{\mathbf{L}^2} - 2\|u(t)\|^2_{\mathbf{L}^2}\big)\\
&= -2\alpha\|xu_x(t)\|^2_{\mathbf{L}^2} + 2\alpha\|u(t)\|^2_{\mathbf{L}^2} + \lambda\int_0^{+\infty} x^2|u|^3 dx.
\end{aligned} \tag{4.17}$$

By the Plansherel theorem using the Fourier splitting method due to [29], we have

$$\begin{aligned}\|xu_x(t)\|^2_{\mathbf{L}^2} &= \|\partial_\xi \xi \widehat{u}(t)\|^2_{\mathbf{L}^2} = \int_{-\infty}^{\infty} |\widehat{u} + \xi \widehat{u}_\xi|^2 |d\xi| \\ &\geq \int_{-\infty}^{\infty} |\xi \widehat{u}_\xi|^2 |d\xi| - \int_{-\infty}^{\infty} |\widehat{u}|^2 |d\xi| \\ &\geq \int_{|\xi|\geq\delta} |\xi \widehat{u}_\xi|^2 d\xi - \|u(t)\|^2_{\mathbf{L}^2} \\ &= \delta^2 \int_{-\infty}^{+\infty} |\widehat{u}_\xi(t,\xi)|^2 d\xi - \delta^2 \int_{|\xi|<\delta} |\widehat{u}_\xi(t,\xi)|^2 d\xi - \|u(t)\|^2_{\mathbf{L}^2} \\ &\geq \delta^2 \|xu(t)\|^2_{\mathbf{L}^2} - 2\delta^3 \sup_{|\xi|\leq\delta} |\widehat{u}_\xi(t,\xi)|^2 - \|u(t)\|^2_{\mathbf{L}^2}\end{aligned}$$

where $\delta > 0$. Thus from (4.17) we have the inequality

$$\begin{aligned}&\frac{d}{dt}\big(\|xu(t)\|^2_{\mathbf{H}^1} - 2\|u(t)\|^2_{\mathbf{L}^2}\big) \\ &\leq -\alpha\delta^2 \|xu(t)\|^2_{\mathbf{H}^1} + 2\alpha\delta^3 \sup_{|\xi|\leq\delta} |\widehat{u}_\xi(t,\xi)|^2 + 2(\alpha+1)\|u(t)\|^2_{\mathbf{L}^2}\end{aligned} \tag{4.18}$$

We choose $\alpha\delta^2 = 2(1+t)^{-1}$ and change $\|u(t)\|^2_{\mathbf{H}^1} = (1+t)^{-2}W(t)$. Then via (4.12),(4.16) we get from (4.18)

$$\frac{d}{dt}W(t) \leq C(1+t)^{\frac{1}{2}}. \tag{4.19}$$

Integration of (4.19) with respect to time yields

$$W(t) \leq \|xu_0\|^2_{\mathbf{H}^1} + C\big((1+t)^{\frac{1}{2}+1} - 1\big).$$

Therefore, we obtain a time decay estimate of the $\mathbf{L}^2$ - norm

$$\|xu(t)\|_{\mathbf{L}^2} \leq C(1+t)^{-1/4}$$

Lemma 4.2 is proved □

By Lemmas 2.2 and 4.2, we have

$$\begin{aligned}\|u(t)\|_{\mathbf{L}^\infty} &\leq \|\mathcal{G}(t)u_0\|_{\mathbf{L}^\infty} + C\int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{-\frac{1}{2}} \|u(\tau)\|^2_{\mathbf{L}^2} d\tau \\ &\quad + C\int_0^{t/2} \langle t-\tau\rangle^{-1} \|x|u|u(\tau)\|_{\mathbf{L}^1} d\tau \\ &\leq C\langle t\rangle^{-1} + C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-1} \langle\tau\rangle^{-1} d\tau \\ &\leq C\langle t\rangle^{-1} \log\langle t\rangle\end{aligned}$$

for all $t > 0$. In the same manner

$$\begin{aligned}
&\|\partial_x^2 u(t)\|_{\mathbf{L}^\infty} \\
&\le C\langle t\rangle^{-2} + \int_0^t \|\partial_x \mathcal{G}(t-\tau)\mathcal{B}|u|u(\tau)\|_{\mathbf{L}^\infty} d\tau \\
&\le C\langle t\rangle^{-2} + C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-2}\|u(\tau)\|_{\mathbf{L}^2}^2 d\tau + C\int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-1}\|u(\tau)\|_{\mathbf{L}^\infty}^2 d\tau \\
&\le C\langle t\rangle^{-2} + C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-2}\langle\tau\rangle^{-\frac{3}{2}} d\tau + C\int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-1}\langle\tau\rangle^{-2}\log^2\langle\tau\rangle d\tau \\
&\le C\langle t\rangle^{-2}\log^3\langle t\rangle
\end{aligned}$$

for all $t > 0$. Denote $f(t,x) = u_{xxt}$. Then by Lemmas 4.2 and 2.2 we have the estimates

$$\begin{aligned}
&\|f(t)\|_{\mathbf{L}^\infty} \\
&= \|\partial_x^2 u_t(t)\|_{\mathbf{L}^\infty} \\
&\le \|\partial_x^2\partial_t \mathcal{G}(t)u_0\|_{\mathbf{L}^\infty} + \|\partial_x^2 \mathcal{B}|u|u(t)\|_{\mathbf{L}^\infty} \\
&\quad + C\int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-1}\|\partial_x^2\mathcal{B}|u|u(\tau)\|_{\mathbf{L}^\infty} d\tau + C\int_0^{t/2} \langle t-\tau\rangle^{-3}\|x|u|u(\tau)\|_{\mathbf{L}^1} d\tau \\
&\le C\langle t\rangle^{-3}\log^4\langle t\rangle + C\int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-1}\langle\tau\rangle^{-3}\log^4\langle\tau\rangle d\tau + C\int_0^{t/2} \langle t-\tau\rangle^{-3}\langle\tau\rangle^{-1} d\tau \\
&\le C\langle t\rangle^{-3}\log^4\langle t\rangle
\end{aligned}$$

for all $t > 0$. We a sufficiently small $\varepsilon > 0$ and consider the following two auxiliary problems

$$\begin{gathered}
U_t - U_{xx} + U^2 = \varepsilon|f|, \quad x \in \mathbb{R}^+,\ t > 0, \\
U(0,x) = \varepsilon|u_0(x)|, \quad x \in \mathbb{R}^+, \\
U(x,0) = 0, \quad t > 0
\end{gathered} \tag{4.20}$$

and

$$\begin{gathered}
V_t - V_{xx} + \varepsilon V^2 = |f|, \quad x \in \mathbb{R}^+,\ t > 0, \\
V(0,x) = |u_0(x)|, \quad x \in \mathbb{R}^+, \\
V(x,0) = 0, \quad t > 0.
\end{gathered} \tag{4.21}$$

Note that problem (4.21) can be reduced to problem (4.20) by the change of variable $V = \varepsilon^{-1}U$. Also note that (4.20) has a sufficiently small initial data and a small force $\varepsilon|u_{xxt}|$. Applying results of paper we obtain an almost optimal time decay estimate

$$\|U(t)\|_{\mathbf{L}^\infty} \le C\langle t\rangle^{-1},$$

hence by Lemma 4.1 we get $\|u(t)\|_{\mathbf{L}^\infty} \le C\langle t\rangle^{-1}$.

Now we estimate the $\mathbf{L}^{1,1+a}$-norm of the solution

$$\begin{aligned}\|u(t)\|_{\mathbf{L}^{1,1+a}} &\le \|\mathcal{G}(t)u_0\|_{\mathbf{L}^{1,1+a}} \\ &\quad + C\int_0^t (\|\mathcal{B}|u|u(\tau)\|_{\mathbf{L}^{1,1+a}} + \langle t-\tau\rangle^{a/2}\|\mathcal{B}|u|u(\tau)\|_{\mathbf{L}^{1,1}})d\tau \\ &\le C\langle t\rangle^{a/2} + C\int_0^t \langle\tau\rangle^{-1}\|u(\tau)\|_{\mathbf{L}^{1,a}}d\tau + C\int_0^t \langle t-\tau\rangle^{\frac{a}{2}}\langle\tau\rangle^{-1}d\tau \\ &\le C\langle t\rangle^{a/2} + C\int_0^t \langle\tau\rangle^{-1}\|u(\tau)\|_{\mathbf{L}^{1,1+a}}d\tau.\end{aligned}$$

Hence by Granwall's inequality we obtain

$$\|u(t)\|_{\mathbf{L}^{1,1+a}} \le C\langle t\rangle^{a/2} \tag{4.22}$$

for all $t>0$. In the same manner we estimate the $\mathbf{L}^{1,1+a}$ - norm of f. By Lemma 2.2 we have

$$\begin{aligned}\|f(t)\|_{\mathbf{L}^{1,1+a}} &= \|u_{xxt}(t)\|_{\mathbf{L}^{1,1+a}} \\ &\le \|\partial_x^2\partial_t\mathcal{G}(t)u_0\|_{\mathbf{L}^{1,1+a}} + \|\partial_x^2\mathcal{B}|u|u(t)\|_{\mathbf{L}^{1,1+a}} \\ &\quad + C\int_0^t \langle t-\tau\rangle^{-1}(\||u|u(\tau)\|_{\mathbf{L}^{1,1+a}} + \langle t-\tau\rangle^{a/2}\||u|u(\tau)\|_{\mathbf{L}^{1,1}})d\tau \\ &\le C\langle t\rangle^{-1+\frac{a}{2}} + C\int_0^t \langle t-\tau\rangle^{-1}\langle\tau\rangle^{\frac{a}{2}-1}d\tau + C\int_0^t \langle t-\tau\rangle^{\frac{a}{2}-1}\langle\tau\rangle^{-1}d\tau \\ &\le C\langle t\rangle^{-1+\frac{a}{2}}.\end{aligned}$$

Thus we can apply the results of paper [21] to get the estimate of the functions $U(t,x)$ and $V(t,x)$. Then by Lemma 4.1 we get an optimal time decay estimate for the solution

$$\|u(t)\|_{\mathbf{L}^\infty} \le C\varepsilon^{-1}\langle t\rangle^{-1}(\log(2+t))^{-1} \tag{4.23}$$

for all $t>0$.

We make a change of the dependent variable $u(t,x)=v(t,x)e^{-\varphi(t)}$ as in the proof of Theorem 1.1. Then we obtain problem (3.1) for new functions $(v(t,x),\varphi(t))$. Now we prove the following estimate

$$\|v(t)\|_{\mathbf{L}^{1,1+a}} \le C\langle t\rangle^{a/2}$$

for all $t>0$. From (3.1) we obtain the integral formula

$$v(t) = \mathcal{G}(t)u_0 + \lambda\int_0^t d\tau\mathcal{G}(t-\tau)\mathcal{B}\Big(|u(\tau)|v(\tau) - \frac{v(\tau)}{\theta}\int_{\mathbb{R}^n}|u(\tau)|v(\tau)dx\Big). \tag{4.24}$$

Using (4.23) and Lemma 2.2 we have

$$\begin{aligned}&\Big\|\int_0^t d\tau\mathcal{G}(t-\tau)\mathcal{B}(|u(\tau)|v(\tau) - \frac{v(\tau)}{\theta}\int_0^{+\infty}|u(\tau)|v(\tau)dx)\Big\|_{\mathbf{L}^{1,1+a}} \\ &\le C\int_0^t \||u(\tau)|v(\tau) - \frac{v(\tau)}{\theta}\int_0^{+\infty}|u(\tau)|v(\tau)dx\|_{\mathbf{L}^{1,1+a}}d\tau \\ &\le C\int_0^t \langle\tau\rangle^{-1}(\log(2+\tau))^{-1}\|v(\tau)\|_{\mathbf{L}^{1,1+a}}d\tau\end{aligned}$$

for all $t > 0$. Therefore, by (4.22) we find

$$\begin{aligned}\|v(t)\|_{\mathbf{L}^{1,1+a}} &\le \|\mathcal{G}(t)v(0)\|_{\mathbf{L}^{1,1+a}} + C\int_0^t \langle\tau\rangle^{-1}(\log(2+\tau))^{-1}\|v(\tau)\|_{\mathbf{L}^{1,1+a}}d\tau \\ &\le C\langle t\rangle^{a/2} + C\int_0^t \langle\tau\rangle^{-1}(\log(2+\tau))^{-1}\|v(\tau)\|_{\mathbf{L}^{1,1+a}}d\tau\end{aligned}$$

for all $t > 0$. Hence by Granwall's inequality we obtain

$$\|v(t)\|_{\mathbf{L}^{1,1+a}} \le C\langle t\rangle^{a/2}$$

for all $t > 0$. In the same way from (4.23) and Lemma 2.2 we get

$$\begin{aligned}\|v(t)\|_{\mathbf{L}^p} &\le \|\mathcal{G}(t)v(0)\|_{\mathbf{L}^p} \\ &\quad + \Big\|\int_0^t d\tau\mathcal{G}(t-\tau)\mathcal{B}(|u(\tau)|v(\tau) - \frac{v(\tau)}{\theta}\int_0^{+\infty}|u(\tau)|v(\tau)dx)\Big\|_{\mathbf{L}^p} \\ &\le C\langle t\rangle^{-\frac{1}{2}-\frac{1}{2}(1-\frac{1}{p})} \\ &\quad + C\int_0^{t/2}(t-\tau)^{-\frac{1}{2}-\frac{1}{2}(1-\frac{1}{p})-\frac{a}{2}}\langle\tau\rangle^{\frac{a}{2}-1}(\log(2+\tau))^{-1}d\tau \\ &\quad + C\int_{\frac{t}{2}}^t \langle\tau\rangle^{-1}(\log(2+\tau))^{-1}\|v(\tau)\|_{\mathbf{L}^p}d\tau \\ &\le C\langle t\rangle^{-\frac{1}{2}-\frac{1}{2}(1-\frac{1}{p})} + \int_{\frac{t}{2}}^t \langle\tau\rangle^{-1}\|v(\tau)\|_{\mathbf{L}^p}d\tau\end{aligned}$$

for all $t > 0$. So by Granwall's inequality we have

$$\|v(t)\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{1}{2}-\frac{1}{2}(1-\frac{1}{p})}$$

for all $t > 0$. Now from (4.24) and Lemma 2.2 we get

$$\begin{aligned}\|v(t) - \mathcal{G}(t)u_0\|_{\mathbf{L}^{1,1}} &\le \int_0^{t/2}(t-\tau)^{-\frac{a}{2}}\Big\||u(\tau)|v(\tau) - \frac{v(\tau)}{\theta}\int_{\mathbb{R}^n}|u(\tau)|v(\tau)dx\Big\|_{\mathbf{L}^{1,1+a}}d\tau \\ &\quad + \int_{\frac{t}{2}}^t \Big\||u(\tau)|v(\tau) - \frac{v(\tau)}{\theta}\int_0^{+\infty}|u(\tau)|v(\tau)dx\Big\|_{\mathbf{L}^{1,1}}d\tau \\ &\le C\int_0^{t/2}(t-\tau)^{-\frac{a}{2}}\langle\tau\rangle^{\frac{a}{2}-1}(\log(2+\tau))^{-1}d\tau \\ &\quad + C\int_{\frac{t}{2}}^t \langle\tau\rangle^{-1}(\log(2+\tau))^{-1}d\tau \\ &\le C\log(2+t)^{-1}\end{aligned} \tag{4.25}$$

for all $t > 0$. Therefore using Lemma3.1 we find for $h(t) = e^{\varphi(t)}$,

$$|h(t) - g(t)| \le C\log g(t),$$

for all $t > 0$. Then from $u(t,x) = e^{-\varphi(t)}v(t,x) = h^{-1}(t)v(t,x)$, we have

$$\begin{aligned}&\|u(t) - \theta e^{-\varphi(t)}G_0(t)\|_{\mathbf{L}^\infty} \\ &\le \|u(t) - e^{-\varphi(t)}\mathcal{G}(t)u_0\|_{\mathbf{L}^\infty} + \|\mathcal{G}(t)u_0 - \theta e^{-\varphi(t)}G_0(t)\|_{\mathbf{L}^\infty} \\ &\le Ct^{-1}g^{-2}(t) + Ct^{-1-\frac{a}{2}}\|u_0\|_{\mathbf{L}^{1,1+a}} \\ &\le Ct^{-1}g^{-2}(t)\end{aligned} \tag{4.26}$$

for all $t > 0$. Also we have

$$\|\theta G_0(t)(h^{-1}(t) - g^{-1}(t))\|_{\mathbf{L}^\infty} \le Ct^{-1} g^{-2}(t)|h(t) - g(t)|,$$

and so by (4.26),

$$\|u(t) - \theta G_0(t) g^{-1}(t)\|_{\mathbf{L}^\infty} \le C(1+t)^{-1} g^{-2}(t) \log g(t).$$

Theorem 1.2 is proved.

Acknowledgements. The author would like to thank the anonymous referee for many useful suggestions and comments. This work is partially supported by CONACyT.

References

[1] C. J. Amick, J. L. Bona, M. E. Schonbek, *Decay of solutions of some nonlinear wave equations* , J. Diff. Eq., **81**, No. 4 (1989), pp. 1-49.

[2] V. Bisognin, *On the asymptotic behavior of the solutions of a nonlinear dispersive system of Benjamin-Bona-Mahony type* , Boll. Un. Mat. Ital. B., **10**, No. 7 (1996), pp. 99-128.

[3] J. L. Bona, L. Luo, *Decay of solutions to nonlinear,dispersive wave equations* , J. Diff. and Int. Eq.,**6**, (1993), pp. 961-980.

[4] J. L. Bona, L. Luo, *More results on the decay of solutions to nonlinear,dispersive wave equations* , Discrete and Contin. Dinamical System, **1**, (1995), pp. 151-193.

[5] A. Constantin, J. Echer, *Way breaking for nonlinear nonlocal shallow water equations* , Acta matematica, **181**, (1998), pp. 229-243.

[6] M. Escobedo, O. Kavian, *Asymptotic behavior of positive solutions of a non-linear heat equation*, Houston Journal of Mathematics **13**, No. 4 (1987), pp. 39-50.

[7] M. Escobedo, O. Kavian, H. Matano, *Large time behavior of solutions of a dissipative nonlinear heat equation,* Comm. Partial Diff. Eqs., **20** (1995), pp. 1427-1452.

[8] V. A. Galaktionov, J. L. Vazquez, *Asymptotic behavior of nonlinear parabolic equations with critical exponents. A dynamical systems approach,* J. Funct. Anal. **100** (1991), no. 2, pp. 435-462.

[9] V. A. Galaktionov, S. P. Kurdyumov, A. A. Samarskii, *On asymptotic eigenfunctions of the Cauchy problem for a nonlinear parabolic equation,* Math. USSR Sbornik **54** (1986), pp. 421-455.

[10] A. Gmira, L. Veron, *Large time behavior of the solutions of a semilinear parabolic equation in* $\mathbb{R}^n$, J. Diff. Eqs. **53** (1984), pp. 258-276.

[11] G. V. Demidenko, S. V. Uspenskii, *Equations and systems that are not solved with respect to the highest derivative,* Nauchnaya Kniga, Novosibirsk, 1998, 437 pp.

[12] I. E. Egorov, S. G. Pyatkov, S. V. Popov, *Nonclassical operator-differential equations,* Nauka, Novosibirsk, 2000. 336 pp.

[13] A. Favini, A. Yagi, *Degenerate differential equations in Banach spaces,* **215**, Marcel Dekker, Inc., New York, 1999, 313 pp.

[14] H. Fujita, *On the blowing-up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. of Tokyo, Sect. I, **13** (1966), pp. 109-124.

[15] H. Gajewski, K. Gröger, K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen,* Akademie-Verlag, Berlin, 1974, 281 pp.

[16] N. Hayashi, E. I. Kaikina, *Nonlinear Theory of Pseudodifferential Equations on a Half-Line,* North-Holland Mathematics Studies, **194**, 340 p, 2004.

[17] N. Hayashi, P. I. Naumkin, *Asymptotics for the Korteweg-de Vries-Burgers equation*, Preprint 2003.

[18] E. I. Kaikina, P. I. Naumkin, I. A. Shishmarev, *The Cauchy problem for Sobolev type equation with power like nonlinearity*, Izvestiya, Ser. Math., **69** (2005), No. 1.

[19] N. Hayashi, E. I. Kaikina, P. I. Naumkin, I. A. Shishmarev, *Asymptotics for the dissipative nonlinear equations*, Springer, **1884,** (2007).

[20] N. Hayashi, E. I. Kaikina, P.I. Naumkin, *Large time behavior of solutions to the dissipative nonlinear Schrödinger equation,* Proceedings of the Royal Soc. Edingburgh, **130**-A (2000), pp. 1029-1043.

[21] N. Hayashi, E. I. Kaikina, P. I. Naumkin, *Global existence and time decay of small solutions to the Landau-Ginzburg type equations,* Journal d'Analyse Mathematique, **90** (2003), pp. 141–173.
[22] G. Karch, *Asymptotic behavior of solutions to some pseudoparabolic equations*, Mat. Meth. in the Applied Sciences, **20** (1997), 271–289.
[23] G. Karch, *Large time behavior of solutions to nonlinear wave equations*, Mat. Meth. in the Applied Sciences, **22** (1999), 1671–1697.
[24] S. Kamin, L. A. Peletier, *Large time behavior of solutions of the porous media equation with absorption,* Israel J. Math. **55** (1986), no.2, pp. 129-146.
[25] O. Kavian, *Remarks on the large time behavior of a nonlinear diffusion equation*, Ann. Inst. Henri Poincaré, Analyse non lin éaire, **4**, No. 5 (1987), pp. 423-452.
[26] E. Mitidieri, S. I. Pokhozhaev, *A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities,* Proc. Steklov Inst. Math. (2001), no. **3** (234), 362 pp.
[27] M. Mei, C. Schmeiser, *Asymptotic profiles of solutions for the BBM-Burgers equations ,* Funkcial.Ekvac., (2001), no. **44** , 151-170.
[28] R. Prado, E. Zuazua, *Asymptotic expansion for the generalized Benjamin-Bona-Majoni-Burgers equation*, J. Diff. and Int. Eq., **15**, (2002), N. 12, pp. 1409-1434.
[29] M. E. Schonbek, *The Fourier splitting method,* Advances in geometric analysis and continuum mechanics, (Stanford, CA, 1993) Internat. Press. Cambridge, MA (1995), pp. 269-274.
[30] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations,* **49**, American Mathematical Society, Providence, RI, 1997, 278 pp.
[31] U. Stefanelli, *On a class of doubly nonlinear nonlocal evolution equations,* Differential Integral Equations, **15** (2002), no. 8, 897–922.
[32] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, **30**, Princeton Univ. Press, Princeton, NJ, 1970.
[33] L. Zhang, *Decay of solutions of the generalized Benjamin-Bona-Majoni equation*, Acta Math. Sinica, New Series, **10**, (1994), pp. 428-438.

Elena I. Kaikina
Instituto de Matemáticas, Universidad Nacional Autónoma de México, Campus Morelia, AP 61-3 (Xangari), Morelia CP 58089, Michoacán, Mexico
E-mail address: ekaikina@matmor.unam.mx