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POSITIVE PERIODIC SOLUTIONS OF NEUTRAL LOGISTIC DIFFERENCE EQUATIONS WITH MULTIPLE DELAYS

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Abstract. Using a fixed point theorem of strict-set-contraction, we some established the existence of positive periodic solutions for the neutral logistic difference equation, with multiple delays,

$$\Delta x(n) = x(n)\Big[a(n) - \sum_{i=1}^{p} a_i(n)x(n-\tau_i(n)) - \sum_{j=1}^{q} c_j(n)\Delta x(n-\sigma_j(n))\Big].$$

1. Introduction

Using the continuation theory for k-set-contractions and the Mawhin's continuation theorem, [2], the existence of positive periodic solutions for the following neutral logistic differential equation with multiple delays

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = N(t)\Big[a(t)-\beta(t)N(t)-\sum_{j=1}^{n} b_j(t)N_j(t-\tau_j(t))-\sum_{j=1}^{n} c_j(t)N_j'(t-\sigma_j(t))\Big] \quad (1.1)$$

are investigated in [3, 4, 11, 12, 13]. Where $a(t)$, $\beta_j(t)$, $b_j(t)$, $c_j(t)$, $\tau_j(t)$, $\gamma_j(t)$ are continuous ω-periodic functions and $a(t) \ge 0$, $\beta_j(t) \ge 0$, $b_j(t) \ge 0$, $c_j(t) \ge 0$ ($j = 1, 2, \dots, n$). For the ecological justification of (1.1), see for example [5, 6, 8, 9, 10].

Given a, b be integers and $a < b$, we employ intervals to denote discrete sets such as $\mathbb{Z}[a, b] = \{a, a+1, \dots, b\}$, $\mathbb{Z}[a, b) = \{a, \dots, b-1\}$, $\mathbb{Z}[a, \infty) = \{a, a+1, \dots\}$, etc. Let $\omega \in \mathbb{Z}[1, \infty)$ be fixed. Throughout this work, we denote the product of $y(n)$ from $n = a$ to $n = b$ by $\prod_{n=a}^{n=b} y(n)$ with the understanding that $\prod_{n=a}^{n=b} y(n) = 1$ for all $a > b$.

The main purpose of this paper is to use a fixed point theorem of strict-set-contraction [1, Theorem 3] to establish the existence of positive periodic solutions of the following neutral logistic difference equation, with multiple delays,

$$\Delta x(n) = x(n)\Big[a(n) - \sum_{i=1}^{p} a_i(n)x(n-\tau_i(n)) - \sum_{j=1}^{q} c_j(n)\Delta x(n-\sigma_j(n))\Big], \qquad (1.2)$$

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where $a, a_i, c_j \in (\mathbb{Z}(-\infty,\infty), \mathbb{R}^+)$ and $\tau_i, \sigma_j \in (\mathbb{Z}(-\infty,\infty), \mathbb{R})$, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, q$ are ω-periodic functions. To the best of our knowledge, this is the first paper to study the existence of periodic solutions of neutral logistic difference equations delays.

For convenience, we introduce the notation

$$\Theta := \prod_{k=0}^{\omega-1}(1 + a(k)), \quad \Gamma = \sum_{s=0}^{\omega-1}\Big[\Theta^{-1}\sum_{i=1}^{p} a_i(s) - \sum_{j=1}^{q} c_j(s)\Big],$$
$$\Pi = \sum_{s=0}^{\omega-1}\Big[\sum_{i=1}^{p} a_i(s) + \sum_{j=1}^{q} c_j(s)\Big], \quad f^M = \max_{n\in\mathbb{Z}[0,\omega-1]}\{f(n)\},$$
$$f^m = \min_{n\in\mathbb{Z}[0,\omega-1]}\{f(n)\},$$

where f is a continuous ω-periodic function. In this paper, we assume that

(H1) $\prod_{k=0}^{\omega-1}(1 + a(k)) > 1$.
(H2) $\Theta\sum_{i=1}^{p} a_i(n) - \sum_{j=1}^{q} c_j(n) \geq 0$.
(H3) $(1 + a^m)\frac{\Gamma}{\Theta(\Theta-1)} \geq \max_{n\in\mathbb{Z}[0,\omega-1]}\Big\{\sum_{i=1}^{p} a_i(n) + \sum_{j=1}^{q} c_j(n)\Big\}$.
(H4) $\frac{\Pi(a^M-1)\Theta}{\Theta-1} \leq \min_{n\in\mathbb{Z}[0,\omega]}\Big\{\Theta^{-1}\sum_{i=1}^{p} a_i(n) - \sum_{j=1}^{q} c_j(n)\Big\}$.
(H5) $\frac{\Theta-1}{\Theta\Gamma}\sum_{j=1}^{q} c_j^M < 1$.

2. Preliminaries

To obtain the existence of a periodic solution of system (1.2), we first make the following preparations:

Let $\mathbb{E}$ be a Banach space and K be a cone in $\mathbb{E}$. The semi-order induced by the cone K is denoted by "$\leq$". That is, $x \leq y$ if and only if $y - x \in K$. In addition, for a bounded subset $A \subset \mathbb{E}$, let $\alpha_{\mathbb{E}}(A)$ denote the (Kuratowski) measure of non-compactness defined by

$$\alpha_E(A) = \inf\big\{\delta > 0 : \text{ there is a finite number of subsets } A_i \subset A$$
$$\text{such that } A = \cup_i A_i \text{ and } \operatorname{diam}(A_i) \leq \delta\big\},$$

where $\operatorname{diam}(\cdot)$ denotes the diameter of the set.

Let $\mathbb{E}, \mathbb{F}$ be two Banach spaces and $D \subset \mathbb{E}$, a continuous and bounded map $\Phi : \bar{\Omega} \to \mathbb{F}$ is called k-set contractive if for any bounded set $S \subset D$ we have

$$\alpha_{\mathbb{F}}(\Phi(S)) \leq k\alpha_{\mathbb{E}}(S).$$

A function Φ is called strict-set-contractive if it is k-set-contractive for some $0 \leq k < 1$.

The following lemma is useful for the proof of our main results of this paper.

Lemma 2.1 ([1, 7]). *Let K be a cone of the real Banach space $\mathbb{E}$ and $K_{r,R} = \{x \in K : r \leq \|x\| \leq R\}$ with $R > r > 0$. Suppose that $\Phi : K_{r,R} \to K$ is strict-set-contractive such that one of the following two conditions is satisfied:*

(i) *$\Phi x \not\leq x$, for all $x \in K$, $\|x\| = r$ and $\Phi x \not\geq x$, for all $x \in K$, $\|x\| = R$.*
(ii) *$\Phi x \not\geq x$, for all $x \in K$, $\|x\| = r$ and $\Phi x \not\leq x$, for all $x \in K$, $\|x\| = R$.*

Then Φ has at least one fixed point in $K_{r,R}$.

Finding an ω-periodic solution of (1.1) is equivalent to finding an ω-periodic solution of the equation

$$x(n)=\sum_{s=n}^{n+\omega-1}G(n,s)x(s)\Big[\sum_{i=1}^{p}a_i(s)x(s-\tau_i(s))+\sum_{j=1}^{q}c_j(s)\Delta x(s-\sigma_j(s))\Big]$$

where

$$G(n,s)=\frac{\prod_{k=s+1}^{n+\omega-1}(1+a(k))}{\prod_{k=0}^{\omega-1}(1+a(k))-1},\quad n\le s\le n+\omega-1.$$

It is easy to see that, for $s\in\mathbb{Z}[k,k+\omega-1]$ we have

$$\frac{1}{\prod_{k=0}^{\omega-1}(1+a(k))-1}\le G(k,s)\le\frac{\prod_{k=0}^{\omega-1}(1+a(k))}{\prod_{k=0}^{\omega-1}(1+a(k))-1}.$$

Let

$$\mathbb{X}=\mathbb{Y}=\{x:\mathbb{Z}(-\infty,\infty)\to\mathbb{R},x(n+\omega)=x(n),n\in\mathbb{Z}(-\infty,\infty)\},$$

$\mathbb{X}$ with the norm $|x|_0=\max_{n\in\mathbb{Z}[0,\omega-1]}\{|x(n)|\},x\in\mathbb{X}$ and $\mathbb{Y}$ with the norm $|y|_1=\max_{n\in\mathbb{Z}[0,\omega-1]}\{|y|_0,|\Delta y|_0\},y\in\mathbb{Y}$. Then $\mathbb{X},\mathbb{Y}$ are Banach spaces. Note that solving (1.2) is equivalent to solving

$$x=\Phi x,\tag{2.1}$$

where

$$(\Phi x)(n)=\sum_{s=n}^{n+\omega-1}G(n,s)x(s)\Big[\sum_{i=1}^{p}a_i(s)x(s-\tau_i(s))+\sum_{j=1}^{q}c_j(s)\Delta x(s-\sigma_j(s))\Big]$$

for $x\in\mathbb{Y}$.

To apply Lemma 2.1 to (1.2), we define the cone K in $\mathbb{Y}$ by

$$K=\{x\in\mathbb{Y}:x(n)\ge 0\text{ and }x(n)\ge\Theta^{-1}|x|_1,n\in\mathbb{Z}[0,\omega-1]\}.\tag{2.2}$$

In what follows, we will give some lemmas concerning K and Φ defined by (2.1) and (2.2), respectively.

Lemma 2.2. *Assume that* (H1)–(H3) *hold.*

(i) *If $a^M\le 1$, then $\Phi:K\to K$ is well defined.*
(ii) *If (H4) holds and $a^M>1$, then $\Phi:K\to K$ is well defined.*

Proof. For each $x\in K$, it is clear that $\Phi x\in\mathbb{Y}$. In view of (2.2), for $n\in\mathbb{Z}(-\infty,\infty)$, we obtain

$$\begin{aligned}&(\Phi x)(n+\omega)\\&=\sum_{s=n+\omega}^{n+2\omega-1}G(n+\omega,s)x(s)\Big[\sum_{i=1}^{p}a_i(s)x(s-\tau_i(s))+\sum_{j=1}^{q}c_j(s)\Delta x(s-\sigma_j(s))\Big]\\&=\sum_{u=n}^{n+\omega-1}G(n+\omega,u+\omega)x(u+\omega)\Big[\sum_{i=1}^{p}a_i(u+\omega)x(u+\omega-\tau_i(u+\omega))\\&\quad+\sum_{j=1}^{q}c_j(u+\omega)\Delta x(u+\omega-\sigma_j(u+\omega))\Big]\\&=(\Phi x)(n).\end{aligned}$$

That is, $(\Phi x)(n+\omega) = (\Phi x)(n)$, $n \in \mathbb{Z}(-\infty,\infty)$. So $\Phi x \in \mathbb{Y}$. In view of (H2), for $x \in K, n \in \mathbb{Z}(-\infty,\infty)$, we have

$$\begin{aligned}
&\sum_{s=n}^{n+\omega-1} G(n,s)x(s)\Big[\sum_{i=1}^{p} a_i(s)x(s-\tau_i(s)) + \sum_{j=1}^{q} c_j(s)\Delta x(s-\sigma_j(s))\Big]\\
&\geq \sum_{s=n}^{n+\omega-1} G(n,s)x(s)\Big[\sum_{i=1}^{p} a_i(s)x(s-\tau_i(s)) - \sum_{j=1}^{q} c_j(s)|\Delta x(s-\sigma_j(s))|\Big] \qquad (2.3)\\
&\geq \sum_{s=n}^{n+\omega-1} G(n,s)x(s)\Big[\sum_{i=1}^{p} a_i(s)\Theta^{-1}|x|_1 - \sum_{j=1}^{q} c_j(s)|x|_1\Big] \geq 0.
\end{aligned}$$

Therefore, for $x \in K, n \in \mathbb{Z}[0,\omega-1]$. We find that

$$|\Phi x|_0 \leq \frac{\Theta}{\Theta-1}\sum_{s=n}^{n+\omega-1} x(s)\Big[\sum_{i=1}^{p} a_i(s)x(s-\tau_i(s)) + \sum_{j=1}^{q} c_j(s)\Delta x(s-\sigma_j(s))\Big]$$

and

$$\begin{aligned}
(\Phi x)(n) &\geq \frac{1}{\Theta-1}\sum_{s=0}^{\omega-1} x(s)\Big[\sum_{i=1}^{p} a_i(s)x(s-\tau_i(s)) + \sum_{j=1}^{q} c_j(s)\Delta x(s-\sigma_j(s))\Big]\\
&= \frac{1}{\Theta-1}\sum_{s=0}^{\omega-1} x(s)\Big[\sum_{i=1}^{p} a_i(s)x(s-\tau_i(s)) + \sum_{j=1}^{q} c_j(s)\Delta x(s-\sigma_j(s))\Big] \qquad (2.4)\\
&\geq \Theta^{-1}|\Phi x|_0.
\end{aligned}$$

Now, we show that $(\Phi x)(n) \geq \Theta^{-1}|(\Phi x)|_1, n \in \mathbb{Z}[0,\omega-1]$. From (2.2), we have

$$\begin{aligned}
&\Delta(\Phi x)(n)\\
&= \Delta(\Phi x)(n+1) - \Delta(\Phi x)(n)\\
&= \sum_{s=n+1}^{n+\omega} G(n+1,s)x(s)\Big[\sum_{i=1}^{p} a_i(s)x(s-\tau_i(s)) + \sum_{j=1}^{q} c_j(s)\Delta x(s-\sigma_j(s))\Big]\\
&\quad - \sum_{s=n}^{n+\omega-1} G(n,s)x(s)\Big[\sum_{i=1}^{p} a_i(s)x(s-\tau_i(s)) + \sum_{j=1}^{q} c_j(s)\Delta x(s-\sigma_j(s))\Big]\\
&= G(n+1,n+\omega)x(n+\omega)\Big[\sum_{i=1}^{p} a_i(n+\omega)x(n+\omega-\tau_i(n+\omega))\\
&\quad + \sum_{j=1}^{q} c_j(n+\omega)\Delta x(n+\omega-\sigma_j(n+\omega))\Big]\\
&\quad + \sum_{s=n}^{n+\omega-1} G(n+1,s)x(s)\Big[\sum_{i=1}^{p} a_i(s)x(s-\tau_i(s)) + \sum_{j=1}^{q} c_j(s)\Delta x(s-\sigma_j(s))\Big]\\
&\quad - G(n+1,n)x(n)\Big[\sum_{i=1}^{p} a_i(n)x(n-\tau_i(n)) + \sum_{j=1}^{q} c_j(n)\Delta x(n-\sigma_j(n))\Big]\\
&\quad - \sum_{s=n}^{n+\omega-1} G(n,s)x(s)\Big[\sum_{i=1}^{p} a_i(s)x(s-\tau_i(s)) + \sum_{j=1}^{q} c_j(s)\Delta x(s-\sigma_j(s))\Big]
\end{aligned}$$

$$
\begin{aligned}
&= [G(n+1,n+\omega) - G(n+1,n)]x(n)\Big[\sum_{i=1}^{p} a_i(n)x(n-\tau_i(n)) \\
&\quad + \sum_{j=1}^{q} c_j(n)\Delta x(n-\sigma_j(n))\Big] \\
&\quad + \sum_{s=n}^{n+\omega-1} [G(n+1,s) - G(n,s)]x(s)\Big[\sum_{i=1}^{p} a_i(s)x(s-\tau_i(s)) \\
&\quad + \sum_{j=1}^{q} c_j(s)\Delta x(s-\sigma_j(s))\Big] \\
&= a(n) \sum_{s=n}^{n+\omega-1} G(n,s)x(s)\Big[\sum_{i=1}^{p} a_i(s)x(s-\tau_i(s)) + \sum_{j=1}^{q} c_j(s)\Delta x(s-\sigma_j(s))\Big] \\
&\quad - x(n)\Big[\sum_{i=1}^{p} a_i(n)x(n-\tau_i(n)) + \sum_{j=1}^{q} c_j(n)\Delta x(n-\sigma_j(n))\Big] \\
&= a(n)(\Phi x)(n) - x(n)\Big[\sum_{i=1}^{p} a_i(n)x(n-\tau_i(n)) + \sum_{j=1}^{q} c_j(n)\Delta x(n-\sigma_j(n))\Big]. \quad (2.5)
\end{aligned}
$$

It follows from the above inequality and (2.3) that if $(\Phi x)(n) \geq 0$, then

$$
(\Delta\Phi x)(n) \leq a(n)(\Phi x)(n) \leq a^M(\Phi x)(n) \leq (\Phi x)(n). \tag{2.6}
$$

On the other hand, from (2.5) and (H3), if $(\Delta\Phi x)(n) < 0$, then

$$
\begin{aligned}
&- \Delta(\Phi x)(n) \\
&= x(n)\Big[\sum_{i=1}^{p} a_i(n)x(n-\tau_i(n)) + \sum_{j=1}^{q} c_j(n)\Delta x(n-\sigma_j(n))\Big] - a(n)(\Phi x)(n) \\
&\leq |x|_1^2\Big[\sum_{i=1}^{p} a_i(n) + \sum_{j=1}^{q} c_j(n)\Big] - a^m(\Phi x)(n) \\
&\leq (1+a^m)\frac{1}{\Theta(\Theta-1)}|x|_1^2 \sum_{s=0}^{\omega-1}\Big[\Theta^{-1}\sum_{i=1}^{p} a_i(s) - \sum_{j=1}^{q} c_j(s)\Big] - a^m(\Phi x)(n) \\
&= (1+a^m) \sum_{s=n}^{n+\omega-1} \frac{1}{\Theta-1}\Theta^{-1}|x|_1\Big[\Theta^{-1}|x|_1\sum_{i=1}^{p} a_i(s) - |x|_1\sum_{j=1}^{q} c_j(s)\Big] - a^m(\Phi x)(n) \\
&\leq (1+a^m) \sum_{s=n}^{n+\omega-1} G(n,s)x(s)\Big[a(s)x(s-\tau_i(s)) - \sum_{j=1}^{q} c_j(s)|\Delta x(s-\sigma_j(s))|\Big] \\
&\quad - a^m(\Phi x)(n) \\
&= (1+a^m)(\Phi x)(n) - a^m(\Phi x)(n) \\
&= (\Phi x)(n).
\end{aligned} \tag{2.7}
$$

It follows from (2.6) and (2.7) that $|(\Delta\Phi x)|_0 \leq |\Phi x|_0$. So $|\Phi x|_1 = |\Phi x|_0$. By (2.4) we have $(\Phi x)(n) \geq \Theta^{-1}|\Phi x|_1$. Hence, $\Phi x \in K$. The proof of part (i) is complete.

Part (ii): In view of the proof of (i), we only need to prove that $(\Delta\Phi x)(n) \geq 0$ implies

$$(\Delta\Phi x)(n) \leq (\Phi x)(n).$$

From (2.3), (2.5), (H2) and (H4), we obtain

$$\begin{aligned}
&(\Delta\Phi x)(n)\\
&\leq a(n)(\Phi x)(n) - \Theta^{-1}|x|_1\Big[\sum_{i=1}^{p} a_i(n)x(n-\tau_i(n)) - \sum_{j=1}^{q} c_j(n)|\Delta x(n-\sigma_j(n))|\Big]\\
&\leq a(n)(\Phi x)(n) - \Theta^{-1}|x|_1^2\Big[\sum_{i=1}^{p}\Theta^{-1}a_i(n) - \sum_{j=1}^{q} c_j(n)\Big]\\
&\leq a^M(\Phi x)(n) - |x|_1^2\frac{a^M-1}{\Theta-1}\sum_{s=n}^{n+\omega-1}\Big[\sum_{i=1}^{p} a_i(s) + \sum_{j=1}^{q} c_j(s)\Big]\\
&\leq a^M(\Phi x)(n) - (a^M-1)\sum_{s=n}^{n+\omega-1}\frac{\Theta}{\Theta-1}\Theta^{-1}|x|_1\Big[\sum_{i=1}^{p} a_i(s)|x|_1 + \sum_{j=1}^{q} c_j(s)|x|_1\Big]\\
&\leq a^M(\Phi x)(n) - (a^M-1)\sum_{s=n}^{n+\omega-1} G(n,s)x(s)\Big[\sum_{i=1}^{p} a_i(s)x(s-\tau_i(s))\\
&\quad+\sum_{j=1}^{q} c_j(s)|\Delta x(s-\sigma_j(s))|\Big]\\
&\leq a^M(\Phi x)(n) - (a^M-1)\sum_{s=n}^{n+\omega-1} G(n,s)x(s)\Big[\sum_{i=1}^{p} a_i(s)x(s-\tau_i(s))\\
&\quad+\sum_{j=1}^{q} c_j(s)\Delta x(s-\sigma_j(s))\Big]\\
&= a^M(\Phi x)(n) - (a^M-1)(\Phi x)(n)\\
&= (\Phi x)(n).
\end{aligned}$$

The proof of (ii) is complete. □

Lemma 2.3. $\Phi(S)$ *is precompact in* $\mathbb{X}$ *for any bounded set* S *in* $\mathbb{Y}$*.*

Proof. Let d be a constant and $S = \{x \in \mathbb{Y} : \|x\|_1 < d\}$ is a bounded set. We prove that $\overline{\Phi S}$ is compact. To do this, we must show that any sequence in ΦS contains a convergent subsequence. Thus, let $\{x_m, m \in \mathbb{Z}[1,\infty)\}$ be a sequence in S. Let us show that $\{\Phi x_m, m \in \mathbb{Z}[1,\infty)\}$ has a convergent subsequence. For convenience, we set

$$M(x(n)) = x(n)\Big[\sum_{i=1}^{p} a_i(n)x(n-\tau_i(n)) + \sum_{j=1}^{q} c_j(n)\Delta x(n-\sigma_j(n))\Big],$$

then

$$(\Phi x)(n) = \sum_{s=n}^{n+\omega-1} G(n,s)x(s)M(x(s)).$$

It is obvious that the sequence $\{M(x_m(0))\}$, $m \in \mathbb{Z}[1,\infty)$, is bounded; so the sequence $\{M(x_m(0))\}$ contains a convergent subsequence. So let the sequence $\{x_{m,0}\}$, be a subsequence of $\{x_m\}$ such that $\{M(x_{m,0}(0))\}$ is convergent.

Again, $\{M(x_{m,0}(1))\}$, $m \in \mathbb{Z}[1,\infty)$, contains a convergent subsequence. So, let $\{x_{m,1}\}$ be a subsequence of $\{x_{m,0}$, such that $\{M(x_{m,1}(1))\}$ is convergent. Observe that $\{x_{m,1}\}$ is a subsequence of $\{x_m\}$ and that $\{M(x_{m,1}(0))\}$ and $\{M(x_{m,1}(1))\}$ are convergent.

Now, $\{M(x_{m,1}(-1)\}$ contains a convergent subsequence. Let $\{x_{-1,m,1}\}$ be a subsequence of $\{x_{m,1}\}$ such that $\{M(x_{-1,m,1}(-1))\}$ is convergent. Also, $\{x_{-1,m,1}\}$ is a subsequence of $\{x_m\}$ and $\{M(x_{-1,m,1}(-1))\}$, $\{M(x_{-1,m,1}(0))\}$, $\{M(x_{-1,m,1}(1))\}$ are convergent.

Continuing in this fashion we find, for each $l \in \mathbb{Z}(-\infty,\infty)$, a subsequence $\{x_{-(l+1),m,(l+1)}\}$, $m \in \mathbb{Z}[1,\infty)\}$ of $\{x_{-l,m,l}, m \in \mathbb{Z}[1,\infty)\}$ such that $\{M(x_{-(l+1),m,(l+1)}(l+1)), m \in \mathbb{Z}[1,\infty)\}$ and $\{M(x_{-(l+1),m,(l+1)}(-(l+1))), m \in \mathbb{Z}[1,\infty)\}$ is convergent. Observe that also the sequences $\{M(x_{-(l+1),m,l+1}(-l)), m \in \mathbb{Z}[1,\infty)\}$, …, $\{M(x_{-(l+1),m,l+1}(0)), m \in \mathbb{Z}[1,\infty)\}$, …, $\{M(x_{-(l+1),m,l+1}(l)), m \in \mathbb{Z}[1,\infty)\}$ are convergent.

Consider now the sequence $\{x_{-u,u,u}, u \in \mathbb{Z}[1,\infty)\}$. Observe that it is a subsequence of $\{x_m, m \in \mathbb{Z}[1,\infty)\}$, and also that $\{M(x_{-u,u,u}(l)), u \in \mathbb{Z}[1,\infty)\}$ is convergent for all $l \in \mathbb{Z}(-\infty,\infty)$. Let us show that $\{\Phi x_{-u,u,u}, u \in \mathbb{Z}[1,\infty)\}$ is a Cauchy sequence.

Since $\{M(x_{-u,u,u}(l)), u \in \mathbb{Z}[1,\infty)\}$ is convergent for all $l \in \mathbb{Z}(-\infty,\infty)$, let $\epsilon > 0$ be given, there exists $u_0 \in \mathbb{Z}[1,\infty)$ such that $e, g \in \mathbb{Z}[1,\infty)$ with $e, g \geq u_0$,

$$\sup_{n\in\mathbb{Z}[0,\omega-1]} |M(x_{-e,e,e}(n)) - M(x_{-g,g,g}(n))| < \frac{\varepsilon(\Theta-1)}{\Theta}.$$

Therefore, if $e, g \in \mathbb{Z}[1,\infty)$ with $e, g \geq u_0$, for all $n \in Z$, we have

$$\begin{aligned} \|(\Phi x_{-e,e,e})(n) - (\Phi x_{-g,g,g})(n)\| &\leq \sum_{u=n}^{n+\omega-1} G(n,u)|M(x_{-e,e,e}(u)) - M(x_{-g,g,g}(u))| \\ &\leq \frac{\Theta}{\Theta-1} \sup_{n\in[0,T-1]} |M(x_{-e,e,e}(n)) - M(x_{-g,g,g}(n))| \\ &< \varepsilon. \end{aligned}$$

This proves that $\{\Phi x_{-u,u,u}, u \in \mathbb{Z}[1,\infty)\}$ is a Cauchy sequence in $\mathbb{X}$, and with this, the proof of Lemma 2.3 is complete. $\square$

Lemma 2.4. *Assume that* (H1)–(H3) *hold and* $R\sum_{j=1}^{q} c_j^M < 1$.

(i) *If* $a^M \leq 1$*, then* $\Phi : K \bigcap \bar{\Omega}_R \to K$ *is strict-set-contractive,*

(ii) *If (H4) holds and* $a^M > 1$*, then* $\Phi : K \bigcap \bar{\Omega}_R \to K$ *is strict-set-contractive,*

where $\Omega_R = \{x \in \mathbb{Y} : |x|_1 < R\}$.

Proof. We prove only (i), since the proof of (ii) is similar. It is easy to see that Φ is continuous and bounded. Now we prove that $\alpha_{\mathbb{Y}}(\Phi(S)) \leq \big(R\sum_{j=1}^{q} c_j^M\big)\alpha_{\mathbb{Y}}(S)$ for any bounded set $S \subset \bar{\Omega}_R$. Let $\eta = \alpha_{\mathbb{Y}}(S)$. Then, for any positive number $\varepsilon < \big(R\sum_{j=1}^{q} c_j^M\big)\eta$, there is a finite family of subsets $\{S_i\}$ satisfying $S = \bigcup_i S_i$ with $\operatorname{diam}(S_i) \leq \eta + \varepsilon$. Therefore

$$|x-y|_1 \leq \eta + \varepsilon \quad \text{for all } x, y \in S_i. \tag{2.8}$$

Since S and S_i are precompact in $\mathbb{X}$, it follows that there is a finite family of subsets $\{S_{ij}\}$ of S_i such that $S_i = \bigcup_j S_{ij}$ and

$$|x - y|_0 \le \varepsilon \quad \text{for all } x, y \in S_{ij}. \tag{2.9}$$

By Lemma 2.3, we know that $\Phi(S)$ is precompact in $\mathbb{X}$. Then, there is a finite family of subsets $\{S_{ijk}\}$ of S_{ij} such that $S_{ij} = \bigcup_k S_{ijk}$ and

$$|\Phi x - \Phi y|_0 \le \varepsilon \quad \text{for all } x, y \in S_{ijk}. \tag{2.10}$$

From (2.3), (2.5) and (2.8)-(2.10) and (H2), for any $x, y \in S_{ijk}$, we obtain

$$\begin{aligned}
&|(\Delta\Phi x) - (\Delta\Phi y)|_0 \\
&= \max_{n\in\mathbb{Z}[0,\omega-1]} \Big\{ \Big| a(n)(\Phi x)(n) - a(n)(\Phi y)(n) \\
&\quad - x(n)\Big[\sum_{i=1}^{p} a_i(n)x(n-\tau_i(n)) + \sum_{j=1}^{q} c_j(n)\Delta x(n-\sigma_j(n))\Big] \\
&\quad + y(n)\Big[\sum_{i=1}^{p} a_i(n)y(n-\tau_i(n)) + \sum_{j=1}^{q} c_j(n)\Delta y(n-\sigma_j(n))\Big]\Big|\Big\} \\
&\le \max_{n\in\mathbb{Z}[0,\omega-1]} \{|a(n)[(\Phi x)(n) - (\Phi y)(n)]|\} \\
&\quad + \max_{n\in\mathbb{Z}[0,\omega-1]} \Big\{ \Big| x(n)\Big[\sum_{i=1}^{p} a_i(n)x(n-\tau_i(n)) + \sum_{j=1}^{q} c_j(n)\Delta x(n-\sigma_j(n))\Big] \\
&\quad - y(n)\Big[\sum_{i=1}^{p} a_i(n)y(n-\tau_i(n)) + \sum_{j=1}^{q} c_j(n)\Delta y(n-\sigma_j(n))\Big]\Big|\Big\} \\
&\le a^M |(\Phi x) - (\Phi y)|_0 \\
&\quad + \max_{n\in\mathbb{Z}[0,\omega-1]} \Big\{ \Big| x(n)\Big[\Big(\sum_{i=1}^{p} a_i(n)x(n-\tau_i(n)) + \sum_{j=1}^{q} c_j(n)\Delta x(n-\sigma_j(n))\Big) \\
&\quad - \Big(\sum_{i=1}^{p} a_i(n)x(n-\tau_i(n)) + \sum_{j=1}^{q} c_j(n)\Delta x(n-\sigma_j(n))\Big)\Big]\Big|\Big\} \\
&\quad + \max_{n\in\mathbb{Z}[0,\omega-1]} \Big\{ \Big| y(n)\Big[\sum_{i=1}^{p} a_i(n)y(n-\tau_i(n)) \\
&\quad + \sum_{j=1}^{q} c_j(n)\Delta y(n-\sigma_j(n))\Big][x(n) - y(n)]\Big|\Big\} \\
&\le a^M\varepsilon + R \max_{n\in\mathbb{Z}[0,\omega-1]} \Big\{ \sum_{i=1}^{p} a_i(n)|x(n-\tau_i(n)) - y(n-\tau_i(n))| \\
&\quad + \sum_{j=1}^{q} c_j(n)|\Delta x(n-\sigma_j(n)) - \Delta y(n-\sigma_j(n))|\Big\} \\
&\quad + \varepsilon \max_{n\in\mathbb{Z}[0,\omega-1]} \Big\{ \sum_{i=1}^{p} a_i(n)y(n-\tau_i(n)) + \sum_{j=1}^{q} c_j(n)|\Delta y(n-\sigma_j(n))|\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\le a^M\varepsilon + R\varepsilon\Big(b^M\Big) + R(\eta+\varepsilon)\Big(\sum_{j=1}^{q} c_j^M\Big) + R\varepsilon\Big(b^M + \sum_{j=1}^{q} c_j^M\Big)\\
&= \Big(R\eta\sum_{j=1}^{q} c_j^M\Big) + \hat{H}\varepsilon,
\end{aligned}
$$

where $\hat{H} = a^M + 2Rb^M + 2R\sum_{j=1}^{q} c_j^M$. From the above inequality and (2.10), we have

$$
|\Phi x - \Phi y|_1 \le \Big(R\sum_{j=1}^{q} c_j^M\Big)\eta + \hat{H}\varepsilon \quad \text{for all } x, y \in S_{ijk}.
$$

Since ε is arbitrary small, it follows that

$$
\alpha_{C_\omega^1}(\Phi(S)) \le \Big(R\sum_{j=1}^{q} c_j^M\Big)\alpha_{C_\omega^1}(S).
$$

Therefore, Φ is strict-set-contractive. The proof of Lemma 2.4 is complete. $\square$

3. Main Result

Our main result of this paper is as follows.

Theorem 3.1. *Assume that* (H1)–(H3), (H5) *hold.*

(i) *If $a^M \le 1$, then system* (1.2) *has at least one positive ω-periodic solution.*
(ii) *If (H4) holds and $a^M > 1$, then system* (1.2) *has at least one positive ω-periodic solution.*

Proof. We only need to prove (i), since the proof of (ii) is similar. Let $R = \frac{\Theta(\Theta-1)}{\Gamma}$ and $0 < r < \frac{(\Theta-1)}{\Theta\Pi}$. Then we have $0 < r < R$. From Lemmas 2.2 and 2.4, we know that Φ is strict-set-contractive on $K_{r,R}$. In view of (2.5), we see that if there exists $x^* \in K$ such that $\Phi x^* = x^*$, then x^* is one positive ω-periodic solution of system (1.2). Now, we shall prove that condition (ii) of Lemma 2.1 hold.

First, we prove that $\Phi x \not\geq x$, for all $x \in K$, $|x|_1 = r$. Otherwise, there exists $x \in K$, $|x|_1 = r$ such that $\Phi x \ge x$. So $|x| > 0$ and $\Phi x - x \in K$, which implies that

$$
(\Phi x)(n) - x(n) \ge \Theta^{-1}|\Phi x - x|_1 \ge 0 \quad \text{for all } t \in [0,\omega]. \tag{3.1}
$$

Moreover, for $t \in [0,\omega]$, we have

$$
\begin{aligned}
(\Phi x)(n) &= \sum_{s=n}^{n+\omega-1} G(n,s)x(s)\Big[\sum_{i=1}^{p} a_i(s)x(s-\tau_i(s)) + \sum_{j=1}^{q} c_j(s)\Delta x(s-\sigma_j(s))\\
&\le \frac{\Theta}{\Theta-1} r|x|_0 \sum_{s=0}^{\omega-1}\Big[\sum_{i=1}^{p} a_i(s) + \sum_{j=1}^{q} c_j(s)\Big]\\
&= \frac{r}{\Theta-1}\Pi|x|_0\\
&< \Theta^{-1}|x|_0.
\end{aligned}
$$

In view of the above inequality and (3.1), we have

$$
|x|_0 \le |\Phi x| < \Theta^{-1}|x|_0 < |x|_0,
$$

which is a contradiction.

Finally, we prove that $\Phi x \nleq x$, for all $x \in K$, $|x|_1 = R$ holds. For this case, we need to prove that
$$\Phi x \nless x \quad x \in K,\, |x|_1 = R.$$
Suppose, for the sake of contradiction, that there exists $x \in K$ and $|x|_1 = R$ such that $\Phi x < x$. Thus $x - \Phi x \in K \setminus \{0\}$. Furthermore, for any $t \in [0,\omega]$, we have
$$x(n) - (\Phi x)(n) \geq \Theta^{-1}|x - \Phi x|_1 > 0. \tag{3.2}$$
In addition, for any $t \in [0,\omega]$, we find
$$\begin{aligned}(\Phi x)(n) &= \sum_{s=n}^{n+\omega-1} G(n,s)x(s)\Big[\sum_{i=1}^{p} a_i(s)x(s-\tau_i(s)) + \sum_{j=1}^{q} c_j(s)\Delta x(s-\sigma_j(s))\\ &\geq \frac{1}{\Theta-1}\Theta^{-1}|x|_1^2 \sum_{s=0}^{\omega-1}\Big[\Theta^{-1}\sum_{i=1}^{p} a_i(s) - \sum_{j=1}^{q} c_j(s)\Big]\\ &= \frac{1}{\Theta(\Theta-1)}\Gamma R^2 = R.\end{aligned}$$
From this inequality and (3.2), we obtain
$$|x| > |\Phi x|_0 \geq R,$$
which is a contradiction. Therefore, conditions (i) and (ii) hold. By Lemma 2.1, we see that Φ has at least one nonzero fixed point in K. Therefore, system (1.2) has at least one positive ω-periodic solution. The proof of Theorem 3.1 is complete. $\square$

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