Electronic Journal of Differential Equations, Vol. 2007(2007), No. 135, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

SPECTRAL STABILITY OF UNDERCOMPRESSIVE SHOCK PROFILE SOLUTIONS OF A MODIFIED KDV-BURGERS EQUATION

JEFF DODD

Abstract. It is shown that certain undercompressive shock profile solutions of the modified Korteweg-de Vries-Burgers equation

$$\partial_t u + \partial_x(u^3) = \partial_x^3 u + \alpha \partial_x^2 u, \quad \alpha \geq 0$$

are spectrally stable when α is sufficiently small, in the sense that their linearized perturbation equations admit no eigenvalues having positive real part except a simple eigenvalue of zero (due to the translation invariance of the linearized perturbation equations). This spectral stability makes it possible to apply a theory of Howard and Zumbrun to immediately deduce the asymptotic orbital stability of these undercompressive shock profiles when α is sufficiently small and positive.

1. Introduction

In 1991, Wu [9] found numerical evidence that certain undercompressive shock solutions of the single conservation law

$$\partial_t u + \partial_x(u^3) = 0 \tag{1.1}$$

admitted smooth shock profile solutions of the following modified Korteweg-de Vries-Burgers equation incorporating dispersion and dissipation:

$$\partial_t u + \partial_x(u^3) = \eta \partial_x^3 u + \mu \partial_x^2 u \tag{1.2}$$

where η and μ are positive real parameters. In 1993, Jacobs, McKinney, and Shearer [3] rigorously characterized these shock profiles. In this paper, we prove a spectral stability result for them of exactly the sort required by a theory of Howard and Zumbrun [2] to ensure orbital asymptotic stability of these shock profiles.

Scaling the dispersion coefficient to unity in (1.2) yields

$$\partial_t u + \partial_x(u^3) = \partial_x^3 u + \alpha \partial_x^2 u \tag{1.3}$$

where $\alpha = \mu/\sqrt{|\eta|}$. Then whenever

$$u(x,t) = \begin{cases} u_- & \text{if } x < st, \\ u_+ & \text{if } x > st \end{cases} \tag{1.4}$$

2000 *Mathematics Subject Classification.* 74J30, 74J40, 35Q53, 35P05.
Key words and phrases. Travelling waves; undercompressive shocks; spectral stability; Evans function.

Submitted July 17, 2007. Published October 13, 2007.

is a shock solution of (1.1), a smooth travelling wave solution $u(x-ct)$ of (1.3) satisfying the boundary conditions

$$u(\pm\infty) = u_\pm, \quad u'(\pm\infty) = u''(\pm\infty) = 0$$

corresponds to a one parameter family of shock profile solutions of (1.2) that converge to the shock as $\eta, \mu \to 0$ with $\mu^2/\eta = \alpha^2$. The central result in [3] is a phase plane analysis showing that for fixed $u_- > 0$ and $-u_- \le u_+ < -u_-/2$, though the shock (1.4) is undercompressive, there is a unique value of α for which such a travelling wave exists, namely $\alpha = (3/\sqrt{2})(u_+ + u_-)$, so that $0 \le \alpha < 3u_-/(2\sqrt{2})$. This travelling wave is given explicitly by

$$u_c(x-ct) = \frac{\sqrt{2}}{6}\alpha + \big(\frac{\sqrt{2}}{6}\alpha - u_-\big)\tanh A(x-ct) \tag{1.5}$$

where $A = \frac{1}{\sqrt{2}}u_- - \frac{\alpha}{6}$.

While the shock profiles (1.5) appear similar to monotone shock profiles of the generalized KdV-Burgers equation

$$\partial_t u + \partial_x u^p = -\partial_x^3 u + \alpha \partial_x^2 u \quad (p > 1,\ \alpha > 0) \tag{1.6}$$

they are amenable neither to the energy method nor the eigenvalue analysis successfully applied to shock profiles of (1.6) in [5] and [7] respectively. Both fail because they rely on the convexity of the nonlinearity $f(u) = u^p$ on the range of u.

Instead we use the Evans Function to analyze the L^2 spectrum of the eigenvalue equation obtained by writing $u(x,t) = u_c(x-ct) + e^{\lambda t}Y(x-ct)$ in (1.3) and linearizing:

$$\partial_y L_c Y = \partial_y[\partial_y^2 + \alpha\partial_y + (c - 3u_c^2(y))]Y = \lambda Y. \tag{1.7}$$

The main result is as follows.

Theorem 1.1 (Nonexistence of unstable eigenvalues). *There exists $\epsilon > 0$ such that for $0 \le \alpha < \epsilon$, the only eigenvalue λ of the linearized perturbation equation* (1.7) *for* (1.3) *with $\operatorname{Re}\lambda \ge 0$ is $\lambda = 0$, which is a simple zero of the Evans function for* (1.7).

In [2], Howard and Zumbrun prove the orbital asymptotic stability of a class of shock profile solutions $u(x-ct)$ of equations of the form

$$u_t + f(u)_x = \alpha u_{xx} + u_{xxx} \quad (\text{where } \alpha > 0 \text{ is constant})$$

that include the shock profiles of (1.3) considered here. Their theorem assumes as a hypothesis exactly the eigenvalue information given in Theorem 1.1. So this paper provides an interesting example where the theory of Howard and Zumbrun can be applied. (Their theorem does not cover the shock profiles considered here for $\alpha = 0$. However, in [1], a different approach is taken that proves orbital asymptotic stability of these $\alpha = 0$ shock profiles in a special sense by exploiting the effects of dispersion. In essence, they are proven to be orbitally asymptotically stable with respect to a weighted norm that decreases as perturbations convect away from the shock profile in the direction indicated by the group velocity associated with linearized perturbation equation.)

2. The Evans Function and Eigenvalues

The Evans function for the eigenvalue equation (1.7) will be defined in reference to the first order system obtained from (1.7) in the standard way:

$$\mathrm{d}y/\mathrm{d}x = A(x,\lambda)y \tag{2.1}$$

where

$$y = \begin{pmatrix} Y(x) \\ \partial_x Y(x) \\ \partial_x^2 Y(x) \end{pmatrix}, \quad A(x,\lambda) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda + \partial_x\left(3u_c^2(x)\right) & 3u_c^2(x) - c & -\alpha \end{bmatrix}$$

and the associated 'transposed system' $\mathrm{d}y/\mathrm{d}x = -A^T(x,\lambda)y$ which we write as a row vector equation for $z = y^T$:

$$\mathrm{d}z/\mathrm{d}x = -zA(x,\lambda). \tag{2.2}$$

Since the following limits exist

$$A^{\pm\infty}(\lambda) = \lim_{x\to\pm\infty} A(x,\lambda) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & 3u_\pm^2 - c & -\alpha \end{bmatrix}$$

it is natural to expect eigenvalues of $A^{+\infty}(\lambda)$ having negative real parts to give rise to solutions of (2.1) which decay exponentially as $x \to +\infty$ and eigenvalues of $A^{-\infty}(\lambda)$ having positive real parts to give rise to solutions of (2.1) which decay exponentially as $x \to -\infty$. (By a proposition of [7], $Y(x) = o(\mathrm{e}^{\nu x})$ if and only if $y(x) = o(\mathrm{e}^{\nu x})$ as $x \to +\infty$, or $x \to -\infty$, for any real ν. So exponential decay of Y is equivalent to exponential decay of y.) Solutions of (2.1) which are square integrable should occur whenever the subspace of solutions which decay as $x \to +\infty$ nontrivially intersects the subspace of solutions which decay as $x \to -\infty$.

The Evans function is designed to detect just such intersections. Here we apply the extensive and systematic account of the Evans function given by Pego and Weinstein [7]. This general theory covers an $n \times n$ system of the form (2.1) on any domain $\Omega \subset \mathbb{C}$ on which the continuous function $A(x,\lambda) : \mathbb{R} \times \Omega \to \mathbb{C}^n \times \mathbb{C}^n$ is analytic in λ for each fixed x and on which $\lim_{x\to\pm\infty} A(x,\lambda) = A^{\pm\infty}(\lambda)$ is attained uniformly and sufficiently rapidly on compact subsets of Ω, provided that $A^{\pm\infty}(\lambda)$ has a unique simple eigenvalue with positive real part for each $\lambda \in \Omega$. In our case, all of these hypotheses are met on any domain $\Omega \subset \mathbb{C}$ except the last, which holds on a particular domain which we now determine.

Lemma 2.1. *For $\alpha \geq 0$ and λ in a neighborhood Ω_α^+ of the open half plane $\{\lambda \mid \operatorname{Re}\lambda > 0\}$, $A^{\pm\infty}(\lambda)$ has a unique simple eigenvalue $\mu_1^\pm(\lambda)$ with positive real part and therefore (counting multiplicities) two eigenvalues $\mu_2^\pm(\lambda)$, $\mu_3^\pm(\lambda)$ with negative real parts.*

Proof. The characteristic equation of $A^{\pm\infty}(\lambda)$ is

$$P_\pm(\mu) = \mu^3 + \alpha\mu^2 + (c - 3u_\pm^2)\mu = \lambda. \tag{2.3}$$

So $A^{\pm\infty}(\lambda)$ has an imaginary eigenvalue if and only if λ lies on the curve

$$S_e^\pm = \{-\alpha\tau^2 + \mathrm{i}(\tau(c - 3u_\pm^2) - \tau^3) \mid \tau \in \mathbb{R}\}.$$

Note that for $\alpha = 0$, $S_e^\pm$ coincide with the imaginary axis. For $\alpha > 0$, $S_e^\pm$ lie in the closed left half plane Re $\lambda \leq 0$ and S_e^+ is to the left of S_e^-. (See Figure 1.)

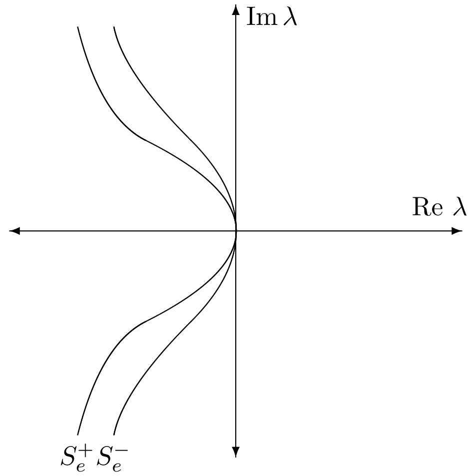

FIGURE 1. The curves $S_e^{\pm}$ for $\alpha > 0$.

Clearly the number of eigenvalues of $A^{\pm\infty}(\lambda)$ having positive real part is constant (counting multiplicities) on the component Ω_α^+ of $\mathbf{C} \setminus (S_e^+ \bigcup S_e^-)$ containing the right half plane. At $\lambda = 0$, the roots of $P_\pm(\mu) - \lambda$ are $\mu = 0$ and $\mu = \frac{1}{2}(\alpha \pm \sqrt{\alpha^2 + 4(3u_\pm^2 - c)})$, one of which is positive and one negative since $3u_\pm^2 - c > 0$. Since $P_\pm(\mu)$ is analytic in μ and λ and $\partial_\mu P_\pm(\mu) \mid_{\mu=0} = (c - 3u_\pm^2) < 0$, the implicit function theorem guarantees that μ is an analytic function of λ for small λ, and in fact real analytic for small real λ. Furthermore, differentiating (2.3) with respect to λ and evaluating at $\lambda = 0$ yields $(c - 3u_\pm^2)\mathrm{d}\mu/\mathrm{d}\lambda = 1$ so at $\lambda = 0$, $\mathrm{d}\mu/\mathrm{d}\lambda < 0$. So for small positive λ, $\mu(\lambda) < 0$, and of course the real parts of the other two roots do not change sign. Thus the conclusion of the theorem holds for small positive λ, and therefore on all of Ω_α^+. □

Having established Lemma 2.1, we can now apply the theory presented in Pego and Weinstein [7] which, for the sake of completeness, we now sketch out. To begin, we may choose right eigenvectors $v^\pm(\lambda)$ and left eigenvectors $w^\pm(\lambda)$ of $A^{\pm\infty}(\lambda)$ corresponding to $\mu_1^\pm(\lambda)$ which are analytic in λ for $\lambda \in \Omega_\alpha^+$ and normalized so that

$$(A^{\pm\infty} - \mu_1^\pm I)v^\pm = 0, \quad w^\pm(A^{\pm\infty} - \mu_1^\pm I) = 0, \quad w^+v^+ = w^-v^- = 1.$$

This can be done explicitly as follows:

$$v^+ = \begin{pmatrix} 1 \\ \mu_1^+ \\ (\mu_1^+)^2 \end{pmatrix}, \quad v^- = \begin{pmatrix} 1 \\ \mu_1^- \\ (\mu_1^-)^2 \end{pmatrix}$$

and

$$w^+ = \big(\mu_1^+(\mu_1^+ + \alpha) + (c - 3u_+^2),\ \mu_1^+ + \alpha,\ 1\big)/P_+'(\mu_1^+),$$
$$w^- = \big(\mu_1^-(\mu_1^- + \alpha) + (c - 3u_-^2),\ \mu_1^- + \alpha,\ 1\big)/P_-'(\mu_1^-)$$

where from (2.3), $P_\pm'(\mu) = 3\mu^2 + 2\alpha\mu + (c - 3u_\pm^2)$. The solutions of (2.1) associated with the eigenvalue μ_1^- and the solutions of the transposed system (2.2) associated with the eigenvalue μ_1^+ can now be described in terms of the eigenvectors v^- and w^+ as follows:

Lemma 2.2. *For each $\lambda \in \Omega_\alpha^+$ there exist unique solutions $\zeta^-(x,\lambda)$ of* (2.1) *and $\eta^+(x,\lambda)$ of* (2.2) *such that*

$$\mathrm{e}^{-\mu_1^-(\lambda)x}\zeta^-(x,\lambda) \to v^-(\lambda) \text{ as } x \to -\infty$$
$$\mathrm{e}^{\mu_1^+(\lambda)x}\eta^+(x,\lambda) \to w^+(\lambda) \text{ as } x \to +\infty.$$

The solution $\zeta^-(x,\lambda)$ spans the space of solutions of (2.1) *which $\to 0$ as $x \to -\infty$, and the solution $\eta^+(x,\lambda)$ spans the space of solutions of* (2.2) *which $\to 0$ as $x \to +\infty$. Both ζ^- and η^+ are analytic in λ.*

For the proof of the above lemma, see [7].

Lemma 2.3. *If $y(x)$ satisfies* (2.1) *and $z(x)$ satisfies* (2.2) *then $z\cdot y$ is independent of x.*

Proof. Note that $\mathrm{d}(z\cdot y)/\mathrm{d}x = (-zA)y + z(Ay) = 0$. □

Definition 2.4 (The Evans function)**.** For $\lambda \in \Omega_\alpha^+$, the Evans function $D(\lambda)$ of (2.1) is the analytic function given by

$$D(\lambda) = \eta^+(x,\lambda)\cdot\zeta^-(x,\lambda)$$

(this product being independent of x by Lemma 2.3).

Theorem 2.5. *For $\lambda \in \Omega_\alpha^+$, λ is an eigenvalue of* (1.7) *if and only if $D(\lambda) = 0$.*

Sketch of proof. It is shown in [7] that for each $\lambda \in \Omega_\alpha^+$ there are two linearly independent solutions $\zeta_2^+(x)$ and $\zeta_3^+(x)$ of (2.1), associated with the eigenvalues μ_2^+ and μ_3^+ of $A^{+\infty}$, which together span the space of solutions of (2.1) which decay (exponentially) as $x \to +\infty$. There is a square integrable solution of (2.1) if and only if $\zeta^- \in \mathrm{sp}(\zeta_2^+,\zeta_3^+)$. But $\mathrm{sp}(\zeta_2^+,\zeta_3^+) = (\eta^+)^\perp$ at each x. (To see this note that $\lim_{x\to+\infty}(\eta^+\cdot\zeta_2^+) = \lim_{x\to+\infty}(\eta^+\cdot\zeta_3^+) = 0$ and both of these quantities are independent of x by Lemma 2.3. Since $\eta^+ \neq 0$ this accounts for all of $(\eta^+)^\perp$.) Thus $\zeta^- \in \mathrm{sp}(\zeta_2^+,\zeta_3^+)$ if and only if $\eta^+\cdot\zeta^- = 0$. □

Lemma 2.6 (Alternative definition of the Evans function)**.** *Suppose that for $\lambda \in \Omega_\alpha^+$, $Y(x)$ satisfies the eigenvalue problem* (1.7) *and the asymptotic condition $Y(x) \sim \mathrm{e}^{\mu_1^-(\lambda)x}$ as $x \to -\infty$. Then $Y(x) \sim D(\lambda)\mathrm{e}^{\mu_1^+(\lambda)x}$ as $x \to +\infty$, unless $D(\lambda) = 0$, in which case $Y(x)\mathrm{e}^{-\mu_1^+(\lambda)x} \to 0$ as $x \to +\infty$.*

Sketch of proof. By the results of [7], the entire solution space of (2.1) is spanned by the solutions ζ_2^+, ζ_3^+ introduced in the proof of Theorem 2.5 and a third solution ζ_1^+, associated with the eigenvalue μ_1^+ of $A^{+\infty}$, satisfying $\lim_{x\to+\infty}\mathrm{e}^{-\mu_1^+x}\zeta_1^+(x) = v^+$. It follows from Lemma 2.2 and the fact that the first component of v^- is 1 that $Y(x)$ is the first component of ζ^-. Writing $\zeta^- = c_1\zeta_1^+ + c_2\zeta_2^+ + c_3\zeta_3^+$, it is clear that

$$\lim_{x\to+\infty}\mathrm{e}^{-\mu_1^+x}\zeta^- = c_1v^+. \tag{2.4}$$

And by Lemma 2.2

$$\lim_{x\to+\infty}\mathrm{e}^{\mu_1^+x}\eta^+ = w^+. \tag{2.5}$$

Multiplying the last two equations yields $c_1 = D(\lambda)$. Noting that the first component of v^+ is 1 and comparing the first components of (2.4) shows the validity of this alternate definition. $\square$

An Evans function can also be defined for a system of the form (2.1) where each of the limiting matrices $A^{\pm\infty}(\lambda)$ has a unique simple eigenvalue $\mu_1^{\pm}(\lambda)$ with negative real part. One way to do this is to convert such a problem to the case of unique simple eigenvalues with positive real parts by making the change of variable $x \to -x$ in the system, or in the original eigenvalue equation if the system came from one, and applying Definition 2.4 or Lemma 2.6 to the converted problem. Alternatively, analogs of Definition 2.4 and Lemma 2.6 may be formulated directly in the natural way (in fact this is how they are given in [7]) and of course yield exactly the same Evans function as Definition 2.4 or Lemma 2.6 when applied to the converted problem. We formulate the analog of Lemma 2.6 in the case of unique simple eigenvalues with negative real parts since we will need it later. In this case, if $W(x)$ satisfies the eigenvalue equation and $W(x) \sim e^{\mu_1^+(\lambda)x}$ as $x \to +\infty$ then $W(x) \sim D(\lambda)e^{\mu_1^-(\lambda)x}$ as $x \to -\infty$, unless $D(\lambda) = 0$ in which case $W(x)e^{-\mu_1^-(\lambda)x} \to 0$ as $x \to -\infty$.

The following additional properties of the Evans function will be used in the sequel.

The extended domain. By the theory of [7], the Evans function for (1.7) can be defined in essentially the same way as above for $\lambda \in \mathbf{C}$ such that the eigenvalue $\mu_1^{\pm}(\lambda)$ remains the unique simple eigenvalue of $A^{\pm\infty}(\lambda)$ having the largest real part, i.e.

$$\operatorname{Re}\mu_1^{\pm}(\lambda) > \max\{\operatorname{Re}\nu \mid \nu \neq \mu_1^{\pm} \text{ and } \nu \in \sigma(A^{\pm\infty}(\lambda))\}.$$

This extends $D(\lambda)$ analytically into a neighborhood of $\overline{\Omega_\alpha^+}$. On the extended domain, $D(\lambda) = 0$ if and only if there is a solution $Y(x)$ of (1.7) satisfying $Y(x) = O(e^{\mu_1^- x})$ as $x \to -\infty$ and $Y(x) = o(e^{\mu_1^+ x})$ as $x \to +\infty$. It turns out (see [7]) that zeros of $D(\lambda)$ to the left of S_e^+ detect solutions which may not decay as $x \to +\infty$ and eigenfunctions for λ to the left of S_e^- may not cause $D(\lambda)$ to be 0. Fortunately the situation is not quite so ambiguous when λ is on S_e^-.

Lemma 2.7. *For $\lambda \in S_e^-$, if λ is an eigenvalue of (1.7) then $D(\lambda) = 0$.*

Sketch of proof. For λ on S_e^-, the condition $Y(x) = O(e^{\mu_1^- x})$ holds for any solution $Y(x)$ of (1.7) which is $o(1)$ as $x \to -\infty$ (see [7, Proposition 1.6]) and clearly the condition $Y(x) = o(e^{\mu_1^+ x})$ holds for any solution $Y(x)$ which is $o(1)$ as $x \to +\infty$ because μ_1^+ has positive real part. So if $\lambda \in S_e^-$ is an eigenvalue of (1.7) then $D(\lambda) = 0$. $\square$

Analytic dependence on parameters. The Evans function for an eigenvalue equation depends analytically on parameters appearing analytically in the eigenvalue equation. In our case, the analytic dependence on α will play a key role in the proof of Theorem 1.1.

The relationship between $D(\lambda)$ and $D(\overline{\lambda})$. Since $A(x,\overline{\lambda}) = \overline{A(x,\lambda)}$, $\zeta^-(x,\overline{\lambda}) = \overline{\zeta^-(x,\lambda)}$ and $\eta^+(x,\overline{\lambda}) = \overline{\eta^+(x,\lambda)}$. So $D(\overline{\lambda}) = \overline{D(\lambda)}$ whenever both of these are defined. (In particular, $D(\lambda)$ is real for real λ. Also, if λ and $\overline{\lambda}$ are both in Ω^+

then λ is an eigenvalue for (1.7) if and only if $\overline{\lambda}$ is too. In this case of course, Y is an eigenfunction for λ if and only if $\overline{Y}$ is an eigenfunction for $\overline{\lambda}$.)

The derivative of $D(\lambda)$. It is shown in [7] for a system of the form (2.1) satisfying the conditions established above that whenever $D(\lambda_0) = 0$,

$$D'(\lambda_0) = \int_{-\infty}^{\infty} \eta^+(x,\lambda_0)\frac{\partial A}{\partial \lambda}(x,\lambda_0)\zeta^-(x,\lambda_0)\ \mathrm{d}x. \tag{2.6}$$

3. The Evans Function for $\alpha = 0$

Theorem 3.1. *When $\alpha = 0$, the Evans function for the eigenvalue equation* (1.7) *obtained by linearizing* (1.3) *around the travelling wave solution*

$$u_c(x - ct) = \sqrt{c}\tanh\big(\sqrt{\frac{c}{2}}(x - ct)\big) \quad (c > 0)$$

is given by

$$D(\lambda) = \frac{\mu_1(\lambda) - \sqrt{2c}}{\mu_1(\lambda) + \sqrt{2c}} \quad (\operatorname{Re}\lambda > 0)$$

where $\mu_1(\lambda)$ is the unique simple root of (2.3) having positive real part. (Note: when $\alpha = 0$, (2.3) is simply $\mu^3 - 2c\mu = \lambda$, and the μ_1^+ and μ_1^- of Lemma 2.1 are equal due to the spatial symmetry of the eigenvalue problem. The μ_1 of this theorem is this common value.)

For the proof of this theorem, we first reduce the problem to an eigenvalue problem for the KdV equation, then use a formula for the Evans function of this KdV problem derived by Pego and Weinstein.

3.1. **Step 1: Reduction to a KdV eigenvalue problem.** It will be convenient to rescale x and t by factors of $1/\sqrt{3}$ and $1/(3\sqrt{3})$ respectively in (1.3). This yields the following evolution equation, travelling wave equation, and eigenvalue equation:

$$\partial_t\tilde{u} + \tilde{u}^2\partial_x\tilde{u} - \partial_x^3\tilde{u} = 0 \tag{3.1}$$

$$\tilde{u}_c\big(x - \frac{c}{3}t\big) = \sqrt{c}\tanh\big(\sqrt{\frac{c}{6}}(x - \frac{c}{3}t)\big) \tag{3.2}$$

$$\partial_y\big(\partial_y^2 - \tilde{u}_c^2 + \frac{c}{3}\big)Y = \lambda Y. \tag{3.3}$$

Proposition 3.2. *Let $w_c(x+\frac{c}{3}t) = -\tilde{u}_c^2(x+\frac{c}{3}t) + \sqrt{6}\partial_y\tilde{u}_c(x+\frac{c}{3}t)$. Then $w(x,t) = w_c(x+\frac{c}{3}t)$ satisfies the KdV equation*

$$\partial_t w + w\partial_x w + \partial_x^3 w = 0. \quad \textit{(KdV)}$$

Proof. We first note that $v(x,t) = i\tilde{u}_c(-x - \frac{c}{3}t)$ satisfies the standard form of mKdV:

$$\partial_t v + v^2\partial_x v + \partial_x^3 v = 0.$$

We then apply the Miura transformation:

$$w(x,t) = v(x,t)^2 \pm (-6)^{\frac{1}{2}} v_x(x,t)$$

which produces a solution w of KdV for any solution v of standard mKdV via the factorization

$$w_t + ww_x + w_{xxx} = (2v \pm (-6)^{\frac{1}{2}}\partial_x)(v_t + v^2v_x + v_{xxx})$$

(see [4]). The proposition follows upon writing w in terms of $\tilde{u}_c(-x-\frac{c}{3}t)$ (using the '+' option in the Miura transformation) and noting that $\tilde{u}_c$ is an odd function. $\square$

Explicit computation shows that

$$w(x,t) = w_c\big(x+\frac{c}{3}t\big) = 2c\,\mathrm{sech}^2\big(\sqrt{\frac{c}{6}}(x+\frac{c}{3}t)\big) - c.$$

(The same computation using the '$-$' option of the Miura transformation produces the trivial solution $w(x,t) = -c$.)

Next we exploit a 'linearized Miura transformation' to transform the eigenvalue equation (3.3) to the corresponding eigenvalue equation for KdV linearized around $w_c(x+\frac{c}{3}t)$:

$$\partial_y\left(-\partial_y^2 - w_c(y) - \frac{c}{3}\right)W = \lambda W. \tag{3.4}$$

Proposition 3.3. *If $Y(y)$ solves the eigenvalue equation* (3.3) *then*

$$W_T(y) = -2\tilde{u}_c(y)Y(y) + \sqrt{6}\partial_y Y(y) \tag{3.5}$$

solves the eigenvalue equation (3.4) with λ replaced by $-\lambda$.

Proof. Writing W_T and w_c in terms of Y and $\tilde{u}_c$ in (3.4) and replacing λ with $-\lambda$ yields the following, to be verified:

$$\begin{aligned}
&-\lambda(\sqrt{6}\partial_y Y - 2\tilde{u}_c Y)\\
&= -\partial_y^3(\sqrt{6}\partial_y Y - 2\tilde{u}_c Y) - \partial_y((-\tilde{u}_c^2 + \sqrt{6}\partial_y\tilde{u}_c)(\sqrt{6}\partial_y Y - 2\tilde{u}_c Y))\\
&\quad - \frac{c}{3}\partial_y(\sqrt{6}\partial_y Y - 2\tilde{u}_c Y).
\end{aligned}$$

This follows in a tedious calculation by appropriately expanding and collecting terms and using the equations satisfied by $\tilde{u}_c$ and Y. $\square$

Proposition 3.4. *If Re $\lambda > 0$, the Evans function $D(\lambda)$ for* (3.3) *is related to the Evans function $D^*(\lambda)$ for (3.4) by*

$$D(\lambda) = D^*(\lambda)\Big(\frac{\mu_1^*(\lambda) + \sqrt{2c/3}}{\mu_1^*(\lambda) - \sqrt{2c/3}}\Big) \tag{3.6}$$

where $\mu_1^*(\lambda)$ is the unique simple root of

$$\mu^3 - \frac{2}{3}c\mu - \lambda = 0 \tag{3.7}$$

having positive real part.

Proof. As $|y| \to \infty$, the eigenvalue equation (3.3) becomes

$$\partial_y\Big(\partial_y^2 - \frac{2}{3}c\Big)Y = \lambda Y$$

whose characteristic roots satisfy (3.7) which is exactly the same polynomial as appeared in Lemma 2.1 in the case $\alpha = 0$ with c replaced by $c/3$. By that result then, if Re $\lambda > 0$ there is a unique simple root $\mu_1^*(\lambda)$ of (3.7) having positive real part. In contrast, as $|y| \to \infty$, the eigenvalue equation (3.4) becomes

$$-\partial_y\Big(\partial_y^2 - \frac{2}{3}c\Big)W = \lambda W$$

for which $-\mu_1^*$ is the unique simple root having negative real part.

By Lemma 2.6 and the subsequent discussion, the Evans function $D^*(\lambda)$ for (3.4) may be defined on the open right half plane as follows: if $W(y)$ solves (3.4) and $W(y) \sim \mathrm{e}^{-\mu_1^* y}$ as $y \to +\infty$ then $W(y) \sim D^*(\lambda)\mathrm{e}^{-\mu_1^* y}$ as $y \to -\infty$.

Now consider a solution $Y(y)$ of (3.3) such that $Y(y) \sim \mathrm{e}^{\mu_1^* y}$ as $y \to -\infty$. Then $Y(y) \sim D(\lambda)\mathrm{e}^{\mu_1^* y}$ as $y \to +\infty$. So from (3.5)

$$\begin{aligned} W_T(y) &\sim (2\sqrt{c} + \sqrt{6}\mu_1^*)\mathrm{e}^{\mu_1^* y} \quad \text{as } y \to -\infty, \\ W_T(y) &\sim (-2\sqrt{c} + \sqrt{6}\mu_1^*)D(\lambda)\mathrm{e}^{\mu_1^* y} \quad \text{as } y \to +\infty. \end{aligned}$$

And finally

$$\begin{aligned} W_T(-y)/(2\sqrt{c} + \sqrt{6}\mu_1^*) &\sim \mathrm{e}^{-\mu_1^* y} \quad \text{as } y \to +\infty, \\ W_T(-y)/(2\sqrt{c} + \sqrt{6}\mu_1^*) &\sim \Big(\frac{-2\sqrt{c} + \sqrt{6}\mu_1^*}{2\sqrt{c} + \sqrt{6}\mu_1^*}\Big)D(\lambda)\mathrm{e}^{-\mu_1^* y} \quad \text{as } y \to -\infty \end{aligned}$$

which after a little rearrangement yields the result. □

3.2. **Step 2: Calculation of $D^*(\lambda)$.** In [8] it is shown that the Evans function for the eigenvalue equation

$$\partial_y\Big(- \partial_y^2 + \hat{c} - 3\hat{c}\,\mathrm{sech}^2\big(\frac{\sqrt{\hat{c}}}{2}y\big)\Big)W = \lambda W \tag{3.8}$$

obtained by linearizing KdV around the soliton solution

$$z_{\hat{c}}(x - \hat{c}t) = 3\hat{c}\,\mathrm{sech}^2\big(\frac{\sqrt{\hat{c}}}{2}(x - \hat{c}t)\big)$$

is given by

$$\hat{D}(\lambda) = \Big(\frac{\hat{\mu}_1(\lambda) + \sqrt{\hat{c}}}{\hat{\mu}_1(\lambda) - \sqrt{\hat{c}}}\Big)^2 \quad (\mathrm{Re}\,\lambda > 0)$$

where $\hat{\mu}_1(\lambda)$ is the unique simple root of

$$-\mu^3 + \hat{c}\mu - \lambda = 0$$

having negative real part. The travelling wave solution $w_c(x + \frac{c}{3}t)$ is related to the soliton solution by

$$w_c\big(x + \frac{c}{3}t\big) = \frac{2}{3}z_{\hat{c}}\Big(\sqrt{\frac{2}{3}}(x + \frac{c}{3}t)\Big) - c.$$

To compare the eigenvalue problems we insert this expression into (3.4), yielding

$$\partial_y\Big(- \partial_y^2 + \frac{2}{3}c - 2c\,\mathrm{sech}^2\big(\sqrt{\frac{c}{6}}y\big)\Big)W = \lambda W$$

which is in fact identical to the eigenvalue problem (3.8) if we take $\hat{c} = \frac{2}{3}c$. So clearly

$$D^*(\lambda) = \Big(\frac{-\mu_1^*(\lambda) + \sqrt{2c/3}}{-\mu_1^*(\lambda) - \sqrt{2c/3}}\Big)^2.$$

Theorem 3.1 now follows by using this expression for $D^*(\lambda)$ in (3.6) and recalling that to obtain $D(\lambda)$ for the original problem, before rescaling, we need to replace c with $3c$ in the resulting formula.

4. The Evans Function for $\alpha > 0$

Here we establish some properties of $D(\lambda)$ which hold for arbitrary $\alpha > 0$. In what follows, $D_\alpha(\lambda)$ denotes the Evans function for (1.7) for a particular value of α.

Lemma 4.1. *The extended domain Ω_α of the Evans function $D_\alpha(\lambda)$ is $\mathbb{C}\backslash(-\infty, \Lambda_\alpha]$ where*

$$\Lambda_\alpha = P_+(\mu_+^*) < 0$$

and μ_+^* is the unique positive real number such that $\frac{\mathrm{d}P_+(\mu)}{\mathrm{d}\mu}\big|_{\mu_+^*} = 0$.

Proof. The Evans function $D_\alpha(\lambda)$ is defined as long as each of

$$P_\pm(\mu) = \mu^3 + \alpha\mu^2 + (c - 3u_\pm^2)\mu = \lambda \tag{4.1}$$

has a unique simple root with largest real part. Fix λ and suppose that $P_\pm(\mu)$ has two real roots with the same real parts: $\mu_1 = a + i\beta_1$ and $\mu_2 = a + i\beta_2$. Inserting these expressions into (4.1) and separating the real and imaginary parts of the resulting equation yields the following for $n = 1$ and 2:

$$\beta_n^2(-3a - \alpha) = \operatorname{Re}\lambda - a^3 - (c - 3u_\pm^2)a - \alpha a^2, \tag{4.2}$$

$$-\beta_n^3 + (3a^2 + 2\alpha a + (c - 3u_\pm^2))\beta_n = \operatorname{Im}\lambda. \tag{4.3}$$

Assuming for the moment that $a \neq -\alpha/3$, (4.2) implies that $\beta_1^2 = \beta_2^2$. So either $\beta_1 = \beta_2$, or $\beta_1 = -\beta_2$. In the latter case, (4.3) implies that $\operatorname{Im}\lambda = 0$. If on the other hand $a = -\alpha/3$ then $3a^2 + 2\alpha a + (c - 3u_\pm^2) = -\alpha^2/3 + (c - 3u_\pm^2) < 0$, so there is only one solution of (4.3) for any given value of $\operatorname{Im}\lambda$, and so $\beta_1 = \beta_2$. Therefore, in any case (4.1) can have two roots with the same real part only if λ is real or if the two roots also have the same imaginary parts, in which case (4.1) has a double root. But it is easy to check that $\mathrm{d}P_\pm/\mathrm{d}\mu = 0$ only for real values of μ. So (4.1) has a double root μ only if μ is real which implies that λ is real as well. In summary then, (4.1) can have two roots with the same real part only for real λ. We already know that $D_\alpha(\lambda)$ is defined for $\lambda > 0$, so the only region which needs further investigation is the negative real axis.

At $\lambda = 0$, the three roots of (4.1) are real and distinct. For small negative λ there are two distinct positive real roots which as λ decreases coalesce at some $\lambda = \Lambda_\pm$ into a single positive root $\mu_\pm^*$ for which $\mathrm{d}P_\pm/\mathrm{d}\mu|_{\mu_\pm^*} = 0$. There is only one such $\mu_\pm^*$ which is positive:

$$\mu_\pm^* = \frac{1}{3}\Big(-\alpha + \sqrt{\alpha^2 + 3(3u_\pm^2 - c)}\Big).$$

The values $\Lambda_\pm$ are given by $\Lambda_\pm = (\mu_\pm^*)^3 + \alpha(\mu_\pm^*)^2 + (c - 3u_\pm^2)\mu_\pm^*$. The lemma now follows by letting $\Lambda_\alpha = \max(\Lambda_+, \Lambda_-)$ and by verifying that $\Lambda_+ > \Lambda_-$ (which is most easily done by sketching the curves $P_\pm(\mu)$). □

In order to localize the changes that may take place in $D_\alpha(\lambda)$ as α increases from 0, it is necessary to have some knowledge of the behavior of $D_\alpha(\lambda)$ for large $|\lambda|$. The following two lemmas provide this.

Lemma 4.2. *Let $\widetilde{\Omega}$ be a subset of Ω_α having positive distance from $(-\infty, \Lambda_\alpha]$. The roots $\mu_1^\pm(\lambda)$, $\mu_2^\pm(\lambda)$, $\mu_3^\pm(\lambda)$ of $P_\pm(\mu)$ satisfy the following:*

(1) *Let* $|\mu_j^\pm - (\lambda^{1/3} - \alpha/3)|$ *denote the minimum over the cube roots* $\lambda^{1/3}$ *of* λ *of the distance between* $\mu_j^\pm$ *and* $\lambda^{1/3} - \alpha/3$. *Then there is a* $C > 0$ *(independent of* λ*) such that for* $\lambda \in \widetilde{\Omega}$,

$$|\mu_j^\pm - (\lambda^{1/3} - \alpha/3)| < C|\lambda|^{-1/3} \quad \text{for } \alpha \geq 0.$$

(2) $\operatorname{Re} \mu_1^\pm(\lambda) \to +\infty$ *uniformly in* $|\lambda|$ *as* $|\lambda| \to \infty$ *in* $\widetilde{\Omega}$.

The proof of the above lemma follows from [7, Lemma 1.20].

Lemma 4.3. *For any* $\beta > 0$, $D_\alpha(\lambda) \to 1$ *uniformly over* $\alpha \in [0, \beta]$ *as* $|\lambda| \to \infty$ *in any region* $\widetilde{\Omega} \subset \mathbb{C}$ *having positive distance from* $\bigcup_{\alpha \in [0,\beta]} (-\infty, \Lambda_\alpha]$.

Proof. Given Lemma 4.2, the proof is the same as that given in [6, Appendix] for the eigenvalue problems associated with the linearized perturbation equations for travelling wave solutions $u = \phi(x - ct)$ of

$$\partial_t u + u^p \partial_x u + \partial_x^3 u = \alpha \partial_x^2 u \quad p \geq 1, \alpha > 0$$

which satisfy the limiting conditions $\phi(y) \to u_- > 0$ as $y \to -\infty$ and $\phi(y) \to 0$ as $y \to +\infty$. □

Lemma 4.4. *For any* $\alpha \geq 0$, *we have* $D'_\alpha(0) > 0$.

Proof. The formula (2.6) for the first derivative at a point where the Evans function vanishes yields

$$D'_\alpha(0) = \int_{-\infty}^{\infty} Z(x) Y(x)\, \mathrm{d}x$$

where

$$\partial_x[\partial_x^2 + \alpha\partial_x + (c - 3u_c^2(x))]Y(x) = 0, \quad Y(x) \sim \mathrm{e}^{\mu_1^- x} \quad \text{as } x \to -\infty \tag{4.4}$$

and Z satisfies the adjoint of the equation for Y:

$$\begin{gathered} -[\partial_x^2 - \alpha\partial_x + (c - 3u_c^2(x))]\partial_x Z(x) = 0, \\ Z(x) \sim \frac{1}{P'_+(\mu_1^+)} \mathrm{e}^{-\mu_1^+ x} \quad \text{as } x \to +\infty. \end{gathered} \tag{4.5}$$

It happens that $\partial_x u_c(x)$ satisfies (4.4), so Y is obtained by properly normalizing this solution, which yields

$$Y(x) = \frac{1}{4} \operatorname{sech}^2(Ax), \quad A = \frac{\mu_1^-}{2}.$$

Next we observe that integrating (4.4) with a 0 boundary condition at $x = -\infty$ yields $L_c Y = [\partial_x^2 + \alpha\partial_x + (c - 3u_c^2(x))]Y = 0$, which Y solves since $Y(x) \to 0$ as $x \to -\infty$. This second order equation differs from the equation for $\partial_x Z$ only in that α is replaced by $-\alpha$. A short calculation shows that

$$(\partial_x^2 - \alpha\partial_x + (c - 3u_c^2(x))\mathrm{e}^{\alpha x} Y = \mathrm{e}^{\alpha x} L_c Y = 0.$$

Since $Y(x) \sim \mathrm{e}^{-\mu_1^- x}$ as $x \to +\infty$ and $\alpha - \mu_1^- < 0$ it follows that $\mathrm{e}^{\alpha x} Y(x) \to 0$ as $x \to +\infty$. So any antiderivative of $\mathrm{e}^{\alpha x} Y(x)$ which decays as $x \to +\infty$ is a multiple of Z. Choosing the appropriate antiderivative and normalizing yields the following:

$$Z(x) = \frac{\mu_1^+}{4P'_+(\mu_1^+)} \int_x^{+\infty} \mathrm{e}^{\alpha t} \operatorname{sech}^2\big(\frac{\mu_1^-}{2} t\big)\, \mathrm{d}t \quad (\mu_1^+ = \mu_1^- - \alpha).$$

Thus

$$D'_\alpha(0) = \frac{\mu_1^+}{16P'_+(\mu_1^+)} \int_{-\infty}^{+\infty} \Big[\int_x^{+\infty} e^{\alpha t} \operatorname{sech}^2\big(\frac{\mu_1^-}{2}t\big)\, dt \Big] \operatorname{sech}^2\big(\frac{\mu_1^-}{2}x\big)\, dx.$$

Integrating by parts to eliminate the inside integral then yields

$$D'_\alpha(0) = \frac{\mu_1^+}{8\mu_1^- P'_+(\mu_1^+)} \int_{-\infty}^{+\infty} e^{\alpha x} \Big(1 + \tanh\big(\frac{\mu_1^-}{2}x\big)\Big) \operatorname{sech}^2\big(\frac{\mu_1^-}{2}x\big)\, dx$$

which is clearly positive since $P'_+(\mu_1^+) > 0$. □

5. Proof of Theorem 1.1

It is clear that the conclusion of Theorem 1.1 holds in the case $\alpha = 0$. For Re $\lambda > 0$, this follows immediately from Theorem 3.1 and the fact that for $\lambda \in \Omega_\alpha^+$, λ is an eigenvalue of (1.7) if and only if $D(\lambda) = 0$. For Re $\lambda = 0$, recall that $D(\lambda)$ may be extended analytically into a neighborhood of the closed right half plane, where by analytic continuation the formula of Theorem 3.1 is still valid. By Lemma 2.7, on the curve S_e^-, which for $\alpha = 0$ is the imaginary axis, it is still true that if λ is an eigenvalue of (1.7) then $D(\lambda) = 0$. But from the formula of Theorem 3.1, $D(\lambda) \neq 0$ for $\lambda \neq 0$.

It remains to finish the proof of Theorem 1.1 for small positive α. To begin, we note that although the parameter α has been real and nonnegative so far, whenever $\lambda \in \Omega_{\alpha_0}$ for some real α_0, $D_\alpha(\lambda)$ may be defined for the eigenvalue equation (1.7) for α in a complex neighborhood of α_0. (The positions of simple roots of $P_\pm(\mu)$ vary analytically in α, and of any roots vary continuously in α, so if $P_\pm(\mu)$ has a unique simple root with largest real part at α_0 the same must hold for α near α_0.) Furthermore, by the general theory (see [7]), $D_\alpha(\lambda)$ is an analytic function of both α and λ. From this analyticity and from Lemmas 4.1 and 4.3, it is easy to show using standard elementary arguments that as α increases from 0 along the real axis, zeros of $D(\lambda)$ may emerge from the interval $(-\infty, \Lambda_\alpha]$, but for small α remain in the left half plane.

Finally we note that by Lemma 4.4, $\lambda = 0$ remains as a simple eigenvalue of $D_\alpha(\lambda)$ for $\alpha > 0$. This completes the proof of Theorem 1.1.

Since it is clear that $D_\alpha(\overline{\lambda}) = \overline{D_\alpha(\lambda)}$ whenever both of these are defined, it follows that a transition to unstable eigenvalues can occur as α increases from 0 only by a pair of complex conjugate eigenvalues crossing the imaginary axis. Unfortunately, the eigenvalue problem for arbitrary $\alpha > 0$ has not proven amenable to further analysis so it is unknown whether or not such a transition ever actually occurs.

Acknowledgments. The results in this paper appeared in my Ph.D. thesis, which was directed by Professor Robert L. Pego at the University of Maryland at College Park. I am very grateful to Professor Pego for bringing this problem to my attention, and for his advice, guidance, and encouragement throughout my work on it. This work was partially supported by the National Science Foundation under grant DMS-9403871.

I would like to thank the anonymous referee for several suggestions which improved the clarity and organization of this paper.

References

[1] J. Dodd, *Convective stability of shock profile solutions of a modified KdV-Burgers equation*, Ph.D. Thesis, University of Maryland at College Park, 1996.

[2] Peter Howard and Kevin Zumbrun, Pointwise estimates and stability for dispersive-diffusive shock waves, *Arch. Rational Mech. Anal.*, **155** (2000), 85–169.

[3] D. Jacobs, B. McKinney, and M. Shearer, Travelling wave solutions of the modified Korteweg - de Vries - Burgers equation, *Journal of Differential Equations*, **116** (1993), 448-467.

[4] R. M. Miura, Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation, *Journal of Mathematical Physics*, **9** (1968), 1202–1204.

[5] R. L. Pego, Remarks on the stability of shock profiles for conservation laws with dissipation, *Transactions of the AMS*, **291** (1985), 353-361.

[6] R. L. Pego, P. Smereka, and M. Weinstein, Oscillatory instability of traveling waves for a KdV-Burgers equation, *Physica-D.*, **67** (1993), 45-65.

[7] R. L. Pego and M.I. Weinstein, Eigenvalues and instabilities of solitary waves, *Phil. Trans. R. Soc. Lond.*, **340** (1992), 47-94.

[8] R. L. Pego and M.I. Weinstein, Asymptotic stability of solitary waves, *Communications in Mathematical Physics*, **164** (1994), 305-349.

[9] C. C. Wu, New theory of MHD shock waves, in "Viscous Profiles and Numerical Methods for Shock Waves," Michael Shearer, ed., SIAM, Philadelphia, 1991.

Jeff Dodd

Department of Mathematical, Computing, and Information Sciences, Jacksonville State University, Jacksonville, AL 36265, USA

E-mail address: jdodd@jsu.edu