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CONVERGENCE OF SOLUTIONS FOR A FIFTH-ORDER NONLINEAR DIFFERENTIAL EQUATION

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Abstract. In this paper, we present sufficient conditions for all solutions of a fifth-order nonlinear differential equation to converge. In this context, two solutions converge to each other if their difference and those of their derivatives up to order four approach zero as time approaches infinity. The nonlinear functions involved are not necessarily differentiable, but satisfy certain increment ratios that lie in the closed sub-interval of the Routh-Hurwitz interval.

1. Introduction

Nonlinear differential equations of higher order have been extensively studied with high degree of generality. In particular, there have been interesting works on asymptotic behaviour, boundedness, periodicity, almost periodicity and stability of solutions for fifth-order nonlinear differential equations. Authors that have worked in this direction include Abou-El-Ela and Sadek [1, 2, 3], Adesina [4, 5, 6], Afuwape and Adesina [9, 10], Chukwu [11, 12], Sadek [14] and Tunc [16, 17, 18, 19, 20], to mention a few. Most of the nonlinear functions involved in these works were assumed to be differentiable, specially, the restoring terms. Specifically, in 1975 and 1976 respectively, Chukwu [11, 12] discussed the boundedness and stability of the solutions of the differential equations

$$x^{(v)} + f_1(x, x', x'', x''', x^{(iv)})x^{(iv)} + bx''' + f_3(x'') + f_4(x') + f_5(x) = p(t) \tag{1.1}$$

and

$$x^{(v)} + ax^{(iv)} + f_2(x''') + cx'' + f_4(x') + f_5(x) = p(t, x, x', x'', x''', x^{(iv)}). \tag{1.2}$$

Later, Yu [21] studied the boundedness and asymptotic stability of the solutions of the differential equation

$$x^{(v)} + \phi(x, x', x'', x''', x^{(iv)})x^{(iv)} + bx''' + h(x'') + g(x') + f(x) = p(t, x, x', x'', x''', x^4). \tag{1.3}$$

Other interesting results on the boundedness and stability of solutions for equations of the form (1.3) were obtained by Abou-El-Ela and Sadek [1], Tiryaki and Tunc [15] and Tunc [16]. In the case where the fifth order differential equations were

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non-autonomous, the asymptotic behaviour of solutions were treated by Abou-El-Ela and Sadek [3], Sadek [14] and Tunc [17, 18, 19, 20]. Some of the results in these works have been generalized to real vector differential equations, see for instance Abou-El-Ela and Sadek [2]. All the above mentioned works were done by using the Lyapunov's second method except for the works of Adesina [4, 5, 6] and Afuwape and Adesina [9, 10], where the frequency domain technique was employed to study some qualitative behaviour of solutions.

However, the problem of convergence of solutions to these equations in which the nonlinear terms are not necessarily required to be differentiable, has so far remained intractable. The purpose of this paper therefore is to tackle this problem. Motivation for this study comes from the works of Afuwape [7, 8] and Ezeilo [13] where sufficient conditions for the convergence of solutions of fourth and third order equations were proved respectively.

Definition Two solutions $x_1(t)$, $x_2(t)$ of the equation (1.4) are said to converge (to each other) if $x_1 - x_2 \to 0$, ${x_1}' - {x_2}' \to 0$, ${x_1}'' - {x_2}'' \to 0$, ${x_1}''' - {x_2}''' \to 0$, ${x_1}^{(iv)} - {x_2}^{(iv)} \to 0$ as $t \to \infty$.

In this paper, we shall investigate the convergence of solutions for equation

$$x^{(v)} + ax^{(iv)} + bx''' + f(x'') + g(x') + h(x) = p(t, x, x', x'', x''', x^{(iv)}), \tag{1.4}$$

where a, b are positive constants, functions f, g, h and p are real valued and continuous in their respective arguments such that the uniqueness theorem is valid, and the solutions are continuously dependent on the initial conditions. Moreover, $f(0) = g(0) = h(0) = 0$. Our results assert the existence of convergence of solutions with the functions f, g, and h not necessarily differentiable. Here, the functions h and g are only required to satisfy the increment ratios

$$\frac{h(\zeta + \eta) - h(\zeta)}{\eta} \in I_0,$$
$$\frac{g(\zeta + \eta) - g(\zeta)}{\eta} \in I_1,$$

where I_0 and I_1 are closed sub-intervals of the Routh-Hurwitz interval. Our results generalize, to fifth-order equations, the results in [7, 8]. Some existing results on fifth-order nonlinear differential equations are also generalized.

2. Assumptions and Main Results

Assumptions:

(1) The function $p(t, x, x', x'', x''', x^{(iv)})$ is equal to $q(t) + r(t, x, x', x'', x''', x^{(iv)})$ with $r(t, 0, 0, 0, 0, 0) = 0$ for all t;

(2) For some positive constants a, b, α, β and Δ_0, $(ab - \alpha)\alpha - a^2\beta > 0$, $(ab - \alpha)\alpha + a\Delta_0 > 0$, $(ab - \alpha) > 0$ and $b^2 > \beta$;

(3) For some positive constants a, b, α, β, Δ_0, Δ_1 K_0 and K_1, the intervals

$$I_0 \equiv \Big[\Delta_0, K_0\big[\frac{[(ab - \alpha)\alpha - a^2\beta]}{a}\big]\Big],$$
$$I_1 \equiv \Big[\Delta_1, K_1\big[\frac{[(ab - \alpha)\alpha + a\Delta_0]}{a^2}\big]\Big]$$

are in the Routh-Hurwitz interval.

The following results are proved.

Theorem 2.1. *In addition to the basic assumptions and 1–3 above, we assume that*

(i) *there are positive constants α, α_0, β and β_0 such that*

$$\alpha \le \frac{f(z_2) - f(z_1)}{z_2 - z_1} \le \alpha_0, \quad z_2 \ne z_1, \tag{2.1}$$

$$\beta \le \frac{g(y_2) - g(y_1)}{y_2 - y_1} \le \beta_0, \quad y_2 \ne y_1; \tag{2.2}$$

(ii) *for any ζ, η, $(\eta \ne 0)$, the increment ratios for h and g satisfy*

$$\frac{h(\zeta + \eta) - h(\zeta)}{\eta} \in I_0,$$

$$\frac{g(\zeta + \eta) - (\zeta)}{\eta} \in I_1;$$

(iii) *there is a continuous function $\phi(t)$ such that*

$$\begin{aligned} &|r(t, x_1, y_1, z_1, u_1, v_1) - r(t, x_2, y_2, z_2, u_2, v_2)| \\ &\le \phi(t)(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2| + |u_1 - u_2| + |v_1 - v_2|) \end{aligned} \tag{2.3}$$

holds for arbitrary $t, x_1, y_1, z_1, u_1, v_1, x_2, y_2, z_2, u_2, v_2$.

Then if there exists a constant D_1 such that

$$\int_0^t \phi^{\varrho}(\tau) d\tau \le D_1 \tag{2.4}$$

for some ϱ with $1 \le \varrho \le 2$, then all solutions of (1.4) *converge.*

Theorem 2.2. *Assume the conditions in the Theorem 2.1 are satisfied. Let $x_1(t)$, $x_2(t)$ be any two solutions of* (1.4)*. Then for each fixed ϱ, $1 \le \varrho \le 2$, there are constants D_2, D_3 and D_4 such that for $t_2 \ge t_1$,*

$$S(t_2) \le D_2 S(t_1) \exp\Big\{ - D_3(t_2 - t_1) + D_4 \int_{t_1}^{t_2} \phi^{\varrho}(\tau) d\tau \Big\}, \tag{2.5}$$

where

$$\begin{aligned} S(t) = (x_2(t) - x_1(t))^2 + ({x_2}'(t) - {x_1}'(t))^2 + ({x_2}''(t) - {x_1}''(t))^2 \\ + ({x_2}'''(t) - {x_1}'''(t))^2 + ({x_2}^{(iv)}(t) - {x_1}^{(iv)}(t))^2. \end{aligned} \tag{2.6}$$

Remark 2.3. If $p = 0$ and the hypotheses (i) and (ii) of the Theorem 2.1 hold for arbitrary $\eta \ne 0$, then the trivial solution of (1.4) is exponentially stable.

Remark 2.4. If $p = 0$ and the hypotheses (i) and (ii) of the Theorem 2.1 hold for arbitrary $\eta \ne 0$, and $\zeta = 0$, then there exists a constant $D_5 > 0$ such that every solution $x(t)$ of (1.4) satisfies

$$|x(t)| \le D_5; \quad |x'(t)| \le D_5; \quad |x''(t)| \le D_5; \quad |x'''(t)| \le D_5; \quad |x^{(iv)}(t)| \le D_5.$$

For the rest of this article, $D_1, D_2, D_3, \dots$ and the D^*'s stand for positive constants. Their identities are preserved throughout this paper.

3. Proof of Main Results

Proof of Theorem 2.2. It is convenient here to consider (1.4) as the equivalent system

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= u, \\ u' &= v + Q(t), \\ v' &= -av - bu - f(z) - g(y) - h(x) + r(t, x, x', x'', x''', x^{(iv)}) - aQ(t), \end{aligned} \tag{3.1}$$

where $Q(t) = \int_0^t q(\tau)d\tau$. Let $x_i(t), y_i(t), z_i(t), u_i(t), v_i(t)$, $(i = 1, 2)$, be two solutions of (1.4), such that inequalities (2.1), (2.2),

$$\Delta_0 \le \frac{h(x_2) - h(x_1)}{x_2 - x_1} \le K_0\big[\frac{[(ab - \alpha)\alpha - a^2\beta]}{a}\big],$$
$$\Delta_1 \le \frac{g(y_2) - g(y_1)}{y_2 - y_1} \le K_1\big[\frac{[(ab - \alpha)\alpha + a\Delta_0]}{a^2}\big]$$

are satisfied. The main tool in the proofs of the convergence theorems will be the function

$$\begin{aligned} 2V =& \beta^2[1 - \epsilon]x^2 + [\alpha^2 + \frac{b\beta(\alpha + \alpha(1 - \epsilon))}{1 - \epsilon} + \frac{a\beta^2\epsilon}{\alpha(1 - \epsilon)}]y^2 \\ &+ [b^2 + \frac{(b^2 - \beta)\epsilon}{1 - \epsilon} + \frac{\epsilon^2\beta}{1 - \epsilon} + \epsilon\beta\frac{(ab - \alpha)}{\alpha(1 - \epsilon)}]z^2 + a^2u^2 + [1 + \frac{\epsilon}{1 - \epsilon}]v^2 \\ &+ 2\alpha\beta(1 - \epsilon)xy + 2b\beta(1 - \epsilon)xz + 2a\beta(1 - \epsilon)xu + 2\beta(1 - \epsilon)xv \\ &+ 2(b\alpha + a\beta\epsilon)yz + 2[a\alpha + \beta\epsilon + \frac{\beta\epsilon}{1 - \epsilon}]yu + 2\alpha yv + 2ab(1 - \epsilon)zu \\ &+ 2[b + \frac{(a\alpha + b)\epsilon}{1 - \epsilon}]zv + 2auv, \end{aligned} \tag{3.2}$$

where $0 < \epsilon < 1$, $ab - \alpha > 0$ and $b^2 > \beta$. Indeed we can rearrange the terms in (3.2) to obtain

$$2V = 2V_1 + 2V_2 + 2V_3 + 2V_4 + 2V_5 + 2V_6, \tag{3.3}$$

where

$$2V_1 = [\beta(1 - \epsilon)x + \alpha y + bz + \frac{au}{2} + v]^2 + \frac{\epsilon^2}{1 - \epsilon}z^2 + 2\frac{(a\beta + b)}{1 - \epsilon}zv + \frac{\epsilon}{2(1 - \epsilon)}v^2;$$
$$2V_2 = \beta^2(1 - \epsilon)\epsilon x^2 + a\beta(1 - \epsilon)xu + \frac{1}{8}a^2u^2;$$
$$2V_3 = b\beta\frac{(\epsilon + \epsilon(1 - \epsilon))}{1 - \epsilon}y^2 + 2[\frac{a\alpha}{2} + \frac{\beta\epsilon}{1 - \epsilon} + \beta\epsilon]yu + \frac{1}{8}a^2u^2;$$
$$2V_4 = \frac{a\beta^2\epsilon}{\alpha(1 - \epsilon)}y^2 + 2a\beta\epsilon yz + \epsilon\beta\frac{(ab - \alpha)}{\alpha(1 - \epsilon)}z^2;$$
$$2V_5 = \frac{(b^2 - \beta)\epsilon}{2(1 - \epsilon)}z^2 + \frac{(ab - 2ab\epsilon)}{2}zu + \frac{1}{8}a^2u^2;$$
$$2V_6 = \frac{a^2u^2}{4} + auv + \frac{\epsilon}{2(1 - \epsilon)}v^2.$$

We note that V_1 is obviously positive definite. This follows from the condition $0 < \epsilon < 1$. Also $V_i, i = 2, 3, \dots, 6$ regarded as quadratic forms in x and u, y and u, y and z, z and u, z and v, u and v respectively is always positive. Let us recall that a real 2×2 matrix

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

is positive definite if and only if it is symmetric, and the elements a_1, a_4 and $a_1a_4 - a_2a_3$ are non negative. Thus we can rearrange the terms in V_2 as

$$\begin{pmatrix} x, & u \end{pmatrix} \begin{pmatrix} \beta^2(1-\epsilon) & a\beta\frac{(1-\epsilon)}{2} \\ a\beta\frac{(1-\epsilon)}{2} & \frac{a^2}{8} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix},$$

from which we have $\frac{2}{3} < \epsilon < 1$ as a condition for the positive semi-definiteness. Similarly, for V_3, we have

$$a^2b\beta\epsilon\frac{(2-\epsilon)}{1-\epsilon} \geq \big[\frac{a\alpha}{2} + \frac{\beta\epsilon}{1-\epsilon} + \beta\epsilon\big]^2$$

as a condition for its positive semi-definiteness. As for V_4 and V_5, we have

$$(ab-\alpha)\beta \geq a\alpha^2(1-\epsilon)^2 \quad \text{and} \quad a^2(b^2-\beta) \geq (ab-2ab\epsilon)^2\frac{(1-\epsilon)}{\epsilon}$$

as conditions for the positive semi-definiteness. The condition for positive semi-definiteness of V_6 is the same as that for V_1. Hence V is positive definite. We can therefore find a constant $D_6 > 0$, such that

$$D_6(x^2 + y^2 + z^2 + u^2 + v^2) \leq V. \tag{3.4}$$

Furthermore, by using the Schwartz inequality $|x||u| \leq \frac{1}{2}(x^2 + u^2)$, then $2|V_2| \leq {D_1}^*(x^2 + u^2)$ for some ${D_1}^* = {D_1}^*(a, \beta, \epsilon) > 0$. Similarly, we obtain the following estimates:

$$\begin{gathered} 2|V_3| \leq {D_2}^*(y^2 + u^2), \quad {D_2}^* = {D_2}^*(a, b, \alpha, \beta, \epsilon) > 0, \\ 2|V_4| \leq {D_3}^*(y^2 + z^2), \quad {D_3}^* = {D_3}^*(a, b, \alpha, \beta, \epsilon) > 0, \\ 2|V_5| \leq {D_4}^*(z^2 + u^2), \quad {D_4}^* = {D_4}^*(a, b, \alpha, \beta, \epsilon) > 0, \\ 2|V_6| \leq {D_5}^*(u^2 + v^2), \quad {D_5}^* = {D_5}^*(a, \epsilon) > 0. \end{gathered}$$

Thus there exists a constant $D_7 > 0$, such that

$$V \leq D_7(x^2 + y^2 + z^2 + u^2 + v^2), \tag{3.5}$$

where

$$D_7 = \max\big\{{D_1}^*; {D_2}^*; {D_3}^*; {D_4}^*; {D_5}^*\big\}.$$

Using inequalities (3.4) and (3.5), we obtain

$$D_6(x^2 + y^2 + z^2 + u^2 + v^2) \leq V \leq D_7(x^2 + y^2 + z^2 + u^2 + v^2). \tag{3.6}$$

□

The following result can be easily verified for $W \equiv V$.

Lemma 3.1. *Let the function* $W(t) = W(x_2 - x_1, y_2 - y_1, z_2 - z_1, u_2 - u_1, v_2 - v_1)$ *be defined by*

$$2W = \beta^2[1-\epsilon](x_2 - x_1)^2 + [\alpha^2 + \frac{b\beta(\alpha + \alpha(1-\epsilon))}{1-\epsilon} + \frac{a\beta^2\epsilon}{\alpha(1-\epsilon)}](y_2 - y_1)^2$$

$$
\begin{aligned}
&+ \big[b^2 + \frac{(b^2-\beta)\epsilon}{1-\epsilon} + \frac{\epsilon^2\beta}{1-\epsilon} + \epsilon\beta\frac{(ab-\alpha)}{\alpha(1-\epsilon)}\big](z_2-z_1)^2 + a^2(u_2-u_1)^2 \\
&+ [1+\frac{\epsilon}{1-\epsilon}](v_2-v_1)^2 + 2\alpha\beta(1-\epsilon)(x_2-x_1)(y_2-y_1) \\
&+ 2b\beta(1-\epsilon)(x_2-x_1)(z_2-z_1) + 2a\beta(1-\epsilon)(x_2-x_1)(u_2-u_1) \\
&+ 2\beta(1-\epsilon)(x_2-x_1)(v_2-v_1) + 2(b\alpha + a\beta\epsilon)(y_2-y_1)(z_2-z_1) \\
&+ 2[a\alpha + \beta\epsilon + \frac{\beta\epsilon}{1-\epsilon}](y_2-y_1)(u_2-u_1) + 2\alpha(y_2-y_1)(v_2-v_1) \\
&+ 2ab(1-\epsilon)(z_2-z_1)(u_2-u_1) + 2[b + \frac{(a\alpha+b)\epsilon}{1-\epsilon}](z_2-z_1)(v_2-v_1) \\
&+ 2a(u_2-u_1)(v_2-v_1),
\end{aligned}
$$

where $0<\epsilon<1$ *and* $W(0,0,0,0,0)=0$*, then there exist finite constants* $D_6>0$*,* $D_7>0$ *such that*

$$
\begin{aligned}
&D_6\big\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(u_2-u_1)^2+(v_2-v_1)^2\big\} \\
&\le W \\
&\le D_7\big\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(u_2-u_1)^2+(v_2-v_1)^2\big\}.
\end{aligned} \tag{3.7}
$$

Proof. These inequalities follows from the verification of W as a Lyapunov function and the fact that the solutions $(x_i, y_i, z_i, u_i, v_i + Q(t))$, $(i = 1, 2)$, satisfy the system (3.1). Then $S(t)$ as defined in (2.6) becomes

$$
\begin{aligned}
S(t) = \big\{&(x_2(t)-x_1(t))^2 + (y_2(t)-y_1(t))^2 + (z_2(t)-z_1(t))^2 \\
&+ (u_2(t)-u_1(t))^2 + (v_2(t)-v_1(t))^2\big\}.
\end{aligned}
$$

□

Next we prove a result on the derivative of $W(t)$ with respect to t.

Lemma 3.2. *Let the hypotheses (i) and (ii) of the Theorem 2.1 hold, then there exist positive constants* D_8 *and* D_9 *such that*

$$
\frac{dW}{dt} \le -2D_8S + D_9S^{1/2}|\theta| \tag{3.8}
$$

where $\theta = r(t, x_2, y_2, z_2, u_2, v_2 + Q) - r(t, x_1, y_1, z_1, u_1, v_1 + Q)$.

Proof. Using the system (3.1), a direct computation of $\frac{dW}{dt}$ gives after simplification

$$
\frac{dW}{dt} = -W_1 + W_2, \tag{3.9}
$$

where

$$
\begin{aligned}
W_1 &= \beta(1-\epsilon)H(x_2,x_1)(x_2-x_1)^2 + \alpha\beta\epsilon(y_2-y_1)^2 \\
&\quad + \frac{1}{\alpha(1-\epsilon)}(b\alpha + a\beta\epsilon)(\alpha(1-\epsilon)-1)(z_2-z_1)^2 \\
&\quad + ab\epsilon(u_2-u_1)^2 + \frac{1}{1-\epsilon}a\epsilon(v_2-v_1)^2 + (G(y_2,y_1)-\beta)[\beta(1-\epsilon)(x_2-x_1) \\
&\quad + \alpha(y_2-y_1) - \frac{b\alpha + a\beta\epsilon}{\alpha(1-\epsilon)}(z_2-z_1) + a(u_2-u_1) + \frac{(v_2-v_1)}{1-\epsilon}](y_2-y_1) \\
&\quad + \big(F(z_2,z_1)-\alpha\big)\Big[\beta(1-\epsilon)(x_2-x_1) + \alpha(y_2-y_1) + a(u_2-u_1)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{(v_2-v_1)}{1-\epsilon}\Big](z_2-z_1)\\
&+ 2(\beta\epsilon+ab\alpha)(y_2-y_1)(u_2-u_1) - \frac{\beta\epsilon(1+(1-\epsilon))}{1-\epsilon}(z_2-z_1)(u_2-u_1)\\
&+ \frac{1}{1-\epsilon}[\alpha - \frac{1}{\alpha}(ab\alpha(1-(1-\epsilon)) - a^2\beta\epsilon)](z_2-z_1)(v_2-v_1);
\end{aligned}
$$

$$
\begin{aligned}
W_2 = \theta(t)[\beta(1-\epsilon)(x_2-x_1) + \alpha(y_2-y_1) - \frac{a\beta\epsilon+b\alpha}{\alpha(1-\epsilon)}(z_2-z_1)\\
+ a(u_2-u_1) + \frac{(v_2-v_1)}{1-\epsilon}];
\end{aligned} \tag{3.10}
$$

$$
\begin{gathered}
F(z_2,z_1) = \frac{f(z_2)-f(z_1)}{z_2-z_1}, \quad z_2 \neq z_1;\\
G(y_2,y_1) = \frac{g(y_2)-g(y_1)}{y_2-y_1}, \quad y_2 \neq y_1;\\
H(x_2,x_1) = \frac{h(x_2)-f(x_1)}{x_2-x_1}, \quad x_2 \neq x_1.
\end{gathered}
$$

Let $\chi_1 = G(y_2,y_1) - \beta$ and $\chi_2 = F(z_2,z_1) - \alpha$. Furthermore let $H(x_2,x_1)$ be denoted simply by H, and define

$$
\sum_{i=1}^{3}\lambda_i = 1; \quad \sum_{i=1}^{7}\mu_i = 1; \quad \sum_{i=1}^{6}\nu_i = 1; \quad \sum_{i=1}^{4}\tau_i = 1; \quad \sum_{i=1}^{3}\gamma_i = 1,
$$

where $\lambda_i > 0$, $\mu_i > 0$, $\nu_i > 0$, $\tau_i > 0$ and $\gamma_i > 0$. Then W_1 can be re-arranged as

$$
W_1 = W_{11} + W_{12} + W_{13}W_{14} + W_{15} + W_{16} + W_{17} + W_{18} + W_{19} + W_{21} + W_{22}, \tag{3.11}
$$

where

$$
\begin{aligned}
W_{11} &= \lambda_1\beta(1-\epsilon)H(x_2-x_1)^2 + \alpha(\mu_1\beta\epsilon + \chi_1)(y_2-y_1)^2\\
&\quad + \frac{(b\alpha+a\beta\epsilon)(\alpha(1-\epsilon)-1)}{\alpha(1-\epsilon)}(z_2-z_1)^2 + \tau_1 ab\epsilon(u_2-u_1)^2\\
&\quad + \gamma_1\frac{a\epsilon}{1-\epsilon}(v_2-v_1)^2;\\
W_{12} &= \lambda_2\beta(1-\epsilon)H(x_2-x_1)^2 + \chi_1\beta(1-\epsilon)(x_2-x_1)(y_2-y_1)\\
&\quad + \mu_2\alpha\beta\epsilon(y_2-y_1)^2;\\
W_{13} &= \lambda_3\beta(1-\epsilon)H(x_2-x_1)^2 + \chi_2\beta(1-\epsilon)(x_2-x_1)(z_2-z_1)\\
&\quad + \nu_2\frac{(b\alpha+a\beta\epsilon)(\alpha(1-\epsilon)-1)}{\alpha(1-\epsilon)}(z_2-z_1)^2;\\
W_{14} &= \mu_3\alpha\beta\epsilon(y_2-y_1)^2 + \frac{1}{\alpha(1-\epsilon)}\chi_1(b\alpha+a\beta\epsilon)(y_2-y_1)(z_2-z_1)\\
&\quad + \nu_3\frac{(b\alpha+a\beta\epsilon)(\alpha(1-\epsilon)-1)}{\alpha(1-\epsilon)}(z_2-z_1)^2;\\
W_{15} &= \mu_4\alpha\beta\epsilon(y_2-y_1)^2 + a\chi_1(y_2-y_1)(u_2-u_1) + \tau_2 ab\epsilon(u_2-u_1)^2;\\
W_{16} &= \mu_5\alpha\beta\epsilon(y_2-y_1)^2 + \frac{1}{1-\epsilon}\chi_1(y_2-y_1)(v_2-v_1) + \frac{1}{1-\epsilon}\gamma_2 a\epsilon(v_2-v_1)^2;
\end{aligned}
$$

$$\begin{aligned} W_{17} &= \nu_4 \frac{(b\alpha + a\beta\epsilon)(\alpha(1-\epsilon)-1)}{\alpha(1-\epsilon)}(z_2-z_1)^2 + \chi_2\alpha(z_2-z_1)(y_2-y_1) \\ &\quad + \mu_6\alpha\beta\epsilon(y_2-y_1)^2; \\ W_{18} &= \nu_5 \frac{(b\alpha + a\beta\epsilon)(\alpha(1-\epsilon)-1)}{\alpha(1-\epsilon)}(z_2-z_1)^2 + a\chi_1(z_2-z_1)(u_2-u_1) \\ &\quad + \tau_3 ab\epsilon(u_2-u_1)^2; \\ W_{19} &= \nu_6 \frac{(b\alpha + a\beta\epsilon)(\alpha(1-\epsilon)-1)}{\alpha(1-\epsilon)}(z_2-z_1)^2 \\ &\quad + \frac{(1+\alpha) - (ab\alpha(1-(1-\epsilon)^2) - a^2\beta\epsilon)}{\alpha(1-\epsilon)}(z_2-z_1)(v_2-v_1) \\ &\quad + \frac{1}{1-\epsilon}\gamma_3 a\epsilon(v_2-v_1)^2; \\ W_{21} &= \tau_4 ab\epsilon(u_2-u_1)^2 + 2(\beta\epsilon + ab\alpha)(u_2-u_1)(y_2-y_1) + \mu_7\alpha\beta\epsilon(y_2-y_1)^2; \\ W_{22} &= \nu_6 \frac{(b\alpha + a\beta\epsilon)(\alpha(1-\epsilon)-1)}{\alpha(1-\epsilon)}(z_2-z_1)^2 - 2\beta\epsilon\frac{(1+(1-\epsilon))}{1-\epsilon}(z_2-z_1)(u_2-u_1) \\ &\quad + \tau_4 ab\epsilon(u_2-u_1)^2. \end{aligned}$$

Since each W_{1i}, $(i = 1, 2, \dots, 9)$, W_{21} and W_{22} are quadratic in their respective variables, then by using the fact that any quadratic of the form $Ap^2 + Bpq + Cq^2$ is non negative if $4AC - B^2 \geq 0$, it follows that

$$\begin{gathered} W_{12} \geq 0 \quad \text{if } H \leq \frac{4}{\Delta_1(1-\epsilon)}\lambda_2\mu_2\alpha\epsilon; \\ W_{13} \geq 0 \quad \text{if } {\chi_2}^2 \leq \frac{4}{\alpha\beta(1-\epsilon)^2}\lambda_3\Delta_0\nu_2(b\alpha + a\beta\epsilon)(\alpha(1-\epsilon)-1); \\ W_{14} \geq 0 \quad \text{if } {\chi_1}^2 \leq 4\mu_3 b\epsilon\nu_3\alpha^2(1-\epsilon)(\alpha(1-\epsilon)-1); \\ W_{15} \geq 0 \quad \text{if } {\chi_1}^2 \leq 4\mu_4\alpha b^2\epsilon^2\tau_2; \\ W_{16} \geq 0 \quad \text{if } {\chi_1}^2 \leq 4\mu_5 a\alpha\beta\epsilon^2\gamma_2; \\ W_{17} \geq 0 \quad \text{if } {\chi_2}^2 \leq \frac{4}{\alpha^2}\nu_4(b\alpha + a\beta\epsilon)(\alpha(1-\epsilon)-1)\mu_6\beta\epsilon; \\ W_{18} \geq 0 \quad \text{if } {\chi_1}^2 \leq \frac{4}{a\alpha(1-\epsilon)}\nu_5(b\alpha + a\beta\epsilon)(\alpha(1-\epsilon)-1)b\tau_3\epsilon. \end{gathered}$$

Thus $W_1 \geq W_{11}$ provided that the above inequalities are satisfied in addition to

$$\begin{aligned} 0 \leq {\chi_1}^2 \leq & 4\min\big\{\mu_3 b\epsilon\nu_3\alpha^2(1-\epsilon)(\alpha(1-\epsilon)-1); \mu_4\alpha b^2\epsilon^2\tau_2; \\ & \mu_5 a\alpha\beta\epsilon^2\gamma_2; \frac{\nu_5}{a\alpha(1-\epsilon)}(b\alpha + a\beta\epsilon)(\alpha(1-\epsilon)-1)b\tau_3\epsilon\big\}; \end{aligned} \tag{3.12}$$

$$0 \leq {\chi_2}^2 \leq \frac{4}{\alpha}(b\alpha + a\beta\epsilon)(\alpha(1-\epsilon)-1)\min\big\{\frac{\Delta_0\nu_2\lambda_3}{\beta(1-\epsilon)^2}; \frac{\nu_4\mu_6\beta\epsilon}{\alpha}\big\}; \tag{3.13}$$

with H lying in

$$I_0 \equiv [\Delta_0, K_0[\frac{[(ab-\alpha)\alpha - a^2\beta]}{a}]] \tag{3.14}$$

and G lying in

$$I_1 \equiv [\Delta_1, K_1[\frac{[(ab-\alpha)\alpha + a\Delta_0]}{a^2}]], \tag{3.15}$$

where I_0 and I_1 are sub-interval of the Routh-Hurwitz intervals $(0, \frac{[(ab-\alpha)\alpha-a^2\beta]}{a})$ and $(0, \frac{[(ab-\alpha)\alpha+a\Delta_0]}{a^2})$ respectively, and

$$K_0 = \frac{a}{(ab-\alpha)\alpha - a^2\beta} \times \frac{4}{\Delta_1(1-\epsilon)}\lambda_2\mu_2\alpha\epsilon;$$

$$K_1 = 4\frac{e}{(ab-\alpha)\alpha - a^2\beta} \times \min\big\{\mu_3 b\epsilon\nu_3\alpha^2(1-\epsilon)(\alpha(1-\epsilon)-1); \mu_4\alpha b^2\epsilon^2\tau_2; \mu_5 a\alpha\beta\epsilon^2\gamma_2; \frac{\nu_5}{a\alpha(1-\epsilon)}(b\alpha + a\beta\epsilon)(\alpha(1-\epsilon)-1)b\tau_3\epsilon\big\}.$$

On choosing

$$2D_8 = \min\big\{\beta(1-\epsilon); \alpha; \frac{-(a\beta\epsilon + b\alpha)}{\alpha(1-\epsilon)}; a; \frac{1}{1-\epsilon}\big\},$$

we have

$$W_1 \geq W_{11} \geq 2D_8 S, \tag{3.16}$$

and if

$$D_9 = \max\big\{\beta(1-\epsilon); \alpha; \frac{-(a\beta\epsilon + b\alpha)}{\alpha(1-\epsilon)}; a; \frac{1}{1-\epsilon}\big\},$$

then

$$W_2 \leq D_9 S^{1/2}|\theta|. \tag{3.17}$$

Combining (3.16) and (3.17) in (3.9), inequality (3.8) is obtained. At last the conclusion to the proof of the Theorem 2.1 will now be given. For this purpose, let ϱ be any constant in the range $1 \leq \varrho \leq 2$ and set $\sigma = 1 - \frac{\varrho}{2}$ so that $0 \leq \sigma \leq \frac{1}{2}$. Then, on rearranging inequality (3.8) we have

$$\frac{dW}{dt} + D_8 S \leq D_9 S^{1/2}|\theta| - D_8 S, \tag{3.18}$$

from which

$$\frac{dW}{dt} + D_8 S \equiv D_{10} S^{\sigma} W^*,$$

where

$$W^* = S^{(\frac{1}{2}-\sigma)}[|\theta| - D_{11}S^{1/2}], \tag{3.19}$$

with $D_{11} = \frac{D_8}{D_{10}}$. If $|\theta| < D_{11}S^{1/2}$, then $W^* < 0$. On the other hand, if $|\theta| \geq D_{11}S^{1/2}$, then the definition of W^* in the equation (3.19) gives at least

$$W^* \leq S^{(\frac{1}{2}-\sigma)}|\theta|$$

and also $S^{1/2} \leq \frac{|\theta|}{D_{11}}$. Thus

$$S^{\frac{1}{2}(1-2\sigma)} \leq [\frac{|\theta|}{D_{11}}]^{(1-2\sigma)},$$

and from this together with W^* follows

$$W^* \leq D_{12}|\theta|^{2(1-\sigma)},$$

where $D_{12} = D_{11}{}^{(\sigma-1)}$. On using the estimate on W^* in inequality (3.18), we obtain

$$\frac{dW}{dt} + D_8 S \leq D_{10}D_{12}S^{\sigma}|\theta|^{2(1-\sigma)} \leq D_{13}S^{\sigma}\phi^{2(1-\sigma)}S^{(1-\sigma)} \tag{3.20}$$

which follows from

$$\begin{aligned}&|r(t,x_1,y_1,z_1,u_1,v_1) - r(t,x_2,y_2,z_2,u_2,v_2)|\\ &\leq \phi(t)(|x_1-x_2| + |y_1-y_2| + |z_1-z_2| + |u_1-u_2| + |v_1-v_2|).\end{aligned}$$

In view of the fact that $\varrho = 2(1-\sigma)$, we obtain

$$\frac{dW}{dt} \leq -D_8 S + D_{13}\phi^{\varrho} S,$$

and on using inequalities (3.7), we have

$$\frac{dW}{dt} + [D_{14} - D_{15}\phi^{\varrho}(t)]W \leq 0, \tag{3.21}$$

for some constants D_{14} and D_{15}. On integrating the estimate (3.21) from t_1 to t_2 ($t_1 \leq t_2$), we obtain

$$W(t_2) \leq W(t_1) \exp\big\{ -D_{14}(t_2 - t_1) + D_{15} \int_{t_1}^{t_2} \phi^{\varrho}(\tau) d\tau \big\}. \tag{3.22}$$

On using Lemma 3.1, we obtain inequality (2.5), with $D_2 = \frac{D_7}{D_6}$; $D_3 = D_{14}$ and $D_4 = D_{15}$. This completes the proof of the Theorem 2.1. □

Proof of Theorem 2.1. The proof follows from the inequality (2.5) and the condition (2.4) on $\phi(t)$. On choosing $D_2 = \frac{D_3}{D_4}$ in inequality (2.5), then as $t = t_2 - t_1 \to \infty$, $S(t) \to 0$ which proves that $x_2 - x_1 \to 0$, $y_2 - y_1 \to 0$, $z_2 - z_1 \to 0$, $u_2 - u_1 \to 0$, $v_2 - v_1 \to 0$ as $t \to \infty$. □

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