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POSITIVE PERIODIC SOLUTIONS OF NEUTRAL LOGISTIC EQUATIONS WITH DISTRIBUTED DELAYS

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Abstract. Using a fixed point theorem of strict-set-contraction, we establish criteria for the existence of positive periodic solutions for the periodic neutral logistic equation, with distributed delays,

$$x'(t)=x(t)\Big[a(t)-\sum_{i=1}^{n}a_i(t)\int_{-T_i}^{0}x(t+\theta)\,\mathrm{d}\mu_i(\theta)-\sum_{j=1}^{m}b_j(t)\int_{-\hat{T}_j}^{0}x'(t+\theta)\,\mathrm{d}\nu_j(\theta)\Big],$$

where the coefficients a, a_i, b_j are continuous and periodic functions, with the same period. The values $T_i, \hat{T}_j$ are positive, and the functions μ_i, ν_j are non-decreasing with $\int_{-T_i}^{0}\mathrm{d}\mu_i=1$ and $\int_{-\hat{T}_j}^{0}\mathrm{d}\nu_j=1$.

1. Introduction

Consider the single species neutral logistic model, with discrete delays,

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t}=N(t)\Big[a(t)-\beta(t)N(t)-\sum_{i=1}^{n}b_i(t)N_i(t-\tau_i(t))-\sum_{j=1}^{m}c_j(t)N_j'(t-\sigma_j(t))\Big], \quad (1.1)$$

where the functions $a(t), \beta(t), b_j(t), c_j(t), \tau_i(t), \sigma_j(t)$ are continuous ω-periodic, and $a(t)\ge 0$, $\beta(t)\ge 0$, $b_i(t)\ge 0$, $c_j(t)\ge 0$ ($i=1,2,\dots,n$, $j=1,2,\dots,m$). An ecological justification of model (1.1) can be found in [3, 4, 6, 10]. Using continuation theory for k-set-contractions, Lu [8], Lu and Ge [9] studied the existence of positive periodic solutions of (1.1). Yang and Cao [11] used Mawhin's continuation theorem [2] to investigated the existence of positive periodic solutions of (1.1). The main results obtained in [6, 9] required $c_j\in C^1, \sigma_j\in C^2$ and $\sigma_j'<1$ ($j=1,2,\dots,n$). To the best of our knowledge, this is the first paper to study the existence of periodic solutions of neutral logistic equations with distributed delays.

The main purpose of this paper is by using a fixed point theorem of strict-set-contraction [1, 5] to establish the existence of positive periodic solutions of the

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following neutral functional differential equation, with distributed delays,

$$x'(t) = x(t)\Big[a(t) - \sum_{i=1}^{n} a_i(t)\int_{-T_i}^{0} x(t+\theta)\,\mathrm{d}\mu_i(\theta) - \sum_{j=1}^{n} b_j(t)\int_{-\hat{T}_j}^{0} x'(t+\theta)\,\mathrm{d}\nu_j(\theta)\Big], \tag{1.2}$$

where $a, a_i, b_j \in C(\mathbb{R}, \mathbb{R}^+)$ are ω-periodic functions, $T_i, \hat{T}_j$ are postive constants, $\mu_i, \nu_j : [-T_i, 0] \to [0, \infty)$ are nondecreasing functions and $\int_{-T_i}^{0} \mathrm{d}\mu_i = 1$, $\int_{-\hat{T}_j}^{0} \mathrm{d}\nu_j = 1$, for $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$. Note that (1.1) is a special case of (1.2). For an ecological justification of (1.1), we refer the reader to [7].

For convenience, we introduce the following notation:

$$\lambda = e^{-\int_0^\omega a(s)\,\mathrm{d}s}, \quad \Delta = \int_0^\omega \Big[\lambda \sum_{i=1}^{n} a_i(s) - \sum_{j=1}^{m} b_j(s)\Big]\,\mathrm{d}s,$$

$$\Pi = \int_0^\omega \Big[\sum_{i=1}^{n} a_i(s) + \sum_{j=1}^{m} b_j(s)\Big]\,\mathrm{d}s, \quad f^M = \max_{t\in[0,\omega]}\{f(t)\},$$

$$f^m = \min_{t\in[0,\omega]}\{f(t)\},$$

where f is a continuous ω-periodic function. Also we introduce the following assumptions:

- (H1) $\lambda := \exp\big(-\int_0^\omega a(s)\,\mathrm{d}s\big) < 1$.
- (H2) $\lambda \sum_{i=1}^{n} a_i(t) - \sum_{j=1}^{m} b_j(t) \ge 0$.
- (H3) $(1 + a^m)\frac{\lambda^2\Delta}{1-\lambda} \ge \max_{t\in[0,\omega]}\big\{\sum_{i=1}^{n} a_i(t) + \sum_{j=1}^{m} b_j(t)\big\}$.
- (H4) $\frac{\Pi(a^M-1)}{\lambda(1-\lambda)} \le \min_{t\in[0,\omega]}\big\{\lambda \sum_{i=1}^{n} a_i(t) - \sum_{j=1}^{m} b_j(t)\big\}$.
- (H5) $\frac{1-\lambda}{\lambda^2\Delta}\big(\sum_{j=1}^{m} b_j^M\big) < 1$.

2. Preliminaries

To obtain the existence of periodic solutions to (1.2), we make the following preparations:

Let E be a Banach space and K be a cone in E. The semi-order induced by the cone K is denoted by "$\le$". That is, $x \le y$ if and only if $y - x \in K$. In addition, for a bounded subset $A \subset E$, let the Kuratowski measure of non-compactness be defined by

$$\alpha_E(A) = \inf\Big\{\delta > 0 : \text{ there is a finite number of subsets } A_i \subset A \\ \text{such that } A = \cup_i A_i \text{ and } \operatorname{diam}(A_i) \le \delta\Big\},$$

where $\operatorname{diam}(A_i)$ denotes the diameter of the set A_i.

Let E, F be two Banach spaces and $D \subset E$, a continuous and bounded mapping $\Phi : \bar{\Omega} \to F$ is called k-set contractive if for every bounded set $S \subset D$ we have

$$\alpha_F(\Phi(S)) \le k\alpha_E(S).$$

The mapping Φ is called strict-set-contractive if it is k-set-contractive for some $0 \le k < 1$. From [1, 2], we cite the following lemma which is useful for the proof of our main result.

Lemma 2.1 ([1, 5]). *Let K be a cone of the real Banach space X and $K_{r,R} = \{x \in K : r \le \|x\| \le R\}$ with $R > r > 0$. Suppose that $\Phi : K_{r,R} \to K$ is strict-set-contractive such that one of the following two conditions is satisfied:*

(i) *($\Phi x \not\leq x$ for all $x \in K$, $\|x\| = r$) and ($\Phi x \not\geq x$ for all $x \in K$, $\|x\| = R$).*
(ii) *($\Phi x \not\geq x$ for all $x \in K$, $\|x\| = r$) and ($\Phi x \not\leq x$ for all $x \in K$, $\|x\| = R$).*

Then Φ has at least one fixed point in $K_{r,R}$.

To apply Lemma 2.1 to (1.2), we set

$$C^0_\omega = \{x \in C^0(\mathbb{R}, \mathbb{R}) : x(t + \omega) = x(t)\}$$

with the norm $|x|_0 = \max_{t\in[0,\omega]}\{|x(t)|\}$, and

$$C^1_\omega = \{x \in C^1(\mathbb{R}, \mathbb{R}) : x(t + \omega) = x(t)\}$$

with the norm $|x|_1 = \max\{|x|_0, |x'|_0\}$. Then C^0_ω and C^1_ω are all Banach spaces.

Define the cone K in C^1_ω by

$$K = \{x \in C^1_\omega : x(t) \ge \lambda|x|_1, t \in [0, \omega]\}. \tag{2.1}$$

Let the mapping Φ be defined by

$$\begin{aligned}(\Phi x)(t) = \int_t^{t+\omega} G(t, s)x(s)\Big[\sum_{i=1}^n a_i(s)\int_{-T_i}^0 x(s + \theta)\,\mathrm{d}\mu_i(\theta)\\
+\sum_{j=1}^m b_j(s)\int_{-\hat{T}_j}^0 x'(s + \theta)\,\mathrm{d}\nu_j(\theta)\Big]\,\mathrm{d}s,\end{aligned} \tag{2.2}$$

where $x \in K, t \in \mathbb{R}$, and

$$G(t, s) = \frac{e^{-\int_t^s a(\theta)\,\mathrm{d}\theta}}{1 - e^{-\int_0^\omega a(\theta)\,\mathrm{d}\theta}}, \quad s \in [t, t + \omega].$$

It is easy to see that $G(t + \omega, s + \omega) = G(t, s)$ and

$$\frac{\lambda}{1-\lambda} \le G(t, s) \le \frac{1}{1-\lambda}, \quad s \in [t, t + \omega].$$

Next, we give some lemmas concerning the K and Φ defined above.

Lemma 2.2. *Assume that (H1)–(H3) hold.*

(i) *If $a^M \le 1$, then $\Phi : K \to K$ is well defined.*
(ii) *If (H4) holds and $a^M > 1$, then $\Phi : K \to K$ is well defined.*

Proof. For any $x \in K$, it is clear that $\Phi x \in C^1(\mathbb{R}, \mathbb{R})$. In view of (2.2), for $t \in \mathbb{R}$, we obtain

$$\begin{aligned}(\Phi x)(t + \omega) &= \int_{t+\omega}^{t+2\omega} G(t + \omega, s)x(s)\Big[\sum_{i=1}^n a_i(s)\int_{-T_i}^0 x(s + \theta)\,\mathrm{d}\mu_i(\theta)\\
&\quad+\sum_{j=1}^m b_j(s)\int_{-\hat{T}_j}^0 x'(s + \theta)\,\mathrm{d}\nu_j(\theta)\Big]\,\mathrm{d}s\\
&= \int_t^{t+\omega} G(t + \omega, u + \omega)x(u + \omega)\Big[a(u + \omega)\int_{-T_i}^0 x(u + \omega + \theta)\,\mathrm{d}\mu_i(\theta)\\
&\quad+ b(u + \omega)\int_{-\hat{T}_j}^0 x'(u + \omega + \theta)\,\mathrm{d}\nu_j(\theta)\Big]\,\mathrm{d}u\end{aligned}$$

$$
\begin{aligned}
&= \int_t^{t+\omega} G(t,u)x(u)\Big[a(u)\int_{-T_i}^0 x(u+\theta)\,\mathrm{d}\mu_i(\theta)\\
&\quad + b(u)\int_{-\hat{T}_j}^0 x'(u+\theta)\,\mathrm{d}\nu_j(\theta)\Big]\,\mathrm{d}u\\
&= (\Phi x)(t).
\end{aligned}
$$

That is, $(\Phi x)(t+\omega) = (\Phi x)(t)$, $t \in \mathbb{R}$. So $\Phi x \in C^1_\omega$. In view of (H2), for $x \in K$, $t \in [0,\omega]$, we have

$$
\begin{aligned}
&\sum_{i=1}^n a_i(t)\int_{-T_i}^0 x(t+\theta)\,\mathrm{d}\mu_i(\theta) + \sum_{j=1}^m b_j(t)\int_{-\hat{T}_j}^0 x'(t+\theta)\,\mathrm{d}\nu_j(\theta)\\
&\ge \sum_{i=1}^n a_i(t)\int_{-T_i}^0 x(t+\theta)\,\mathrm{d}\mu_i(\theta) - \sum_{j=1}^m b_j(t)\int_{-\hat{T}_j}^0 |x'(t+\theta)|\,\mathrm{d}\nu_j(\theta)\\
&\ge \sum_{i=1}^n a_i(t)\int_{-T_i}^0 \lambda|x|_1\,\mathrm{d}\mu_i(\theta) - \sum_{j=1}^m b_j(t)\int_{-\hat{T}_j}^0 |x|_1\,\mathrm{d}\nu_j(\theta)\\
&\ge \lambda\sum_{i=1}^n a_i(t) - \sum_{j=1}^m b_j(t)]|x|_1\,\mathrm{d}\nu_j(\theta) \ge 0.
\end{aligned} \tag{2.3}
$$

Therefore, for $x \in K$, $t \in [0,\omega]$, we find

$$
\begin{aligned}
|\Phi x|_0 \le \frac{1}{1-\lambda}\int_0^\omega x(s)\Big[\sum_{i=1}^n a_i(s)\int_{-T_i}^0 x(s+\theta)\,\mathrm{d}\mu_i(\theta)\\
+ \sum_{j=1}^m b_j(s)\int_{-\hat{T}_j}^0 x'(s+\theta)\,\mathrm{d}\nu_j(\theta)\Big]\,\mathrm{d}s
\end{aligned}
$$

and

$$
\begin{aligned}
(\Phi x)(t) &\ge \frac{\lambda}{1-\lambda}\int_t^{t+\omega} x(s)\Big[\sum_{i=1}^n a_i(s)\int_{-T_i}^0 x(s+\theta)\,\mathrm{d}\mu_i(\theta)\\
&\quad + \sum_{j=1}^m b_j(s)\int_{-\hat{T}_j}^0 x'(s+\theta)\,\mathrm{d}\nu_j(\theta)\Big]\,\mathrm{d}s\\
&= \frac{\lambda}{1-\lambda}\int_0^{\omega} x(s)\Big[\sum_{i=1}^n a_i(s)\int_{-T_i}^0 x(s+\theta)\,\mathrm{d}\mu_i(\theta)\\
&\quad + \sum_{j=1}^m b_j(s)\int_{-\hat{T}_j}^0 x'(s+\theta)\,\mathrm{d}\nu_j(\theta)\Big]\,\mathrm{d}s\\
&\ge \lambda|\Phi x|_0.
\end{aligned} \tag{2.4}
$$

Now, we show that $(\Phi x)'(t) \ge \lambda|(\Phi x)'|_0, t \in [0,\omega]$. From (2.2), we have

$$
\begin{aligned}
(\Phi x)'(t) = G(t,t+\omega)x(t+\omega)\Big[a(t+\omega)\int_{-T_i}^0 x(t+\omega+\theta)\,\mathrm{d}\mu_i(\theta)\\
+ b(t+\omega)\int_{-\hat{T}_j}^0 x'(t+\omega+\theta)\,\mathrm{d}\nu_j(\theta)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- G(t,t)x(t)\Big[\sum_{i=1}^{n} a_i(t)\int_{-T_i}^{0} x(t+\theta)\,\mathrm{d}\mu_i(\theta) \\
&+ \sum_{j=1}^{m} b_j(t)\int_{-\hat{T}_j}^{0} x'(t+\theta)\,\mathrm{d}\nu_j(\theta)\Big] + a(t)(\Phi x)(t) \\
&= a(t)(\Phi x)(t) - x(t)\Big[\sum_{i=1}^{n} a_i(t)\int_{-T_i}^{0} x(t+\theta)\,\mathrm{d}\mu_i(\theta) \\
&+ \sum_{j=1}^{m} b_j(t)\int_{-\hat{T}_j}^{0} x'(t+\theta)\,\mathrm{d}\nu_j(\theta)\Big].
\end{aligned} \tag{2.5}
$$

It follows from (2.3) and (2.5) that if $(\Phi x)'(t) \ge 0$, then

$$
(\Phi x)'(t) \le a(t)(\Phi x)(t) \le a^M(\Phi x)(t) \le (\Phi x)(t). \tag{2.6}
$$

On the other hand, from (2.4), (2.5) and (H3), if $(\Phi x)'(t) < 0$, then

$$
\begin{aligned}
&-(\Phi x)'(t) \\
&= x(t)\Big[\sum_{i=1}^{n} a_i(t)\int_{-T_i}^{0} x(t+\theta)\,\mathrm{d}\mu_i(\theta) + \sum_{j=1}^{m} b_j(t)\int_{-\hat{T}_j}^{0} x'(t+\theta)\,\mathrm{d}\nu_j(\theta)\Big] \\
&\quad - a(t)(\Phi x)(t) \\
&\le |x|_1^2\Big[\sum_{i=1}^{n} a_i(t) + \sum_{j=1}^{m} b_j(t)\Big] - a^m(\Phi x)(t) \\
&\le (1+a^m)\frac{\lambda^2}{1-\lambda}|x|_1^2\int_0^{\omega}\Big[\lambda\sum_{i=1}^{n} a_i(s) + \sum_{j=1}^{m} b_j(s)\Big]\,\mathrm{d}s - a^m(\Phi x)(t) \\
&= (1+a^m)\int_0^{\omega}\frac{\lambda}{1-\lambda}\lambda|x|_1\Big[\lambda|x|_1\sum_{i=1}^{n} a_i(s) - |x|_1\sum_{j=1}^{m} b_j(s)\Big]\,\mathrm{d}s - a^m(\Phi x)(t) \\
&\le (1+a^m)\int_t^{t+\omega} G(t,s)x(s)\Big[\sum_{i=1}^{n} a_i(s)\int_{-T_1}^{0} x(s+\theta)\,\mathrm{d}\mu_i(\theta) \\
&\quad - \sum_{j=1}^{m} b_j(s)\int_{-T_2}^{0} |x'(\theta+s)|\,\mathrm{d}\nu_j(\theta)\Big]\,\mathrm{d}s - a^m(\Phi x)(t) \\
&\le (1+a^m)\int_t^{t+\omega} G(t,s)x(s)\Big[\sum_{i=1}^{n} a_i(s)\int_{-T_1}^{0} x(s+\theta)\,\mathrm{d}\mu_i(\theta) \\
&\quad + \sum_{j=1}^{m} b_j(s)\int_{-T_2}^{0} x'(\theta+s)\,\mathrm{d}\nu_j(\theta)\Big]\,\mathrm{d}s - a^m(\Phi x)(t) \\
&= (1+a^m)(\Phi_i x)(t) - a^m(\Phi x)(t) \\
&= (\Phi x)(t).
\end{aligned}
$$

It follows from the above inequality and (2.6) that $|(\Phi x)'|_0 \le |\Phi x|_0$. So $|\Phi x|_1 = |\Phi x|_0$. By (2.2) we have $(\Phi x)(t) \ge \lambda|\Phi x|_1$. Hence, $\Phi x \in K$. The proof of (i) is complete.

(ii) In view of the proof of (i), we only need to prove that $(\Phi x)'(t) \geq 0$ implies

$$(\Phi x)'(t) \leq (\Phi x)(t).$$

From (2.3), (2.5), (H2) and (H4), we obtain

$$\begin{aligned}
(\Phi_i x)'(t) &\leq a(t)(\Phi x)(t) - \lambda |x|_1 \Big[\sum_{i=1}^{n} a_i(t) \int_{-T_i}^{0} x(t+\theta)\, \mathrm{d}\mu_i(\theta) \\
&\quad - \sum_{j=1}^{m} b_j(t) \int_{-\hat{T}_j}^{0} |x'(t+\theta)|\, \mathrm{d}\nu_j(\theta) \Big] \\
&\leq a(t)(\Phi x)(t) - \lambda |x|_1^2 \Big[\lambda \sum_{i=1}^{n} a_i(t) - \sum_{j=1}^{m} b_j(t) \Big] \\
&\leq a^M (\Phi x)(t) - \lambda |x|_1^2 \frac{a^M - 1}{\lambda(1-\lambda)} \int_0^{\omega} \Big[\sum_{i=1}^{n} a_i(s) + \sum_{j=1}^{m} b_j(s) \Big]\, \mathrm{d}s \\
&\leq a^M (\Phi x)(t) - (a^M - 1) \int_t^{t+\omega} \frac{1}{1-\lambda} |x|_1 \Big[\sum_{i=1}^{n} a_i(s)|x|_1 + \sum_{j=1}^{m} b_j(s)|x|_1 \Big]\, \mathrm{d}s \\
&\leq a^M (\Phi x)(t) - (a^M - 1) \int_t^{t+\omega} G(t,s) x(s) \Big[\sum_{i=1}^{n} a_i(t) \int_{-T_i}^{0} x(t+\theta)\, \mathrm{d}\mu_i(\theta) \\
&\quad + \sum_{j=1}^{m} b_j(t) \int_{-\hat{T}_j}^{0} |x'(t+\theta)|\, \mathrm{d}\nu_j(\theta) \Big]\, \mathrm{d}s \\
&\leq a^M (\Phi x)(t) - (a^M - 1) \int_t^{t+\omega} G(t,s) x(s) \Big[\sum_{i=1}^{n} a_i(t) \int_{-T_i}^{0} x(t+\theta)\, \mathrm{d}\mu_i(\theta) \\
&\quad + \sum_{j=1}^{m} b_j(t) \int_{-\hat{T}_j}^{0} x'(t+\theta)\, \mathrm{d}\nu_j(\theta) \Big]\, \mathrm{d}s \\
&= a^M (\Phi x)(t) - (a^M - 1)(\Phi x)(t) \\
&= (\Phi x)(t).
\end{aligned}$$

The proof of (ii) is complete. $\square$

Lemma 2.3. *Assume that (H1)–(H3) hold and $R \sum_{j=1}^{m} b_j^M < 1$.*

(i) *If $a^M \leq 1$, then $\Phi : K \bigcap \bar{\Omega}_R \to K$ is strict-set-contractive,*
(ii) *If (H4) holds and $a^M > 1$, then $\Phi : K \bigcap \bar{\Omega}_R \to K$ is strict-set-contractive,*

where $\Omega_R = \{x \in C_\omega^1 : |x|_1 < R\}$.

Proof. We only need to prove (i), since the proof of (ii) is similar. It is easy to see that Φ is continuous and bounded. Now we prove that $\alpha_{C_\omega^1}(\Phi(S)) \leq \big(R \sum_{j=1}^{m} b_j^M\big) \alpha_{C_\omega^1}(S)$ for any bounded set $S \subset \bar{\Omega}_R$. Let $\eta = \alpha_{C_\omega^1}(S)$. Then, for any positive number $\varepsilon < \big(R \sum_{j=1}^{m} b_j^M\big)\eta$, there is a finite family of subsets $\{S_i\}$ satisfying $S = \bigcup_i S_i$ with $\mathrm{diam}(S_i) \leq \eta + \varepsilon$. Therefore,

$$|x - y|_1 \leq \eta + \varepsilon \quad \text{for all } x, y \in S_i. \tag{2.7}$$

As S and S_i are precompact in C^0_ω, it follows that there is a finite family of subsets $\{S_{ij}\}$ of S_i such that $S_i = \bigcup_j S_{ij}$ and

$$|x-y|_0 \le \varepsilon \quad \text{for all } x,y\in S_{ij}. \tag{2.8}$$

In addition, for any $x\in S$ and $t\in[0,\omega]$, we have

$$\begin{aligned}
|(\Phi x)(t)| &= \int_t^{t+\omega} G(t,s)x(s)\Big[\sum_{i=1}^n a_i(t)\int_{-T_i}^0 x(t+\theta)\,\mathrm{d}\mu_i(\theta)\\
&\quad + \sum_{j=1}^m b_j(t)\int_{-\hat{T}_j}^0 x'(t+\theta)\,\mathrm{d}\nu_j(\theta)\Big]\,\mathrm{d}s\\
&\le \frac{R^2}{1-\lambda}\int_t^{t+\omega}\Big[\sum_{i=1}^n a_i(s)+\sum_{j=1}^m b_j(s)\Big]\,\mathrm{d}s := H
\end{aligned}$$

and

$$\begin{aligned}
|(\Phi x)'(t)| &= \Big|a(t)(\Phi x)(t) - x(t)\Big[\sum_{i=1}^n a_i(s)\int_{-T_i}^0 x(s+\theta)\,\mathrm{d}\mu_i(\theta)\\
&\quad + \sum_{j=1}^m b_j(s)\int_{-\hat{T}_j}^0 x'(s+\theta)\,\mathrm{d}\nu_j(\theta)\Big]\Big|\\
&\le a^M H + R^2\sum_{j=1}^m (a^M+b^M).
\end{aligned}$$

Applying the Arzela-Ascoli Theorem, we know that $\Phi(S)$ is precompact in C^0_ω. Then, there is a finite family of subsets $\{S_{ijk}\}$ of S_{ij} such that $S_{ij}=\bigcup_k S_{ijk}$ and

$$|\Phi x - \Phi y|_0 \le \varepsilon \quad \text{for all} x,y\in S_{ijk}. \tag{2.9}$$

From (2.3), (2.5) and (2.7)-(2.9) and (H2), for any $x,y\in S_{ijk}$, we obtain

$$\begin{aligned}
&|(\Phi x)' - (\Phi y)'|_0\\
&= \max_{t\in[0,\omega]}\Big\{\Big|a(t)(\Phi x)(t) - a(t)(\Phi y)(t)\\
&\quad - x(t)\Big[\sum_{i=1}^n a_i(t)\int_{-T_i}^0 x(t+\theta)\,\mathrm{d}\mu_i(\theta) + \sum_{j=1}^m b_j(t)\int_{-\hat{T}_j}^0 x'(t+\theta)\,\mathrm{d}\nu_j(\theta)\Big]\\
&\quad + y(t)\Big[\sum_{i=1}^n a_i(t)\int_{-T_i}^0 y(t+\theta)\,\mathrm{d}\mu_i(\theta) + \sum_{j=1}^m b_j(t)\int_{-\hat{T}_j}^0 y'(t+\theta)\,\mathrm{d}\nu_j(\theta)\Big]\Big|\Big\}\\
&\le \max_{t\in[0,\omega]}\{|a(t)[(\Phi x)(t) - (\Phi y)(t)]|\}\\
&\quad + \max_{t\in[0,\omega]}\Big\{\Big|x(t)\Big[\sum_{i=1}^n a_i(t)\int_{-T_i}^0 x(t+\theta)\,\mathrm{d}\mu_i(\theta) + \sum_{j=1}^m b_j(t)\int_{-\hat{T}_j}^0 x'(t+\theta)\,\mathrm{d}\nu_j(\theta)\Big]\\
&\quad - y(t)\Big[\sum_{i=1}^n a_i(t)\int_{-T_i}^0 y(t+\theta)\,\mathrm{d}\mu_i(\theta) + \sum_{j=1}^m b_j(t)\int_{-\hat{T}_j}^0 y'(t+\theta)\,\mathrm{d}\nu_j(\theta)\Big]\Big|\Big\}\\
&\le a^M|(\Phi x) - (\Phi y)|_0
\end{aligned}$$

$$
\begin{aligned}
&+ \max_{t\in[0,\omega]} \Big\{ \Big| x(t) \Big[\Big(\sum_{i=1}^{n} a_i(t) \int_{-T_i}^{0} x(t+\theta)\,\mathrm{d}\mu_i(\theta) \\
&+ \sum_{j=1}^{m} b_j(t) \int_{-\hat{T}_j}^{0} x'(t+\theta)\,\mathrm{d}\nu_j(\theta) \Big) \\
&- \Big(\sum_{i=1}^{n} a_i(t) \int_{-T_i}^{0} y(t+\theta)\,\mathrm{d}\mu_i(\theta) + \sum_{j=1}^{m} b_j(t) \int_{-\hat{T}_j}^{0} y'(t+\theta)\,\mathrm{d}\nu_j(\theta) \Big) \Big] \Big| \Big\} \\
&+ \max_{t\in[0,\omega]} \Big\{ \Big| y(t) \Big[\sum_{i=1}^{n} a_i(t) \int_{-T_i}^{0} y(t+\theta)\,\mathrm{d}\mu_i(\theta) \\
&+ \sum_{j=1}^{m} b_j(t) \int_{-\hat{T}_j}^{0} y'(t+\theta)\,\mathrm{d}\nu_j(\theta) \Big] [x(t)-y(t)] \Big| \Big\} \\
\le\;& a^M \varepsilon + R \max_{t\in[0,\omega]} \Big\{ \sum_{i=1}^{n} a_i(t) \int_{T_i}^{0} |x(s+\theta) - y(s+\omega)|\,\mathrm{d}\mu_i(s) \\
&+ \sum_{j=1}^{m} b_j(t) \int_{\hat{T}_j}^{0} |x'(s+\theta) - y'(s+\omega)|\,\mathrm{d}\nu_j(s) \\
&+ \varepsilon \max_{t\in[0,\omega]} \Big\{ \sum_{i=1}^{n} a_i(t) \int_{-T_i}^{0} y(t+\theta)\,\mathrm{d}\mu_i(\theta) + \sum_{j=1}^{m} b_j(t) \int_{-\hat{T}_j}^{0} y'(t+\theta)\,\mathrm{d}\nu_j(\theta) \Big\} \\
\le\;& a^M\varepsilon + R\varepsilon \sum_{i=1}^{n} a_i^M + R(\eta+\varepsilon) \sum_{j=1}^{m} b_j^M + R\varepsilon \sum_{i=1}^{n} a_i^M + R\varepsilon \sum_{j=1}^{m} b_j^M \\
=\;& R\eta \sum_{j=1}^{m} b_j^M + \hat{H}\varepsilon,
\end{aligned}
$$

where $\hat{H} = a^M + 2R\sum_{i=1}^{n} a_i^M + 2R\sum_{j=1}^{n} b_j^M$. From the above equation and (2.9), we have

$$
|\Phi x - \Phi y|_1 \le \Big(R \sum_{j=1}^{m} b_j^M \Big) \eta + \hat{H}\varepsilon \quad \text{for all } x, y \in S_{ijk}.
$$

Since ε is arbitrary small, it follows that

$$
\alpha_{C^1_\omega}(\Phi(S)) \le \Big(R \sum_{j=1}^{m} b_j^M \Big) \alpha_{C^1_\omega}(S).
$$

Therefore, Φ is strict-set-contractive. The proof of Lemma 2.3 is complete. □

3. Main Result

Our main result of this paper is as follows.

Theorem 3.1. *Assume that (H1)–(H3), (H5) hold.*

(i) *If $a^M \le 1$, then system* (1.2) *has at least one positive ω-periodic solution.*
(ii) *If (H4) holds and $a^M > 1$, then system* (1.2) *has at least one positive ω-periodic solution.*

Proof. We only need to prove (i), since the proof of (ii) is similar. Let $R = \frac{1-\lambda}{\lambda^2\Delta}$ and $0 < r < \frac{\lambda(1-\lambda)}{\Pi}$. Then we have $0 < r < R$. From Lemma 2.2 and 2.3, we know that Φ is strict-set-contractive on $K_{r,R}$. In view of (2.5), we see that if there exists $x^* \in K$ such that $\Phi x^* = x^*$, then x^* is one positive ω-periodic solution of (1.2). Now, we shall prove that condition (ii) of Lemma 2.1 hold.

First, we prove that $\Phi x \not\geq x$, for all $x \in K$, $|x|_1 = r$. Otherwise, there exists $x \in K$, $|x|_1 = r$ such that $\Phi x \geq x$. So $|x| > 0$ and $\Phi x - x \in K$, which implies that

$$(\Phi x)(t) - x(t) \geq \lambda |\Phi x - x|_1 \geq 0 \quad \text{for all } t \in [0,\omega]. \tag{3.1}$$

Moreover, for $t \in [0,\omega]$, we have

$$\begin{aligned}
(\Phi x)(t) &= \int_t^{t+\omega} G(t,s)x(s)\Big[\sum_{i=1}^n a_i(s)\int_{-T_i}^0 x(s+\theta)\,\mathrm{d}\mu_i(\theta) \\
&\quad + \sum_{j=1}^m b_j(s)\int_{-\hat{T}_j}^0 x'(s+\theta)\,\mathrm{d}\nu_j(\theta)\Big]\,\mathrm{d}s \\
&\leq \frac{1}{1-\lambda} r|x|_0 \int_0^\omega \Big[\sum_{i=1}^n a_i(s) + \sum_{j=1}^m b_j(s)\Big]\,\mathrm{d}s \\
&= \frac{1}{1-\lambda}\Pi|x|_0 < \lambda|x|_0.
\end{aligned} \tag{3.2}$$

In view of (3.1) and (3.2), we have

$$|x|_0 \leq |\Phi x|_0 < \lambda|x|_0 < |x|_0,$$

which is a contradiction. Finally, we prove that $\Phi x \not\leq x$ for all $x \in K$, $|x|_1 = R$ also holds. For this case, we only need to prove that

$$\Phi x \not< x \quad x \in K,\ |x|_1 = R.$$

Suppose, for the sake of contradiction, that there exists $x \in K$ and $|x|_1 = R$ such that $\Phi x < x$. Thus $x - \Phi x \in K \setminus \{0\}$. Furthermore, for any $t \in [0,\omega]$, we have

$$x(t) - (\Phi x)(t) \geq \lambda|x - \Phi x|_1 > 0. \tag{3.3}$$

In addition, for any $t \in [0,\omega]$, we find

$$\begin{aligned}
(\Phi x)(t) &= \int_t^{t+\omega} G(t,s)x(s)\Big[\sum_{i=1}^n a_i(s)\int_{-T_i}^0 x(s+\theta)\,\mathrm{d}\mu_i(\theta) \\
&\quad + \sum_{j=1}^m b_j(s)\int_{-\hat{T}_j}^0 x'(s+\theta)\,\mathrm{d}\nu_j(\theta)\Big]\,\mathrm{d}s \\
&\geq \frac{\lambda}{1-\lambda}|x|_0^2 \int_0^\omega \Big[\lambda\sum_{i=1}^n a_i(s) - \sum_{j=1}^m b_j(s)\Big]\,\mathrm{d}s \\
&= \frac{\lambda^2}{1-\lambda}\Delta R^2 = R.
\end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we obtain $|x| > |\Phi x|_0 \geq R$, which is a contradiction. Therefore, conditions (i) and (ii) hold. By Lemma 2.1, we see that Φ has at least one nonzero fixed point in K. Therefore, (1.2) has at least one positive ω-periodic solution. The proof of Theorem 3.1 is complete. □

We remark that from the proof of our results, if some (or all) $\hat{T}_j (j = 1, 2, \dots, n)$ are replaced by ∞ the conclusion of Theorem 3.1 remains valid.

As an example of (1.1), consider the equation

$$x'(t) = x(t)\Big[\frac{1+\cos t}{4\pi} - (5-2\sin t)\int_{-1}^{0} x(t+\theta)\mathrm{d}\theta - \frac{1-\sin t}{20}\int_{-1}^{0} x'(t+\theta)\,\mathrm{d}\theta\Big]. \quad (3.5)$$

Obviously,

$$a(t) = \frac{1+\cos t}{4\pi}, \quad a_1(t) = 5 - 2\sin t, \quad b_1(t) = \frac{1-\sin t}{20}.$$

Furthermore, we have

$$\lambda = e^{-\frac{1}{2}}, \max_{0\le t\le 2\pi}\{\lambda a_1(t) - b_1(t)\} > 1.7 > 0,$$

$$\Delta = \int_0^{2\pi} [\lambda a_1(s) - b_1(s)]\,\mathrm{d}s = 10\pi e^{-\frac{1}{2}} - \frac{1}{10}\pi > 18,$$

$$\max_{0\le t\le 2\pi}\{a_1(t) + b_1(t)\} = \frac{71}{10},$$

$$(1 + a^m)\frac{\lambda^2}{1-\lambda}\Delta > 16 > \frac{71}{10} > \max_{0\le t\le 2\pi}\{a_1(t) + b_1(t)\},$$

$$\frac{1-\lambda}{\lambda^2\Delta} b_1^M < 5.95 \times 10^{-3} < 1.$$

Hence, (H1)–(H3), (H5) hold and $a^M \le 1$. According to Theorem 3.1, system (3.5) has at least one positive 2π-periodic solution.

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