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ON THE ESSENTIAL SPECTRA OF MATRIX OPERATORS AND APPLICATIONS

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Abstract. In this paper, we investigate the essential spectra of some matrix operators on Banach spaces. The results obtained are used for describing the essential spectra of differential operators.

1. Introduction

This paper is devoted to the study of the essential spectra of 2×2 matrix operators of the form

$$L_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

considered on the product Banach space $X := X_1 \times X_2$. In general, the operators occurring in the representation L_0 are unbounded. The operator A acts on the Banach space X_1 and has the domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space X_2, and the intertwining operator B (resp. C) is defined on the domain $\mathcal{D}(B)$ (resp. $\mathcal{D}(C)$) and acts from X_2 to X_1 (resp. from X_1 to X_2). Below, we shall assume that $\mathcal{D}(A) \subset \mathcal{D}(C)$ and $\mathcal{D}(B) \subset \mathcal{D}(D)$, and then the matrix operator L_0 defines a linear operator in X with domain $\mathcal{D}(A) \times \mathcal{D}(B)$.

One of the problems in the study of such operators is that in general L_0 is not closed or even closable, even if its entries are closed. Important results concerning the spectral theory of this type of operators have been obtained during the last years. One of these works, the paper by Atkinson and all [3] concerns the essential spectrum of such an operator. First, they give some sufficient conditions under which L_0 is closable and describe its closure which we shall denote by L. Second, they study the Wolf essential spectrum of a matrix differential operator in the case where the operators are defined on a bounded domain $\Omega \subset \mathbb{R}^n$.

The first main purpose of this paper is in a generalization of the results of [3] on the essential spectrum of the closure L to the case where the assumptions of [3] concerning the resolvent of the entry A are weakened. In [3] it is assumed that the resolvent $(A - \lambda I)^{-1}$ for some (and hence for all) λ in the resolvent set $\rho(A)$ of A is a compact operator on X_1; whereas in our paper we assume that only $(A - \lambda I)^{-1}$, $\lambda \in \rho(A)$ belongs to a two-sided closed ideal $\mathcal{I}(X_1) \subset \mathcal{F}(X_1)$ of $\mathcal{L}(X_1)$ where $\mathcal{F}(X_1)$ is the set of Fredholm perturbation on X_1 (see Definition 2.1) and $\mathcal{L}(X_1)$ denotes

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the Banach algebra of all bounded linear operators on X_1. We shall show under certain additional assumptions that the study of the essential spectrum of L will be reduced to that of the Schur complement.

$$D - \overline{C(A-\mu I)^{-1}B}$$

for some $\lambda \in \rho(A)$, where $\overline{C(A-\mu I)^{-1}B}$ denotes the closure of the operator $C(A-\mu I)^{-1}B$. Note that the condition made on the resolvent of A in [3] fails if A is an elliptic operator on a domain of infinite measure. Then, this condition is more restrictive in the applications, and the class of entries of the matrix L_0 that we consider here is essentially more general than in [3].

There are several, and in general, non-equivalent definitions of the essential spectrum of a closed operator on a Banach space. Through all this paper we are concerned with six of them (see Section 2). The second main purpose of this paper is to study and characterize the essential spectrum of L in all cases. In particular, the Wolf essential spectrum studied in [3] is included. Then, the aim of this work is to pursue the analysis of the Wolf essential spectrum started in [3]. Indeed, we extend the results obtained in [3] to a large class of operators and at the same time to the six essential spectra. Therefore the results obtained in [3] turn out to be a particular case of the results proven in this paper.

Let us conclude this introduction with some historical comments and some bibliographical references, which do not intend to be complete. The problem of the essential spectrum of differential operators of mixed order, which appears in mathematical physics was studied by many authors. Among such works we can quote for example [30, 39, 38, 1, 9]. The particular case of symmetric block differential operators with Dirichlet boundary conditions for A, the essential spectrum was studied in [1, 9]. An example from magnetohydrodynamics can be found in Section 5 in [3].

For Agmon, Douglis and Nirenberg elliptic system [2], the most general results were obtained by Grubb and Geymonat [15]. We recall, the abstract model L_0 was introduced in [3]. This model clarifies the essence of the problem and allows us to uncover details that have not been noticed before, even for concrete problems, for example, the condition that the matrix operators admit a closure and the description of the domain of this closure. Recently, in Kurasov and Nabako [32], it has been proven that the essential spectrum of self-adjoint operator associated with matrix differential operator appearing in problems of magnetic hydrodynamics, consists of two branches. The first one is called regularity spectrum and the second branch is called singularity spectrum which appears due to singularity of the coefficients.

Our paper is organised as follows: In Section 2, we introduce the algebraic framework in which our investigation will be done. The analysis is based on the concept of Fredholm perturbations. In Section 3, we introduce a general hypotheses on different entries of the operators matrix L. In Section 4, we investigate some results concerning essential spectra of L. The main results of this section are Theorems 4.1 and 4.3 which contain a general description of different types of the essential spectra of the operator L. In the end of Section 4 we give some sufficient conditions to verify our hypothesis. Finally, in Section 5 we apply the results obtained in Sections 4 to study the essential spectra of an example where A, B and C are ordinary differential operators on spaces of vector functions and D is a multiplication operator.

2. Preliminary Results

Let X and Y be Banach spaces and let A be an operator from X into Y. We denote by $\mathcal{D}(A) \subset X$ its domain and $R(A) \subset Y$ its range. We denote by $\mathcal{C}(X,Y)$ (resp. $\mathcal{L}(X,Y)$) the set of all closed, densely defined linear operators (resp. the Banach algebra of all bounded linear operators) from X into Y. For $A \in \mathcal{C}(X,Y)$, by $\sigma(A)$, $\rho(A)$ and $N(A)$ we denote the spectrum, the resolvent set and the null space of A respectively. The nullity, $\alpha(A)$, of A is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in Y. The set of upper semi-Fredholm operators from X into Y is defined by

$$\Phi_+(X,Y) = \{A \in \mathcal{C}(X,Y)\text{s uch that } \alpha(A) < \infty \text{ and } R(A) \text{ is closed in } Y\},$$

the set of lower semi-Fredholm operators from X into Y is defined by

$$\Phi_-(X,Y) = \{A \in \mathcal{C}(X,Y) : \beta(A) < \infty \text{ and } R(A) \text{ is closed in } Y\},$$

the set of semi-Fredholm operators from X into Y is defined by

$$\Phi_\pm(X,Y) = \Phi_+(X,Y) \cup \Phi_-(X,Y),$$

the set of Fredholm operators from X into Y is defined by

$$\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y),$$

the set of bounded Fredholm operators from X into Y is defined by

$$\Phi^b(X,Y) = \Phi(X,Y) \cap \mathcal{L}(X,Y),$$

and the set Φ_A is defined by

$$\Phi_A = \{\lambda \in \mathbb{C} : \lambda - A \in \Phi(X,Y)\}.$$

If $A \in \Phi(X,Y)$, the number $i(A) = \alpha(A) - \beta(A)$ is called the index of A. The subset of all compact operators of $\mathcal{L}(X,Y)$ is denoted by $\mathcal{K}(X,Y)$. If $X = Y$ then $\mathcal{L}(X,Y)$, $\mathcal{K}(X,Y)$, $\mathcal{C}(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_\pm(X,Y)$, $\Phi(X,Y)$ and $\Phi^b(X,Y)$ are replaced, respectively, by $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\mathcal{C}(X)$, $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_\pm(X)$, $\Phi(X)$ and $\Phi^b(X)$.

Definition 2.1. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X,Y)$. F is called a Fredholm perturbation if $U + F \in \Phi^b(X,Y)$ whenever $U \in \Phi^b(X,Y)$.

The set of Fredholm perturbations is denoted by $\mathcal{F}^b(X,Y)$. This class of operators is introduced and investigated in [10]. In particular, it is shown that $\mathcal{F}^b(X,Y)$ is a closed subset of $\mathcal{L}(X,Y)$ and if $X = Y$, then $\mathcal{F}^b(X) := \mathcal{F}^b(X,X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.

Proposition 2.2 ([10, pp. 69-70])**.** *Let X, Y, Z be Banach spaces. If at least one of the sets $\Phi^b(X,Y)$ or $\Phi^b(Y,Z)$ is not empty, then*

(i) *$F \in \mathcal{F}^b(X,Y)$, $A \in \mathcal{L}(Y,Z)$ imply $AF \in \mathcal{F}^b(X,Z)$.*
(ii) *$F \in \mathcal{F}^b(Y,Z)$, $A \in \mathcal{L}(X,Y)$ imply $FA \in \mathcal{F}^b(X,Z)$.*

Definition 2.3. Let X be a Banach space and $R \in \mathcal{L}(X)$. R is said to be a Riesz operator if $\Phi_R = \mathbb{C} \setminus \{0\}$.

For further information on the family of Riesz operators we refer to [4, 27] and the references therein.

Remark 2.4. (i) In [44], it is proved that $\mathcal{F}^b(X)$ is the largest ideal of $\mathcal{L}(X)$ contained in the family of Riesz operators.

(ii) Let X and Y be two Banach spaces. If in Definition 2.1 we replace $\Phi^b(X,Y)$ by $\Phi(X,Y)$ we obtain the set $\mathcal{F}(X,Y)$.

Definition 2.5. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X,Y)$. Then F is called an upper (resp. lower) Fredholm perturbation if $U+F \in \Phi_+^b(X,Y) := \Phi_+(X,Y) \cap \mathcal{L}(X,Y)$ (resp. $\Phi_-^b(X,Y) := \Phi_-(X,Y) \cap \mathcal{L}(X,Y)$) whenever $U \in \Phi_+^b(X,Y)$ (resp. $\Phi_-^b(X,Y)$).

The sets of upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}_+^b(X,Y)$ and $\mathcal{F}_-^b(X,Y)$, respectively. In [11], it is shown that $\mathcal{F}_+^b(X,Y)$ and $\mathcal{F}_-^b(X,Y)$ are closed subsets of $\mathcal{L}(X,Y)$, and if $X = Y$, then $\mathcal{F}_+^b(X) := \mathcal{F}_+^b(X,X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.

Remark 2.6. Let X and Y be two Banach spaces. If in Definition 2.5 we replace $\Phi_+^b(X,Y)$ (resp. $\Phi_-^b(X,Y)$) by $\Phi_+(X,Y)$ (resp. $\Phi_-(X,Y)$) we obtain the set $\mathcal{F}_+(X,Y)$ (resp. $\mathcal{F}_-(X,Y)$).

Definition 2.7. An operator $A \in \mathcal{L}(X,Y)$ is said to be weakly compact if $A(B)$ is relatively weakly compact in Y for every bounded subset $B \subset X$.

The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X,Y)$. If $X = Y$, the family of weakly compact operators on X, $\mathcal{W}(X) := \mathcal{W}(X,X)$, is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (cf. [8, 12]).

Definition 2.8. Let X and Y be two Banach spaces. An operator $A \in \mathcal{L}(X,Y)$ is called strictly singular if, for every infinite-dimensional subspace M, the restriction of A to M is not a homeomorphism.

Let $\mathcal{S}(X,Y)$ denote the set of strictly singular operators from X into Y. The concept of strictly singular operators was introduced in the pioneering paper by Kato [28] as a generalization of the notion of compact operators. For a detailed study of the properties of strictly singular operators we refer to [12, 28]. For our own use, let us recall the following four facts. The set $\mathcal{S}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$, if $X = Y$, $\mathcal{S}(X) := \mathcal{S}(X,X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$. If X is a Hilbert space then $\mathcal{K}(X) = \mathcal{S}(X)$. The class of weakly compact operators on L_1-spaces (resp. $C(K)$-spaces with K a compact Haussdorff space) is nothing else but the family of strictly singular operators on L_1-spaces (resp. $C(K)$-spaces) (see [41, Theorem 1]).

Let X be a Banach space. If N is a closed subspace of X, we denote by π_N^X the quotient map $X \to X/N$. The codimension of N, $\operatorname{codim}(N)$, is defined as the dimension of the vector space X/N.

Definition 2.9. Let X and Y be two Banach spaces and $S \in \mathcal{L}(X,Y)$. S is said to be strictly cosingular operator from X into Y, if there exists no closed subspace N of Y with $\operatorname{codim}(N) = \infty$ such that $\pi_N^Y S : X \to Y/N$ is surjective.

Let $C\mathcal{S}(X,Y)$ denote the set of strictly cosingular operators from X into Y. This class of operators was introduced by Pelczynski [41]. It forms a closed subspace of $\mathcal{L}(X,Y)$ which is, $C\mathcal{S}(X) := C\mathcal{S}(X,X)$, a closed two-sided ideal of $\mathcal{L}(X)$ if $X = Y$ (cf. [46]).

Definition 2.10. A Banach space X is said to have the Dunford-Pettis property (for short property DP) if for each Banach space Y every weakly compact operator $T : X \to Y$ takes weakly compact sets in X into norm compact sets of Y.

It is well known that any L_1 space has the DP property [7]. Also, if Ω is a compact Hausdorff space, $C(\Omega)$ has the DP property [14]. For further examples we refer to [6] or [8, p. 494, 497, 508, 511]. Note that the DP property is not preserved under conjugation. However, if X is a Banach space whose dual has the DP property then X has the DP property (see [14]). For more information we refer to the paper by Diestel [6] which contains a survey and exposition of the Dunford-Pettis property and related topics.

The following identity was established in [35, Lemma 2.3 (ii)].

Lemma 2.11 ([35]). *Let X be an arbitrary Banach space. Then*
$$\mathcal{F}(X) = \mathcal{F}^b(X),$$
where $\mathcal{F}(X) := \mathcal{F}(X, X)$.

An immediate consequence of this result is that $\mathcal{F}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.

Remark 2.12. Let X and Y be two Banach spaces. In contrast to the result of Lemma 2.11, the fact that $\mathcal{F}(X, Y)$ is equal or not to $\mathcal{F}^b(X, Y)$ seems to be unknown.

In general, we have the following inclusions:
$$\mathcal{K}(X) \subset \mathcal{S}(X) \subset \mathcal{F}^b_+(X) \subset \mathcal{F}(X),$$
$$\mathcal{K}(X) \subset C\mathcal{S}(X) \subset \mathcal{F}^b_-(X) \subset \mathcal{F}(X)$$
where $\mathcal{F}^b_-(X) = \mathcal{F}^b_-(X, X)$.

We say that X is weakly compactly generating (w.c.g.) if the linear span of some weakly compact subset is dense in X. For more details and results we refer to [6]. In particular, all separable and all reflexive Banach spaces are w.c.g. as well as $L_1(\Omega, d\mu)$ if (Ω, μ) is σ-finite.

It is proved in [47] that if X is a w.c.g. Banach space then
$$\mathcal{F}_+(X) = \mathcal{S}(X) \quad \text{and} \quad \mathcal{F}_-(X) = C\mathcal{S}(X).$$
We say that X is subprojective, if given any closed infinite-dimensional subspace M of X, there exists a closed and finite dimensional subspace $N \subset M$ and a continuous projection from X onto N. Clearly any Hilbert space is subprojective. The spaces c_0, l_p, $(1 \le p < \infty)$, and L_p $(2 \le p < \infty)$, are also subprojective (cf. [48]).

We say that X is superprojective if every subspace V having infinite codimension in X is contained in a closed subspace W having infinite codimension in X and such that there is a bounded projection from X to W. The spaces l_p, $(1 < p < \infty)$, and L_p $(1 < p \le 2)$, are superprojective (cf. [48]).

Let X be a w.c.g. Banach space. If X is superprojective (resp. subprojective) then $\mathcal{S}(X) \subset C\mathcal{S}(X)$ (resp. $C\mathcal{S}(X) \subset \mathcal{S}(X)$). So, $\mathcal{S}(X) \subset \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$ (resp. $C\mathcal{S}(X) \subset \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$), where $\mathcal{F}_+(X) := \mathcal{F}_+(X, X)$ and $\mathcal{F}_-(X) := \mathcal{F}_-(X, X)$.

Let (Ω, Σ, μ) be a positive measure space and let X_p denote the spaces $L_p(\Omega, d\mu)$ with $1 \le p < \infty$. Since $p \in [1, \infty)$ the spaces X_p are w.c.g. and consequently we have $\mathcal{F}_+(X_p) = \mathcal{S}(X_p)$ and $\mathcal{F}_-(X_p) = C\mathcal{S}(X_p)$. In X_p and in $C(E)$ (the Banach space of continuous scalar-valued function on E with the supremum norm) provided that E is a compact Hausdorff space we have $\mathcal{S}(X_p) = C\mathcal{S}(X_p) = \mathcal{F}(X_p)$ (cf. [47]).

For a self-adjoint operator A on a Hilbert space X, the essential spectrum of A is the set of limit points of the spectrum of A (with eigenvalues counted according to their multiplicities), i.e., all points of the spectrum except isolated eigenvalues of finite multiplicity (see, for example, [49, 50]).

If X is a Banach space and $A \in \mathcal{C}(X)$, there are many ways to define the essential spectrum. We recall six of them:

- $\sigma_{e1}(A) := \{\lambda \in \mathbb{C} : \lambda - A \notin \Phi_+(X)\} = \mathbb{C} \setminus \Phi_{+A}$,
- $\sigma_{e2}(A) := \{\lambda \in \mathbb{C} : \lambda - A \notin \Phi_-(X)\} = \mathbb{C} \setminus \Phi_{-A}$,
- $\sigma_{e3}(A) = \{\lambda \in \mathbb{C} : \lambda - A \notin \Phi_\pm(X)\} = \mathbb{C} \setminus \Phi_{\pm A}$,
- $\sigma_{e4}(A) := \{\lambda \in \mathbb{C} : \lambda - A \notin \Phi(X)\} = \mathbb{C} \setminus \Phi_A$,
- $\sigma_{e5}(A) := \mathbb{C} \setminus \rho_5(A)$,
- $\sigma_{e6}(A) := \mathbb{C} \setminus \rho_6(A)$.

where $\rho_5(A) := \{\lambda \in \Phi_A : i(\lambda - A) = 0\}$ and

$$\rho_6(A) := \{\lambda \in \rho_5(A) \text{ such that scalars near } \lambda \text{ are in } \rho(A)\}.$$

The subsets $\sigma_{e1}(.)$ and $\sigma_{e2}(.)$ are the Gustafson and Weidmann essential spectra [16]. $\sigma_{e3}(.)$ is the Kato essential spectrum [29]. $\sigma_{e4}(.)$ is the Wolf essential spectrum [13, 50, 42, 49]. $\sigma_{e5}(.)$ is the Schechter essential spectrum [13, 16, 42, 43], and $\sigma_{e6}(.)$ denotes the Browder essential spectrum [16, 27, 40, 42]. Note that all these sets are closed and in general satisfy the following inclusions

$$\sigma_{e1}(A) \cap \sigma_{e2}(A) = \sigma_{e3}(A) \subset \sigma_{e4}(A) \subset \sigma_{e5}(A) \subset \sigma_{e6}(A).$$

Note that, if A is a self-adjoint operator on a Hilbert space, then

$$\sigma_{e1}(A) = \sigma_{e2}(A) = \sigma_{e3}(A) = \sigma_{e4}(A) = \sigma_{e5}(A) = \sigma_{e6}(A).$$

The essential spectra were studied by many authors. Now, the main question is about the classes of stability of the essential spectra by a perturbation. Motivated by a problem concerning the spectrum of the transport operator posed in [33], Latrach and Jeribi [36] proved that the Shechter essential spectrum is stable with the perturbations by weakly compact operators on L_1-spaces. Then, Jeribi, Latrach and Dehici have extended the results about the classes of stability to strictly singular operators on L_p-spaces (see [18, 19, 20, 26, 37]), to weakly compact operators on Dunford-Pettis spaces (see [21, 34]), and to Fredholm perturbation operators on Banach spaces (see [25, 22, 24, 35]). The Wolf essential spectrum for the N-body problem, was studied in [5].

Remark 2.13. If $\lambda \in \sigma C(A)$ (the continuous spectrum of A) then $R(\lambda - A)$ is not closed (otherwise $\lambda \in \rho(A)$ see [45, Lemma 5.1 p. 179]). Therefore $\lambda \in \sigma_{ei}(A)$, $i = 1, \dots, 6$. Consequently we have $\sigma C(A) \subset \cap_{i=1}^{6} \sigma_{ei}(A)$. If the spectrum of A is purely continuous then $\sigma(A) = \sigma C(A) = \sigma_{ei}(A)$ $i = 1, \dots, 6$.

3. General Hypotheses

The aim of this section is to present some hypotheses which we need in the sequel. We begin with hypotheses which assure the closedness of the operator L_0. Let X_1 and X_2 be Banach spaces. In the product space $X := X_1 \times X_2$ we consider an operator formally defined by a matrix

$$L_0 := \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{3.1}$$

Our assumptions are as follows: in general, the operators occurring in the representation L_0 are unbounded, A acts on the space X_1 and has the domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on X_2, and the intertwining operators B and C are defined on the domains $\mathcal{D}(B)$ and $\mathcal{D}(C)$, respectively, and acts between these spaces.

In what follows, we assume that the following conditions hold.

- (H1) A is a densely defined operator in X_1 with nonempty resolvent set $\rho(A)$.
- (H2) B and C are densely defined closable operators from X_2 into X_1 and from X_1 into X_2, respectively, and $\mathcal{D}(C) \supset \mathcal{D}(A)$.
- (H3) For some $\mu \in \rho(A)$ the operator $(A-\mu I)^{-1}B$ is bounded on its domain $\mathcal{D}(B)$.
- (H4) $\mathcal{D}(B) \subset \mathcal{D}(D)$.
- (H5) For some $\mu \in \rho(A)$ the operator $D - C(A-\mu I)^{-1}B$ is closable. We denote its closure by $S(\mu)$.

For a better understanding, we give some comments to explain these hypothesis.

Remark 3.1. (i) From the fact that $\mathcal{D}(C) \supset \mathcal{D}(A)$ and the closed graph theorem we infer that for each $\mu \in \rho(A)$ the operator $G(\mu) := C(A-\mu I)^{-1}$ is defined on X_1 and is bounded.

(ii) If the hypothesis (H3) holds for some $\mu \in \rho(A)$ then it holds for all $\mu \in \rho(A)$. Indeed, let $\mu_0 \in \rho(A)$ be such that the operator $(A-\mu_0 I)^{-1}B$ is bounded. Then, for arbitrary $\mu \in \rho(A)$ the relation

$$(A-\mu I)^{-1}B = (A-\mu_0 I)^{-1}B + (\mu-\mu_0)(A-\mu I)^{-1}(A-\mu_0 I)^{-1}B \tag{3.2}$$

shows that $(A-\mu I)^{-1}B$ is also bounded.

(iii) We denote the closure of $(A-\mu I)^{-1}B$ by $F(\mu)$. Then the relation (3.2) implies

$$F(\mu) = F(\mu_0) + (\mu-\mu_0)(A-\mu I)^{-1}F(\mu_0).$$

(iv) The fact that $\mu \in \rho(A)$ implies that the operator $C(A-\mu I)^{-1}$ is defined everywhere on X_1 and hence the operator $C(A-\mu I)^{-1}B$ is defined on $\mathcal{D}(B)$.

(v) According to assumption (H4), the operator $D - C(A-\mu I)^{-1}B$ is defined on $\mathcal{D}(B)$.

(vi) If hypothesis (H5) holds for some $\mu \in \rho(A)$ then it holds for all $\mu \in \rho(A)$.

(vii) With the matrix defined in (3.1), we associate the operator $L_0 : \mathcal{D}(L_0) \subset X$, with $\mathcal{D}(L_0) = \mathcal{D}(A) \times \mathcal{D}(B)$ and

$$L_0 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{pmatrix}$$

The operator L_0 can be factored in the Frobenius-Shur sense:

$$L_0 - \mu I = \begin{pmatrix} I & 0 \\ G(\mu) & I \end{pmatrix} \begin{pmatrix} A-\mu I & 0 \\ 0 & D - C(A-\mu I)^{-1}B - \mu I \end{pmatrix} \begin{pmatrix} I & F(\mu) \\ 0 & I \end{pmatrix}$$

We recall now a result from [3] regarding the closability of the operator L_0.

Theorem 3.2 ([3, Theorem 1.1]). *Let hypotheses (H1)–(H5) be satisfied and let $\mu \in \rho(A)$. Then L_0 is closable and its closure L is given by*

$$L = \mu I + \begin{pmatrix} I & 0 \\ G(\mu) & I \end{pmatrix} \begin{pmatrix} A - \mu I & 0 \\ 0 & S(\mu) - \mu I \end{pmatrix} \begin{pmatrix} I & F(\mu) \\ 0 & I \end{pmatrix},$$

or equivalently, $L : \mathcal{D}(L) \subset X \to X$ with $\mathcal{D}(L) = \Big\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X : x_1 + F(\mu)x_2 \in \mathcal{D}(A)$ and $x_2 \in \mathcal{D}(S(\mu))\Big\}$ and

$$L \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A(x_1 + F(\mu)x_2) - \mu F(\mu)x_2 \\ C(x_1 + F(\mu)x_2) + S(\mu)x_2 \end{pmatrix}$$

Remark 3.3. Using the hypotheses (H2) and (H3) we infer that, for $\mu,\ \mu_0 \in \rho(A)$, the difference

$$\big(D - C(A-\mu)^{-1}B\big) - \big(D - C(A-\mu_0)^{-1}B\big) = (\mu-\mu_0)C(A-\mu)^{-1}(A-\mu_0)^{-1}B \quad (3.3)$$

is a bounded operator. Therefore if the operator $D - C(A - \mu I)^{-1}B$ is closable for some $\mu \in \rho(A)$ then it is closable for all $\mu \in \rho(A)$. Since the operator L is the closure of L_0, it does not depend on the choice of the point $\mu \in \rho(A)$ in its description above.

For the rest of this article, $\mathcal{I}(X)$ will denote an arbitrary nonzero two-sided closed ideal of $\mathcal{L}(X)$ satisfying the condition

$$\mathcal{I}(X) \subset \mathcal{F}(X).$$

Using Lemma 2.11 and [11, Proposition 4, p. 70] we deduce that

$$\mathcal{K}(X) \subset \mathcal{I}(X),$$

where $\mathcal{K}(X)$ stands for the ideal of compact operators. Hence the ideal of compact operators is the minimal subset of $\mathcal{L}(X)$ (in the sense of the inclusion) for which the results of this paper are valid.

We conclude this section with the following hypotheses:

- (H6) For some $\mu \in \rho(A)$ the resolvent $(A - \mu I)^{-1} \in \mathcal{I}(X_1)$.
- (H7) For some $\mu \in \rho(A)$ the operator $G(\mu)F(\mu) := \overline{C(A - \mu I)^{-2}B} \in \mathcal{I}(X_2)$.

Remark 3.4. Note that the assumptions (H1)–(H6) do not imply (H7) even if the operator $C(A - \mu I)^{-1}B$ is bounded (see [3, p. 8] where $\mathcal{I}(X) = \mathcal{K}(X)$ is the ideal of compact operators).

4. Main Results

In this section we present some results concerning the essential spectra of the operator L. We begin with the following preparatory result which is crucial for our purposes. It describes a sufficient condition for assumption (H1)–(H6) ti imply (H7).

Theorem 4.1. *Under the assumptions (H1)–(H6), condition (H7) is satisfied if and only if the operator $S(\mu)$ admits the representation*

$$S(\mu) = S_0 + M(\mu) \quad \mu \in \rho(A) \qquad (4.1)$$

with a closed operator S_0, which is independent of μ, and an operator $M(\mu) \in \mathcal{I}(X_2)$. In this case S_0 can be chosen to be $S(\mu_0)$ for any $\mu_0 \in \rho(A)$, and $M(\mu)$ depends holomorphically on μ in $\rho(A)$.

Proof. If (H7) is satisfied, by (3.3), we write $S(\mu)$ in the form

$$\begin{aligned} S(\mu) &= S(\mu_0) + (\mu-\mu_0)G(\mu)F(\mu_0) \\ &= S(\mu_0) + (\mu-\mu_0)G(\mu_0)F(\mu_0) + (\mu-\mu_0)^2 G(\mu_0)(A-\mu I)^{-1}F(\mu_0). \end{aligned}$$

From assumptions (H6) and (H7) and Proposition 2.2 we can deduce that the representation (4.1) follows, with $S_0 = S(\mu_0)$.

Conversely, equation (4.1) implies that

$$S(\mu) - S_0 = (M(\mu) - M(\mu_0))_{|\mathcal{D}(S_0)}. \tag{4.2}$$

On the other hand,

$$S(\mu) - S(\mu_0) = (\mu-\mu_0)G(\mu)F(\mu_0)_{|\mathcal{D}(S_0)}. \tag{4.3}$$

So, using (4.2) and (4.3), we deduce that

$$G(\mu)F(\mu_0) = (\mu-\mu_0)^{-1}(M(\mu) - M(\mu_0)).$$

If $\mu \to \mu_0$ the operator $G(\mu)F(\mu_0)$ tends to $G(\mu_0)F(\mu_0)$ in the operator norm topology which mean that $(\mu-\mu_0)^{-1}(M(\mu)-M(\mu_0))$ also converges to $G(\mu)F(\mu_0)$ in the operator norm topology. Furthermore, $(M(\mu)-M(\mu_0)) \in \mathcal{I}(X_2)$ and $\mathcal{I}(X_2)$ is a closed sided ideal of $\mathcal{L}(X_2)$. So, $G(\mu_0)F(\mu_0) \in \mathcal{I}(X_2)$ and this completes the proof. □

For $\mu \in \rho(A)$, we introduce the following matrix operators which we shall need in the sequel:

$$\mathbf{G}(\mu) := \begin{pmatrix} I & 0 \\ G(\mu) & I \end{pmatrix},$$

$$\mathbf{D}(\mu) := \begin{pmatrix} A-\mu I & 0 \\ 0 & S_0 + M(\mu) - \mu I \end{pmatrix},$$

$$\mathbf{F}(\mu) := \begin{pmatrix} I & F(\mu) \\ 0 & I \end{pmatrix}.$$

The following remark will be used for proving the next Theorem 4.3.

Remark 4.2. (a) Using Theorems 3.2 and 4.1 we can write the operator L in the form

$$L - \mu I = \mathbf{G}(\mu)\mathbf{D}(\mu)\mathbf{F}(\mu). \tag{4.4}$$

(b) If $\mu \in \rho(A)$, then

- (i) $\alpha(A-\mu I) = \beta(A-\mu I) = 0$.
- (ii) $\alpha(\mathbf{D}(\mu)) = \alpha(S_0 + M(\mu) - \mu I)$.
- (iii) $\beta(\mathbf{D}(\mu)) = \beta(S_0 + M(\mu) - \mu I)$.

We now state the main result of this paper.

Theorem 4.3. *Assume hypotheses (H1)–(H7). Then*

- (i) $\sigma_{ei}(L) = \sigma_{ei}(S_0)$ *for* $i = 4, 5$. *Moreover, if* $C\sigma_{e5}(L)$ *[the complement of* $\sigma_{e5}(L)$*] is connected and neither* $\rho(S_0)$ *nor* $\rho(S(\mu))$ *is empty, then*

$$\sigma_{e6}(L) = \sigma_{e6}(S_0).$$

- (ii) *If* $\mathcal{I}(X_2) \subset \mathcal{F}_+(X_2)$ *then* $\sigma_{e1}(L) = \sigma_{e1}(S_0)$.
- (iii) *If* $\mathcal{I}(X_2) \subset \mathcal{F}_-(X_2)$ *or* $[\mathcal{I}(X_2)]^* \subset \mathcal{F}_+(X_2^*)$, *then* $\sigma_{e2}(L) = \sigma_{e2}(S_0)$.
- (iii) *If* $\mathcal{I}(X_2) \subset \mathcal{F}_+(X_2) \cap \mathcal{F}_-(X_2)$ *then* $\sigma_{e3}(L) = \sigma_{e3}(S_0)$.

Proof. (i) First assume that $\lambda \in \rho(A)$. It is clear that $\mathbf{F}(\lambda)$ is a bijection from $\mathcal{D}(L)$ onto $\mathcal{D}(\mathbf{D}(\lambda)) = \mathcal{D}(A) \times \mathcal{D}(S_0)$ and $\mathbf{G}(\lambda)$ is a bijection from X onto X. Therefore,

$$\alpha(L - \lambda I) = \alpha(\mathbf{D}(\lambda)), \tag{4.5}$$

$$\beta(L - \lambda I) = \beta(\mathbf{D}(\lambda)). \tag{4.6}$$

By Remark 4.2 (b) (ii)-(iii) taking into account (4.5), (4.6), we obtain

$$\alpha(L - \lambda I) = \alpha(S_0 + M(\lambda) - \lambda I), \tag{4.7}$$

$$\beta(L - \lambda I) = \beta(S_0 + M(\lambda) - \lambda I). \tag{4.8}$$

Since $M(\lambda) \in \mathcal{I}(X_2)$, the numbers $\alpha(L - \lambda I)$ and $\beta(L - \lambda I)$ are finite if and only if $\alpha(S_0 - \lambda I)$ and $\beta(S_0 - \lambda I)$ are finite. Consequently, $L - \lambda I$ is a Fredholm operator if and only if $S_0 - \lambda I$ is a Fredholm operator and, if this is the case, $i(L - \lambda I) = i(S_0 - \lambda I)$.

Let $\lambda \notin \rho(A)$. By hypothesis (H6), the spectrum of A is discrete. Therefore, λ is an isolated eigenvalue of A. Let

$$P_\lambda = -\frac{1}{2\pi i} \int_{|\lambda - \xi| = \varepsilon} (A - \xi I)^{-1} d\xi$$

be the Riesz projection with ε sufficiently small. Then $\lambda \in \rho(A_\lambda)$ where A_λ is the finite-dimensional perturbation of A given by

$$A_\lambda := A(I - P_\lambda) + \delta P_\lambda, \quad \delta \neq \lambda.$$

Now, for $\mu \in \rho(A_\lambda)$, we have

$$\overline{D - C(A_\lambda - \mu I)^{-1} B} = S_0 + M_\lambda(\mu)$$

where the operator S_0 is introduced in (4.1) and operator $M_\lambda(\mu)$ is an operator from $\mathcal{I}(X_2)$. Let L_λ be the closure of the operator

$$\begin{pmatrix} A_\lambda & B \\ C & D \end{pmatrix}.$$

Since L_λ is a finite-dimensional perturbation of L, $L - \lambda I$ is a Fredholm operator on X if and only if $L_\lambda - \lambda I$ is a Fredholm operator on X. Now, with the first part of the present proof, we conclude that $\lambda \in \sigma_{ei}(L_\lambda)$ if and only if $\lambda \in \sigma_{ei}(S_0)$ $i = 4, 5$.

The proof of statement (i) for $i = 6$ is essentially the same as that of the last assertion of [35, Theorem 3.1]. This completes the proof of part (i).

(ii) Let $\lambda \in \rho(A)$. Using the fact that $\mathcal{I}(X_2) \subset \mathcal{F}_+(X_2)$ and [35, Lemma Lemma 2.2 (i)] we deduce that $\alpha(S_0 + M(\lambda) - \lambda I)$ is finite if and only if $\alpha(S_0 - \lambda I)$ is finite. Hence, by (4.7) $\alpha(L - \lambda I)$ is finite if and only if $\alpha(S_0 - \lambda I)$ is finite.

Let $\lambda \notin \rho(A)$. Since $\mathcal{F}_+(X_1) \subset \mathcal{F}(X_1)$, then by hypothesis (H6) we deduce that λ is an isolated eigenvalue of A. Now, the rest of the proof of (ii) may be done in a way similar to that in (i). This completes the proof of (ii).

(iii) If $\mathcal{I}(X_2) \subset \mathcal{F}_-(X_2)$ then the proof is the same as in (ii). It suffices to use (4.8) and [35, Lemma 2.2 (ii)].

If $[\mathcal{I}(X_2)]^* \subset \mathcal{F}_+(X_2^*)$, the result follows from the fact that $\alpha(S_0^* + M(\lambda)^* - \bar{\lambda} I) = \beta(S_0 + M(\lambda) - \lambda I)$ (see [12] or [29]).

(iv) Assertion (iv) follows from [35, Proposition 3.1 (iv)]. The proof is complete. □

Remark 4.4. Let X be a Banach space having the Dunford-Pettis property. It is proved in [34] that the ideal of weakly compact operators, $\mathcal{W}(X)$, leaves invariant the sets $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_\pm(X)$ and $\Phi(X)$ under additive perturbations, i.e., $\mathcal{W}(X) \subset \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$. Hence for $\mathcal{I}(X) = \mathcal{W}(X)$, the assertions of the Theorem 4.3 are valid.

We deduce the following result.

Corollary 4.5. *If the operator D is everywhere defined and bounded and $C(A - \mu_0 I)^{-1}B$ (for some and hence for all $\mu_0 \in \rho(A)$) is bounded, then S_0 can be chosen to be*

$$S_0 = D - \overline{C(A - \mu_0 I)^{-1}B}.$$

In particular, under these assumptions, $\sigma_{ei}(L) \neq \emptyset$, $i := 1 \dots 6$, if $\dim(X_2) = \infty$.

In applications (see Section 5), hypotheses (H3), (H6) and the boundedness of the operator $C(A - \mu I)^{-1}B$ are not easy to verify. Now, we give some sufficient conditions which imply the above assumptions that are easier to check. To this end, we introduce the following concept.

Definition 4.6. The resolvent of the operator A is said to have a ray of minimal growth if there exists some $\theta \in [0, 2\pi)$ such that

$$\gamma_\theta := \{\lambda \in \mathbb{C} : \lambda = te^{i\theta},\ t \in \mathbb{R}^+\} \subset \rho(A)$$

and there is a positive constant M such that

$$\|(A - \lambda I)^{-1}\| \leq \frac{M}{1 + |\lambda|} \text{ holds for all } \lambda \in \gamma_\theta. \tag{4.9}$$

The domain $\mathcal{D}(A)$ of A is equipped with the graph norm topology i.e., $\|x\|_1 = \|x\| + \|Ax\|$; hence $\mathcal{D}(A)$ is a Banach space. Let $X_{1,1}$ denote $(\mathcal{D}(A), \|.\|_1)$. If A is an operator whose resolvent has a ray of minimal growth, then the intermediate spaces

$$X_{1,\theta} = \mathcal{D}(A^\theta), \quad 0 \leq \theta \leq 1$$

between X_1 and $X_{1,1} = \mathcal{D}(A)$ with the norm $\|x\|_\theta = \|x\| + \|A^\theta x\|$ are well defined and the same holds for the intermediate spaces $X^*_{1,\theta}$ between $X^*_{1,1} = \mathcal{D}(A^*)$ and X^*_1.

Proposition 4.7 ([3, Proposition 3.1])**.** *If the operators A and B satisfy properties (H1), (H2), then the assumption (H3) holds if and only if $\mathcal{D}(B^*) \supset \mathcal{D}(A^*)$.*

Theorem 4.8. *Let X be a Banach space and let $\mathcal{I}(X)$ denote an arbitrary nonzero two-sided ideal of $\mathcal{L}(X)$ contained in $\mathcal{F}(X)$. Let $T \in \mathcal{C}(X)$ be such that $\rho(T) \neq \emptyset$. Then $(\lambda - T)^{-1} \in \mathcal{I}(X)$ for some $\lambda \in \rho(T)$ if and only if the embedding of $\mathcal{D}(T)$ into X is in $\mathcal{I}(X)$.*

Remark 4.9. Note that for $\mathcal{I}(X) = \mathcal{K}(X)$, Theorem 4.8 is nothing but the classical spectral theorem for compact operators.

Proof of Theorem 4.8. Let $\lambda \in \rho(T)$ such that $(\lambda - T)^{-1} \in \mathcal{I}(X)$. The operator $\lambda - T : \mathcal{D}(T) \to X$ is an isomorphism when the domain, $\mathcal{D}(T)$, of the operator T is equipped with the graph norm. By using the fact that $(\lambda - T)^{-1} \in \mathcal{I}(X)$ and writing the embedding j of $\mathcal{D}(T)$ into X as $j := (\lambda - T)^{-1}(\lambda - T)$ with $\mathcal{D}(j) := \mathcal{D}(T)$ we deduce that $j \in \mathcal{I}(X)$.

Inversely, let $\lambda \in \rho(T)$. Then we can write $(\lambda - T)^{-1} = j \circ (\lambda - T)^{-1}$ where $j : \mathcal{D}(T) \to X$ is in $\mathcal{I}(X)$ then $(\lambda - T)^{-1}$ is in $\mathcal{I}(X)$ as the compose of a continuous map $(\lambda - T)^{-1}$ and a map j in $\mathcal{I}(X)$. □

Lemma 4.10. *Let conditions (H1), (H2), (H6) be satisfied and assume that the resolvent of A has a ray of minimal growth. Furthermore, assume that for some $\theta \in (0, 1)$ the following inclusions hold:*

$$\mathcal{D}(B^*) \supset \mathcal{D}((A^*)^\theta) := \mathcal{D}((A^\theta)^*), \tag{4.10}$$

$$\mathcal{D}(\overline{C}) \supset \mathcal{D}(A^{1-\theta}). \tag{4.11}$$

Then also conditions (H3) and (H7) are fulfilled and the operator $C(A - \mu I)^{-1}B$ is bounded for $\mu \in \rho(A)$.

Proof. It follows from the properties of fractional powers [31, Chapter 4] that $X^*_{1,\theta} \supset X^*_{1,1} = \mathcal{D}(A^*)$ for any $\theta \in [0, 1)$. Moreover, the embedding of $X^*_{1,1}$ into $X^*_{1,\theta}$ is a Fredholm perturbation since $0 \in \rho(A)$ by (4.9) and $A^{-1} \in \mathcal{I}(X_1)$ (see Theorem 4.8). Using Proposition 4.7 we conclude that (H3) holds if and only if $\mathcal{D}(B^*) \supset \mathcal{D}(A^*)$. Thus, (H3) derives from (4.10).

Now, write the operator $CA^{-1}B$ in the form

$$CA^{-1}B = \overline{C}A^{-(1-\theta)}A^{-\theta}B. \tag{4.12}$$

The operator $\overline{C}A^{-(1-\theta)}$ is bounded on X_1 by (4.11). The operator $A^{-\theta}B$ is bounded on $\mathcal{D}(B)$ by (4.10) and Proposition 4.7. Hence, the boundedness of the operator $CA^{-1}B$ follows from (4.12). Finally, by writing

$$CA^{-2}B = \overline{C}A^{-(1-\theta)}A^{-1}(A^{-\theta}B)$$

the hypothesis (H7) follows from the fact that $A^{-1} \in \mathcal{I}(X_1)$ and Proposition 2.2. □

Let X be a Banach space and let T be a closed operator on X. By $\Delta^0(T)$ we denote the maximal open subset of $\mathbb{C}$ where the resolvent $(T - \lambda I)^{-1}$ is finitely meromorphic, i.e., it is meromorphic on $\Delta^0(T)$ and all the coefficients in the principal parts of the Laurent expansions at the poles are of finite rank.

Remark 4.11. The set $\Delta^0(T)$ is the union of all components w of Φ_T for which $w \cap \rho(T) \neq \emptyset$ [11, Lemma 2.1].

Using representation (4.4) we can prove the following result.

Corollary 4.12. *Under assumptions (H1)–(H6), the set $\Delta^0(L)$ is the union of all components w of Φ_{S_0} such that for some $\mu \in w$ the operator $S(\mu) - \mu I$ maps X_2 bijectively onto itself.*

We give now, a sufficient condition for the fact that $\Delta^0(L)$ contains the unbounded component of $\Delta(S_0)$, denoted by $\Delta^0_{ext}(S_0)$.

Corollary 4.13. *Let conditions (H1), (H2, (H6) hold. Assume that the resolvent of A has a ray of minimal growth. Assume, in addition, that for some $\theta \in (0, 1)$ the inclusions* (4.10) *and* (4.11) *hold and the operator D is bounded. Then the inclusion*

$$\Delta^0(L) \supset \Delta^0_{ext}(S_0)$$

holds. In particular, if $\Delta(S_0)$ is simply connected, then the equality $\Delta^0(L) = \Delta^0(S_0)$ holds.

Proof. Let $\lambda \in \gamma_\theta$, $0 < \theta < 1$, and consider the identity:

$$S(\lambda) - \lambda I = -\lambda I + S(0) + [\overline{C}A^{-(1-\theta)}][\lambda(A-\lambda)^{-1}]\,[\overline{A^{-\theta}B}].$$

By Lemma 4.10 the operators $S(0)$, $\overline{C}A^{-(1-\theta)}$ and $\overline{A^{-\theta}B}$ are everywhere defined and bounded. Hence the operator $S(\lambda) - \lambda I$ has a bounded inverse for all $\lambda \in \gamma_\theta$ with $|\lambda|$ sufficiently large. By Corollary 4.12 the unbounded component $\Delta^0_{ext}(S_0)$ of $\Delta(S_0)$ is a component of $\Delta^0(L)$. □

5. An Example

The aim of this section is in the analysis of the essential spectra of the operator defined in (3.1) where A, B, C are ordinary differential operators in spaces of vector functions and D is a multiplication operator defined as follows: The operator $A : \mathcal{D}(A) \subset X_1 \to X_1$ is defined on $\mathcal{D}(A) = \{\varphi \in (H^l_p)^n : U\varphi = 0\}$, as

$$\varphi \mapsto A\varphi(x) = \sum_{k=0}^{l} a_k(x)\varphi^{(l-k)}(x)$$

where $X_1 := (L_p(0,1))^n$, $p > 1$, $n \in \mathbb{N}$, $l > 0$, and a_i, $0 \le i \le l$ are $n \times n$ matrix functions with sufficiently smooth entries and $\det a_0(x) \neq 0$; $(H^l_p)^n := (H^l_p(0,1))^n$ is a Sobolev space of n-vector functions. The domain $\mathcal{D}(A)$ is supposed to be given by general boundary conditions

$$U(\varphi) := U_0 \begin{pmatrix} \varphi(0) \\ \varphi'(0) \\ \vdots \\ \varphi^{(l-1)}(0) \end{pmatrix} + U_1 \begin{pmatrix} \varphi(1) \\ \varphi'(1) \\ \vdots \\ \varphi^{(l-1)}(1) \end{pmatrix} = 0 \tag{5.1}$$

where U_0 and U_1 are $nl \times nl$ matrices.

The operator $B : \mathcal{D}(B) \subset X_2 \to X_1$ is defined on $\mathcal{D}(B) = \{\varphi \in (H^s_p)^m : \hat{U}\varphi = 0\}$, as

$$\varphi \mapsto B\varphi(x) = \sum_{k=0}^{s} b_k(x)\varphi^{(s-k)}(x).$$

where $X_2 := (L_p(0,1))^m$, $m \in \mathbb{N}$, $0 \le s \le l$; b_i, $0 \le i \le s$ are $n \times m$ matrix functions with sufficiently smooth entries. The system of boundary conditions $U^*(v) = 0$ ($v \in H^s_q$ with $q = \frac{p}{p-1}$) is the adjoint of the system (5.1). We take all boundary conditions of order $\le s-1$ in the system of boundary conditions $U^*(v) = 0$ and denote the corresponding subsystem of linear forms by $\hat{U}^*(\cdot)$. The domain $\mathcal{D}(B)$ is chosen, on the one hand, to satisfy the condition $\mathcal{D}(B^*) \supset \mathcal{D}(A^*)$ and, on the other hand, to be as large as possible in order to cover examples which are interesting in applications.

The operator $C : \mathcal{D}(C) \subset X_1 \to X_2$ is defined on $\mathcal{D}(C) = \{\varphi \in (H^h_p)^n : U\varphi = 0\}$, as

$$\varphi \mapsto C\varphi(x) = \sum_{k=0}^{h} c_k(x)\varphi^{(h-k)}(x),$$

where $0 \le h \le l$, $s + h = l$ and c_i, $0 \le i \le h$ are $m \times n$ matrix functions with sufficiently smooth entries.

The operator $D : X_2 \to X_2$ is defined as

$$\varphi \mapsto D\varphi(x) = d(x)\varphi(x),$$

where d is an $m \times m$ matrix function which is assumed to be measurable and essentially bounded (hence D is a bounded operator on X_2).

For more details we refer the reader to [3]. The next proposition contains conditions that we need in the sequel.

Proposition 5.1. *With the above notation, we have the following:*

(i) [3, Proposition 4.1] *B is closable and the inclusion $\mathcal{D}(B^*) \supset \mathcal{D}((A^*)^{\frac{s}{l}})$ holds.*

(ii) [3, Proposition 4.2] *C is closable and $\mathcal{D}(\bar{C}) \supset \mathcal{D}((A^*)^{\frac{h}{l}})$*

(iii) [3, Theorem 4.3] *For $\mu \in \rho(A)$ the operator $D - C(A - \mu I)^{-1}B$ which is defined on $\mathcal{D}(B)$, admits a bounded closure $S(\mu) = S_0 + K(\mu)$, where S_0 is the operator of multiplication by the function $d - c_0 a_0^{-1} b_0$ and $K(\mu)$ a compact operator in X_2.*

Theorem 5.2. *Let L_0 the operator defined as above and let L denote the closure of L_0. Then*

$$\sigma_{ei}(L) = \{\lambda \in \mathbb{C} : \operatorname{ess\,inf} |\det[d(x) - c_0(x)a_0^{-1}(x)b_0(x) - \lambda I]| = 0\} \quad i = 1, \dots, 6. \tag{5.2}$$

Moreover, if the complement of this set is connected, then this complement coincides with the domain of finite meromorphy of the operator function $(L - \lambda I)^{-1}$.

Proof. Using Theorem 4.3 and Proposition 5.1 (iii) we deduce that $\sigma_{ei}(L) = \sigma_{ei}(S_0)$, $i = 1, \dots, 6$ where S_0 is the operator of multiplication by the matrix function $d - c_0 a_0^{-1} b_0$. On the other hand, it is shown in [17] that the spectrum of S_0 is purely continuous and is given by the expression on the right-hand side of (5.2). Now the result follows from Remark 2.13. $\square$

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