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REGULARIZATION FOR EVOLUTION EQUATIONS IN HILBERT SPACES INVOLVING MONOTONE OPERATORS VIA THE SEMI-FLOWS METHOD

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Abstract. In a Hilbert space H consider the equation

$$\frac{d}{dt}x(t) + T(t)x(t) + \alpha(t)x(t) = f(t), \quad t \geq 0,$$

where the family of operators $T(t)$, $t \geq 0$ converges in a certain sense to a monotone operator S, the function α vanishes at infinity and the function f converges to a point h. In this paper we provide sufficient conditions that guarantee the fact that full limiting functions of any solution of the equation are points of the orthogonality set $\mathcal{O}(h;S)$ of S at h, namely the set of all $x \in H$ such that $\langle Sx - h, x - z\rangle = 0$, for all $z \in S^{-1}(h)$. If the set $\mathcal{O}(h;S)$ is a singleton, then the original solution converges to a solution of the algebraic equation $Sz = h$. Our problem is faced by using the semi-flow theory and it extends to various directions the works [1, 12].

1. Introduction

Let H be a (real) Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let $S : H \to H$ be a given operator. The operator S is said to be *monotone*, if it satisfies

$$\langle Sx_1 - Sx_2, x_1 - x_2\rangle \geq 0, \quad \text{for} \quad \text{all} \quad x_1, x_2 \in H.$$

The operator S is called *strictly monotone*, if the above inequality is strict for all $x_1 \neq x_2$.

Definition 1.1. Let $S : H \to H$ be a monotone operator and let $h \in \mathcal{R}(S)$, the range of S. We define the orthogonality of S at h to be the set

$$\mathcal{O}(h;S) := \{x \in H : \langle Sx - h, x - z\rangle = 0,\ z \in S^{-1}(h)\}.$$

It is clear that, in general,

$$S^{-1}(h) \subseteq \mathcal{O}(h;S); \tag{1.1}$$

and if, for instance, S is a strictly monotone operator (hence $S^{-1}(h)$ is a singleton), then this relation holds as equality. Some illustrative examples will be given later.

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In this paper we investigate when the orthogonality set attracts all solutions of a class of so called (S,h)-admissible differential equations in the space H. Special role in our approach plays the minimizer of S, namely a point x^* such that

$$\|x^*\| = \inf\{\|x\| : x \in S^{-1}(h)\}.$$

In the literature one can find a great number of works dealing with continuous, or discrete paths (i.e. iterates generating sequences) converging strongly to a point of $S^{-1}(h)$, see, e.g., [2 - 11, 13, 16, 17, 21 - 30, 32 - 34, 40, 42, 46, 47, 51, 54, 57, 59, 60]. Applications of operators and problems generating the situation can be found elsewhere, see, e.g, [20, 56].

Moreover there are approaches of the problem where a point of $S^{-1}(h)$ is approximated by the solutions of a suitably formulated ordinary differential equation related to the pair (S,h). We term it (S,h)-admissible equations. More precisely, consider the evolution equation in H of the form

$$\frac{d}{dt}x(t) + T(t)x(t) + \alpha(t)x(t) = f(t), \quad t \geq 0, \tag{1.2}$$

with arbitrary initial value, where $\alpha(t)$, $t \geq t_0$, is a convex, positive, decreasing real valued function. In case the family of operators $T(t)$ approaches S in a certain sense, the perturbation $f(t)$ approaches h and the function α satisfies at least the conditions

$$\lim_{t\to+\infty} \alpha(t) = 0, \quad \int_{t_0}^{+\infty} \alpha(t)dt = +\infty, \quad \lim_{t\to+\infty} \frac{\alpha'(t)}{\alpha^2(t)} = 0, \tag{1.3}$$

Alber [1] and Alber-Ryazantseva [12] followed methods of regularization in essentially ill posed problems [48, p.180] have shown that any solution of (1.2) converges strongly to the point x^*.

Our results are proved by setting rather mild conditions on the coefficients and especially to the scalar function α appeared in (1.2), which are weaker than (1.3). Indeed, we need only the function α vanished at $+\infty$ and being decreasing and satisfying an integral-like condition weaker than the two last conditions in (1.3), see the remark following Lemma 3.11 below. However, we show convergence to the orthogonality set, while in [12] it is proved convergence to the minimizer.

Here let us consider for a moment a point z in $S^{-1}(h)$, i.e. a point that satisfies the equation

$$Sz = h.$$

Then the constant function $u(t) := z$, $t \in \mathbb{R}$ satisfies the ordinary differential equation

$$\dot{u} + Su(t) = h. \tag{1.4}$$

In this work we put the problem of approximation in a new frame by considering the autonomous ordinary differential equation (1.4) as the full limiting equation of (1.2), generated by a semi-flow with phase-space the cartesian product of four spaces. According to this semi-flow each point of this space consists of translations of the solution x and of the coefficients T, α, f, which appear in equation (1.2). These translations produce semi-dynamical systems similar to the Bebutov dynamical system (see, e.g., [55] p. 81) and it is widely used elsewhere [15, 36 - 38, 50, 52, 53]. For the discrete version of this semi-dynamical system one can consult e.g. [39].

The limiting properties of the solutions of (1.2) are described by the solutions of (1.4). For instance, if the latter admits solutions, which are constant functions, then all solutions of (1.2) converge, as the time increases, to such a constant function.

In the references stated above it was also shown how to use the semi-flow theory to study the asymptotic behavior of solutions of nonautonomous ordinary differential equations, Volterra integral equations and more general operator equations in the the space of continuous functions. Applications to stability theory can be found e.g. in [38, 52]. For more applications, details, and references, see e.g. [50, 52], [45].

Next we shall recall some facts from the limiting equations theory, as it is developed in [14, 36, 37. 52, 53], [45, Appendix], etc. We shall not follow the lines of any of these papers, but instead we prefer to combine ideas from them to establish our semi-flow. The basic idea is to take into consideration the continuous changes of the coefficients in the equation as the time progresses and then to get asymptotic properties of the solutions via the behavior of the solutions of the full limiting equations.

The major difficulty in the semi-flow approach is to define the appropriate topology in the phase space. First we will construct the shifting family of the original evolution equation in the sense of nonautonomous differential equation [45, 53] and then, following the classical method of limiting equations, we will investigate the convergence of the trajectories to their ω-limit sets. The most general facts for dynamical systems generated by Volterra-integral operators, or abstract operators, (see, e.g. [50, 36], etc.]) are not used here.

2. Some auxiliary facts

Before presenting our main results we shall recall some auxiliary facts about the geometric properties of Hilbert spaces. To begin with we first give illustrative examples for the orthogonality of an operator in Hilbert spaces.

Examples. (a) Let for any x in the plane Rx be the point obtained by rotating x counterclockwise by a right angle. Define $Sx := x - Rx$ and observe that $S0 = 0$ and S is a monotone operator having orthogonality at any h the singleton $S^{-1}(h)$.

(b) In $H := l^2$ define the operator

$$B : (x_1, x_2, x_3, x_4, \dots) \to (x_2, -x_1, x_3, x_4, \dots).$$

Then B is an isometry and so the linear operator $Sx := x - Bx$ is monotone. Observe that

$$S^{-1}(0) = \{0\},$$
$$\mathcal{O}(0; S) = \{(0, 0, x_2, x_3, \dots) : \sum_{i=2}^{+\infty} x_i^2 < +\infty\}.$$

(c) Let C be a convex closed subset of a Hilbert space H and let Sx be the point of the minimum distance of x from C :

$$Sx := \{z \in C : \|z - x\| = \inf\{\|y - x\| : y \in C\}\}.$$

It is well known that S defines an operator with domain H, which, as it is shown in [35, p. 42], is monotone.

We claim that it holds

$$\mathcal{O}(h; S) = S^{-1}(h), \tag{2.1}$$

for all $h \in C$. Before we prove this fact, we shall shift the point h to the origin as follows: Take any point $h \in C$. Defining the set

$$C' := \{u = x - h : \quad x \in C\}$$

and the operator

$$S' : H \to C' : u \to S(u + h) - h,$$

it is clear that $0 \in C'$ and it holds $Sx = h$, if and only if $S'x' = 0$, where $x' := x - h$. In this case we have

$$\begin{aligned}
0 = S'x' &= Sx - h \\
&= \{z \in C : \|z - x\| = \inf\{\|y - x\| : y \in C\}\} - h \\
&= \{u := z - h \in C' : \|z - x\| = \inf\{\|y - x\| : y \in C\}\} \\
&= \{u := z - h \in C' : \|u - (x - h)\| = \inf\{\|v - (x - h)\| : v \in C'\}\} \\
&= \{u \in C' : \|u - x'\| = \inf\{\|v - x'\| : v \in C'\}\}.
\end{aligned}$$

This shows that $S'x'$ is the point at the minimum distance of x' from C'.

Now it is clear that relation (2.1) holds if and only if

$$\mathcal{O}(0; S') = S'^{-1}(0). \tag{2.2}$$

Because of (1.1), in order to show (2.2) it is enough to prove that

$$\mathcal{O}(0; S') \subseteq S'^{-1}(0). \tag{2.3}$$

Take a point $z \in C'$ with $S'z = 0$ and consider any $u \in \mathcal{O}(0; S')$, namely a point such that

$$\langle S'u, u - z\rangle = 0. \tag{2.4}$$

Then, on one hand, we have

$$\|S'u - u\|^2 \le \|u\|^2$$

(notice that $0 \in C'$), which gives

$$\|S'u\|^2 - 2\langle S'u, u\rangle \le 0 \tag{2.5}$$

and on the other hand

$$\|z\|^2 \le \|S'u - z\|^2,$$

since 0 and $S'u$ are points of the set C'. The latter implies that

$$0 \le \|S'u\|^2 - 2\langle S'u, z\rangle, \tag{2.6}$$

Relations (2.4)–(2.6) imply that

$$\|S'u\|^2 = 2\langle S'u, z\rangle,$$

from which, we get

$$\|S'u - z\| = \|z\|.$$

The latter, clearly, says that $S'u = S'z = 0$ and so our claim is proved.

Now we present some well known notions which have been introduced in connection with nonlinear problems in functional analysis (see, e.g., [18, 19, 20, 41, 49, 56]).

An operator S is *hemicontinuous at a certain point* x_0, if for any sequence (x_n) converging to x_0 along a line, the sequence (Sx_n) converges weakly to Sx_0. That is, $\langle S(x_0 + t_n x) - Sx_0, y\rangle \to 0$, as $t_n \to 0$, for all $x, y \in H$. An operator S is *demicontinuous at a certain point* x_0, if for any sequence (x_n) converging to x_0, the sequence (Sx_n) converges weakly to Sx_0. This means that, if $\lim x_n = x_0$, then for each $u \in H$ it holds

$$\lim\langle Sx_n - Sx_0, u\rangle = 0.$$

A consequence of this fact is the following: Let $x : I \to H$ be a continuous function, where I is any subset of the real line. Then, for each $u \in H$, the function

$$t \to \langle Sx(t), u\rangle$$

is continuous.

An operator S is *bounded*, if it maps bounded sets into bounded sets. An operator S is *maximal monotone,* if it has no proper monotone extensions.

Lemma 2.1 ([35, p. 48])**.** *If $S : H \to H$ is a monotone hemicontinuous operator, then S is maximal monotone.*

Actually we need an operator S which is at least demicontinuous. Then it is hemicontinuous and so by the previous lemma it is maximal monotone. The latter property guarantees special properties for the operator S as in the following result:

Lemma 2.2 ([35, p. 55])**.** *If $S : H \to H$ be a monotone hemicontinuous operator, then the range $\mathcal{R}(I + S)$ of the operator $I + S$ is the entire space H. (Here I stands for the identity on H.)*

Let $\theta > 0$ be given. If $S : H \to H$ is a hemicontinuous operator, then, clearly, $\frac{1}{\theta}S$ does so. Thus, from the previous lemma, it follows that the range of the operator $\theta I + S$ is the entire space H. For our purpose we need the following well known result (stated for general set-relations) as [23, Lemma 1]. See, also, [51].

Lemma 2.3. *If S is a maximal monotone operator, then for each $\theta > 0$, there exists a unique $y_\theta \in H$ for which*

$$0 = \theta y_\theta + Sy_\theta.$$

Also, if $0 \in \mathcal{R}(S)$, then the strong limit

$$\lim_{\theta\to 0^+} y_\theta = x^*$$

exists and is the point of $A := S^{-1}(0)$ closest to 0.

The first part of the lemma was given in [49], but for continuous monotone operators.

Now assume that $S : H \to H$ is a demicontinuous operator and let $h \in \mathcal{R}(S)$. Define the set

$$A := S^{-1}(h). \tag{2.7}$$

The following result which characterizes the minimizer of the set A is known, but for completeness of this work we shall give the proof here.

Lemma 2.4. *Under the conditions above, the set A is convex, (strongly) closed and there is a unique point x^* in A such that*

$$\|x^*\| = \min\{\|x\| : x \in A\}.$$

Proof. First we claim that a point x is an element of the set A if and only if the inequality

$$\langle Sz - h, z - x\rangle \geq 0 \tag{2.8}$$

holds for all $z \in H$. Indeed, if $Sx = h$, then, for all $z \in H$, we have

$$\langle Sz - h, z - x\rangle = \langle Sz - Sx, z - x\rangle \geq 0,$$

by the monotonicity of S.

The inverse: Assume that x is a point in H satisfying (2.8) for all z. Let $s \in (0, 1)$ and define $w := h - Sx$. Consider the point $z := x + sw$ and observe that (2.8) gives

$$\langle S(x + sw) - h, w\rangle \geq 0.$$

Letting $s \to 0$, by the hemicontinuity of the operator S, we get $\langle Sx - h, w\rangle \geq 0$. Thus it follows that

$$-\|Sx - h\|^2 = \langle Sx - h, h - Sx\rangle = \langle Sx - h, w\rangle \geq 0,$$

which implies that $Sx = h$. Hence $x \in A$. By using relation (2.8) we can easily show the convexity of A.

We show the closedeness of the set A. Indeed, let (x_n) be a sequence in A converging (strongly) to a point $x \in H$. If x does not belong to A, according to our claim above, there is a $z \in H$ such that

$$\langle Sz - h, z - x\rangle \quad < -\varepsilon,$$

for some $\varepsilon > 0$. From this relation we get $\|Sz-h\| > 0$ and moreover, for all indices n

$$\begin{aligned}\varepsilon &< -\langle Sz - h, z - x\rangle = -\langle Sz - h, z - x_n\rangle - \langle Sz - h, x_n - x\rangle \\ &= -\langle Sz - Sx_n, z - x_n\rangle - \langle Sz - h, x_n - x\rangle \\ &\leq -\langle Sz - h, x_n - x\rangle \leq \|Sz - h\|\|x_n - x\|,\end{aligned} \tag{2.9}$$

contrary to the fact that $x_n \to x$. Therefore the set A is closed.

Finally, we show uniqueness. To this end, assume that there are two points $x* \neq y*$ such that

$$\|x^*\| = \|y^*\| = \min\{\|x\| : x \in A\}.$$

Here we proceed as in the proof of the main claim in Example (a). If $x* = 0$, then, clearly, $y^* = 0$, too, which is false. So, assume that $x^* \neq 0$. Consider the set

$$A_0 := \{\frac{x}{\|x^*\|} : x \in A\}.$$

Then the points

$$x_0^* := \frac{x^*}{\|x^*\|}, \quad y_0^* := \frac{y^*}{\|x^*\|}$$

are different and satisfy

$$\|x_0^*\| = \|y_0^*\| = 1.$$

Since H is a Hilbert space, it is strictly convex [31, p. 31], thus by standard properties of such spaces [31, p. 41] we get

$$\|x_0^* + y_0^*\| < 2.$$

Since A_0 is a convex set, we have $\frac{x_0^*+y_0^*}{2} \in A_0$. Hence, by the minimality of x_0^*, we get

$$1 = \|x_0^*\| \leq \|\frac{x_0^* + y_0^*}{2}\| \leq \frac{\|x_0^* + y_0^*\|}{2} < 1,$$

a contradiction. The proof of lemma is complete. □

Remark 2.5. If a monotone operator S is demicontinuous, then, it is hemicontinuous and, from Lemma 2.1, S is maximal monotone. Thus from Lemma 2.3 it follows that the operator equation

$$Sx + \lambda x = h \tag{2.10}$$

admits a solution x_λ which approaches x^* as λ tends to zero.

To proceed we make the following assumptions:

- (H1) $\alpha : [0, +\infty) \to (0, +\infty)$ is a decreasing continuous function such that $\lim_{t\to+\infty} \alpha(t) = 0$.
- (H2) $f : [0, +\infty) \to H$ is a continuous function such that $\lim_{t\to+\infty} f(t) = h$.

Using Remark 2.5, it follows that there is a unique point $z(t)$ in H satisfying the relation

$$Sz(t) + \alpha(t)z(t) = h, \tag{2.11}$$

for $t \geq 0$.

Lemma 2.6. *The function $t \mapsto z(t)$ satisfying equation in* (2.11) *is continuous on* $[0, +\infty)$ *and it converges to x^* as t tends to $+\infty$. In particular, z is bounded.*

Proof. For t_0 fixed and any $t \geq 0$, we obtain from (2.11) that

$$Sz(t_0) - Sz(t) = \alpha(t)[z(t) - z(t_0)] + [\alpha(t) - \alpha(t_0)]z(t_0).$$

Multiplying both sides with $z(t) - z(t_0)$ and using monotonicity of the operator S we get

$$\|z(t) - z(t_0)\| \leq \frac{|\alpha(t) - \alpha(t_0)|}{\alpha(t)} \|z(t_0)\|,$$

which, clearly, implies the continuity of z at t_0. The fact that $z(t)$ converges to x^* follows from Remark 2.5. □

Lemma 2.7. *Let f, α be functions satisfying the assumptions* (H2) *and* (H1) *and moreover*

$$\sup_{t\geq 0} \frac{\|f(t) - h\|}{\alpha(t)} < +\infty. \tag{2.12}$$

If $z_\alpha(t)$ is the (unique) point satisfying the algebraic equation

$$Sz_\alpha(t) + \alpha(t)z_\alpha(t) = f(t), \tag{2.13}$$

for all $t \geq 0$, then the function $t \mapsto z_\alpha(t)$ is continuous and bounded.

Proof. Continuity follows as that in Lemma 2.6. To show boundedness, we subtract relation (2.11) from (2.13) and obtain

$$[Sz_\alpha(t) - Sz(t)] + \alpha(t)[z_\alpha(t) - z(t)] = f(t) - h.$$

Therefore we have

$$\langle Sz_\alpha(t) - Sz(t), z_\alpha(t) - z(t)\rangle + \alpha(t)\|z_\alpha(t) - z(t)\|^2 = \langle f(t) - h, z_\alpha(t) - z(t)\rangle,$$

which, because of the monotonicity of the operator S, gives

$$\alpha(t)\|z_\alpha(t) - z(t)\|^2 \leq \|f(t) - h\|\|z_\alpha(t) - z(t)\|.$$

Let U be the set of all t such that $z_\alpha(t) = z(t)$. Then, because of (2.12) there is a positive real number M such that

$$\|z_\alpha(t) - z(t)\| \leq M,$$

for all $t \notin U$ we obtain. This and boundedness of z proves the result. □

3. Some semi-flow facts

Let us denote by $\mathcal{M}$ the set of all demicontinuous operators with domain the Hilbert space H and range in H. We endow the set $\mathcal{M}$ by the following convergence structure: A sequence (S_n) converges to a certain S, and write it as $S_n \to S$, if it holds

$$\lim \|S_n z - Sz\| = 0,$$

uniformly for z in bounded sets.

To unify things we use the symbol E to denote either the real line $\mathbb{R}$, or the Hilbert space H, or the space $\mathcal{M}$. Then we shall denote by $C(\mathbb{R}, E)$ and by $C(\mathbb{R}^+, E)$ the sets of all continuous functions defined on the real line and on $[0, +\infty)$ respectively and with values in E. We furnish these sets with the topology of uniform convergence on compact intervals.

First we shall present briefly the shifting semi-flow on $C(\mathbb{R}^+, E)$ in a formal way, although it was previously used implicitly by many authors (see, e.g., [36, 50] and the references therein). This is actually the Bebutov dynamical system (see, e.g., [55, p. 81]), where the functions are restricted on $\mathbb{R}^+$.

We shall assume that the reader is familiar with all basic notions on dynamical systems (namely, ω-limit set, α-limit set, invariance, compactness, stationary points, etc). The classical book [55] provides a good basis of the theory of dynamical systems on metric spaces.

For any $u \in C(\mathbb{R}^+, E)$ and $t \geq 0$ we define the translation of u by t to be the function $s \to u(t + s)$, $s \in \mathbb{R}^+$. Then consider the mapping

$$p_E(t, u)(\cdot) = u(t + \cdot) : [0, +\infty) \times C(\mathbb{R}^+, E) \to C(\mathbb{R}^+, E)$$

and we show the following result.

Lemma 3.1. *Under the hypotheses above the mapping p_E defines a semi-flow with phase-space the set $C(\mathbb{R}^+, E)$.*

Proof. First we see that it holds

$$p_E(0, u)(\cdot) = u(0 + \cdot) = u(\cdot).$$

Also, the semi-group property holds, since we have

$$p_E(t, p_E(r, u)(\cdot))(s) = p_E(r, u)(t + s) = u(r + t + s) = p_E(t + r, u)(s).$$

Finally, to show continuity of $p_E(t, u)$ with respect to the pair (t, u) we let (t_n, u_n) be a sequence in $[0, +\infty) \times C(\mathbb{R}^+, E)$ converging to (t_0, u_0). This means that (t_n) converges to t_0, thus the set $N := \{t_0, t_n : n = 1, 2, \dots\}$ is compact and the sequence (u_n) converges to the function u_0 uniformly on compact intervals.

Now, for any compact interval I of $\mathbb{R}^+$ and any $s \in I$, it holds

$$\begin{aligned}
&\|p_E(t_n, u_n)(s) - p_E(t_0, u_0)(s)\| \\
&= \|u_n(t_n + s) - u_0(t_0 + s)\| \\
&\leq \|u_n(t_n + s) - u_0(t_n + s)\| + \|u_0(t_n + s) - u_0(t_0 + s)\| \\
&\leq \sup_{\theta \in N+I} \|u_n(\theta) - u_0(\theta)\| + \|u_0(t_n + s) - u_0(t_0 + s)\|.
\end{aligned}$$

Here $N + I$ is the (compact) set $\{r + t : r \in N,\ t \in I\}$. Hence continuity of p_E follows from the previous relation, since both quantities in the right side tent to zero as $n \to +\infty$. $\square$

It is clear that a point u is a rest (or, stationary) point with respect the semi-flow, if and only if the function u is constant. A motion is called *compact* if its trajectory lies in a compact set.

Lemma 3.2. *Assume that u is a uniformly continuous function in $C(\mathbb{R}^+, E)$ with relatively compact range in case $E = H$ and with bounded range in case $E = \mathbb{R}$. Then the ω-limit set of u is nonempty, compact, connected and invariant.*

Proof. Take any sequence (t_n) in $\mathbb{R}^+$, a compact set I of positive reals and consider the family

$$\mathcal{G} := \{u(t_n + s) : s \in I, \quad n = 1, 2, \dots\}.$$

Then, applying a generalization of the Ascoli's theorem (see, e.g., [58, p. 290], we see that $\mathcal{G}$ is relatively compact. This implies that the ω-limit set is nonempty. The other properties follow as for the Bebutov dynamical system exhibited in [55, pp. 28-32]. $\square$

Usually, a function u satisfying the assumptions of the previous lemma as well as its motion, are called compact. Notice that Langrange stable motions in [55] correspond to compact motions in our case.

Corollary 3.3. *Let $u \in C(\mathbb{R}^+, E)$ be such that the limit $\lim_{t\to+\infty} u(t)$ exists in E. Then the trajectory of the point u is compact.*

Proof. The result follows from the previous lemma and the fact that the existence of the limit and the continuity of u implies uniform continuity on $\mathbb{R}^+$, as well as the relative compactness of the range of u. $\square$

Definition 3.4. A function $\bar{u} \in C(\mathbb{R}^+, E)$ is said to be a limiting function of u, if $\bar{u} \in \omega(u)$.

For any $u \in C(\mathbb{R}^+, E)$ and any limiting function $\bar{u}$ of u, there is a full orbit Q through $\bar{u}$, which lies in $\omega(u)$. Thus a flow is generated in $\omega(u)$. The latter is the *invariance property* in dynamical systems [45]. To prove the above fact one can use the following lemma adopted to our case, which is known in the theory of semi-flows and its proof can be succeeded by using a Cantor type diagonalization argument.

Lemma 3.5. *Let $p_E(t, u)$ be a (positively) compact motion with ω-limit set $\omega(u)$. Then for each $\bar{u} \in \omega(u)$, there is a continuous function $Q : \mathbb{R} \to \omega(u)$, which satisfies the following properties:*
a) $Q(0) = \bar{u}$, and
b) $p_E(t, Q(s)) = Q(t + s)$, for all $s \in \mathbb{R}$ and $t \geq 0$.

In the sequel we say that Q is a *full limiting orbit* of u, if Q is a full orbit through a point $\bar{u} \in \omega(u)$.

Definition 3.6. A function $\hat{u} \in C(\mathbb{R}, E)$ is said to be a full limiting function of u, if it can be written in the form $\hat{u}(t) = Q(t)(0)$, for a certain full limiting orbit Q of u.

Let $C_{un}(\mathbb{R}^+, E)$ be the set of all uniformly continuous functions of $C(\mathbb{R}^+, E)$ with relatively compact range. (If $E = \mathbb{R}$, then boundedness is sufficient for relative compactness.) Then the set of all full limiting functions of any $u \in C_{un}(\mathbb{R}^+, E)$ is a nonempty, compact, connected and invariant subset of $C_{un}(\mathbb{R}^+, E)$. Indeed, the set of full limiting functions of u is the ω-limit set of any continuation $z \in C_{un}(\mathbb{R}, E)$

of u (for instance, define $z(t) := u(0)$, if $t < 0$, and $z(t) := u(t)$, if $t \geq 0$) with respect to the Bebutov dynamical system (see, e.g. [55, p. 81], and therefore it has the above properties. Notice that in [55] the above properties are referred to metric spaces and are proved by using sequential approach. This approach con be used even in our case, where E is the space H or the space $\mathcal{M}$.

In the sequel we shall use the subspace $C_{\text{mon}}(\mathbb{R}^+, \mathcal{M})$ of all functions $T \in C(\mathbb{R}^+, \mathcal{M})$ such that

$$\langle T(t)z - T(t)y, z - y\rangle \geq 0,$$

for all nonnegative reals $t \geq 0$. It is not hard to see that the space $C_{\text{mon}}(\mathbb{R}^+, \mathcal{M})$ is a closed subset of $C(\mathbb{R}^+, \mathcal{M})$.

Now consider the differential equation (1.2), where $T \in \mathcal{C}_{\text{mon}}(\mathbb{R}, \mathcal{M})$, $\alpha \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $f \in C(\mathbb{R}, H)$.

Definition 3.7. An equation of the form (1.2) is called (S, h)-admissible if any bounded solution of (1.2) has range in a compact set, the coefficients α, f satisfy the conditions (H1), (H2) and the function T satisfies the following condition:

(H3) The function $T(t)$ converges to S as $t \to +\infty$ with respect to the convergence structure of $\mathcal{M}$.

Lemma 3.8. *If x, y are two solutions of* (1.2)*, then they have the same domain, $\mathcal{D}$, say. Also if $x(0) = y(0)$, then $x(t) = y(t)$, for all t in $\mathcal{D}$.*

Proof. Assume that x, y are two solutions of (1.2) defined on an interval of the form $[0, t_0)$. Then, for all $t \in [0, t_0)$, we have

$$x'(t) - y'(t) + T(t)x(t) - T(t)y(t) + \alpha(t)(x(t) - y(t)) = 0.$$

Multiplying both sides by the factor $x(t) - y(t)$ we get

$$\langle x'(t) - y'(t), x(t) - y(t)\rangle \leq -\alpha(t)\langle x(t) - y(t), x(t) - y(t)\rangle, \tag{3.1}$$

for all $t \in [0, t_0)$, or

$$\frac{1}{2}\frac{d}{dt}\|x(t) - y(t)\|^2 \leq -\alpha(t)\|x(t) - y(t)\|^2,$$

from which we get

$$\|x(t) - y(t)\| \leq \|x(0) - y(0)\| \exp\Big(\int_0^t \alpha(s)ds\Big), \quad t \in [0, t_0). \tag{3.2}$$

This relation implies that the solutions x, y have the same interval of domain and moreover, if $x(0) = y(0)$, then $x(t) = y(t)$, for all t, hence the uniqueness. □

Lemma 3.9. *Consider the function $T \in C_{\text{mon}}(\mathbb{R}, \mathcal{M})$ satisfying assumption* (H3) *and let us assume that α satisfies assumption $(H1)$. Moreover suppose that*

$$\sup_{t\geq 0}\frac{\|T(t)z - Sz\|}{\alpha(t)} < +\infty \tag{3.3}$$

holds uniformly for z in bounded sets. Then the (unique) solution y_α of the functional equation

$$T(t)y_\alpha(t) + \alpha(t)y_\alpha(t) = f(t), \quad t \geq 0, \tag{3.4}$$

is defined for all $t \geq 0$ and it is bounded.

Proof. Let $z_\alpha(t)$ be the solution of the algebraic equation
$$Sz_\alpha(t) + \alpha(t)z_\alpha(t) = f(t). \tag{3.5}$$
Subtracting relation (3.5) from (3.4) we have
$$[T(t)y_\alpha(t) - T(t)z_\alpha(t)] + [T(t)z_\alpha(t) - Sz_\alpha(t)] + \alpha(t)[y_\alpha(t) - z_\alpha(t)] = 0$$
and so
$$\langle T(t)z_\alpha(t) - Sz_\alpha(t), y_\alpha(t) - z_\alpha(t)\rangle + \alpha(t)\|y_\alpha(t) - z_\alpha(t)\|^2 \le 0,$$
because of the monotonicity of the operator $T(t)$. Thus we have
$$\alpha(t)\|y_\alpha(t) - z_\alpha(t)\|^2 \le \|T(t)z_\alpha(t) - Sz_\alpha(t)\|\|y_\alpha(t) - z_\alpha(t)\|.$$
By using the fact that the set $\{z_\alpha(t) : t \ge 0\}$ is bounded (Lemma 2.7), from (3.3) the result follows. □

Lemma 3.10. *Under the conditions* (2.12) *and* (3.3)*, for each* $\tau > 0$ *any solution of the differential equation*
$$\frac{d}{dt}z(t;\tau) + T(t)z(t;\tau) + \alpha(\tau)z(t;\tau) = f(t) \tag{3.6}$$
is defined on the entire interval $[0, +\infty)$ *and it is bounded uniformly for all* $\tau \ge 0$.

Proof. Fix a $\tau \ge 0$ and let $y_\alpha(\tau)$ be the solution of the algebraic equation (3.4) for $t = \tau$. From (3.4) and (3.6) we obtain
$$\begin{aligned}
&z'(t;\tau) + [T(t)z(t;\tau) - T(t)y_\alpha(\tau)]\\
&+ [T(t)y_\alpha(\tau) - T(\tau)y_\alpha(\tau)] + \alpha(\tau)[z(t;\tau) - y_\alpha(\tau)]\\
&= f(t) - f(\tau).
\end{aligned}$$
Multiply both sides with the factor $z(t;\tau) - y_\alpha(\tau)$ and use the monotonicity of the operator $T(t)$ to obtain
$$\begin{aligned}
&\langle z'(t;\tau), z(t;\tau) - y_\alpha(\tau)\rangle + \langle T(t)y_\alpha(\tau) - T(\tau)y_\alpha(\tau), z(t;\tau) - y_\alpha(\tau)\rangle\\
&+ \alpha(\tau)\|z(t;\tau) - y_\alpha(\tau)\|^2\\
&\le \langle f(t) - f(\tau), z(t;\tau) - y_\alpha(\tau)\rangle.
\end{aligned}$$
From this relation we obtain
$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|z(t;\tau) - y_\alpha(\tau)\|^2 &\le \|T(t)y_\alpha(\tau) - T(\tau)y_\alpha(\tau)\|\|z(t;\tau) - y_\alpha(\tau)\|\\
&\quad - \alpha(\tau)\|z(t;\tau) - y_\alpha(\tau)\|^2\\
&\quad + \|f(t) - f(\tau)\|\|z(t;\tau) - y_\alpha(\tau)\|.
\end{aligned} \tag{3.7}$$
Therefore for all t for which (3.7) holds, the (nonnegative) function $\psi(t;\tau) := \|z(t;\tau) - y_\alpha(\tau)\|$ satisfies the inequality
$$\psi'(t;\tau) \le -\alpha(\tau)\psi(t;\tau) + \|T(t)y_\alpha(\tau) - T(\tau)y_\alpha(\tau)\| + \|f(t) - f(\tau)\|. \tag{3.8}$$
Thus by using standard comparison arguments for differential inequalities (see, e.g. [44, p. 15]) we get that $\psi(t;\tau) \le r(t;\tau)$, for all $t \ge 0$, for which the solution $r(t;\tau)$ of the differential equation
$$r'(t;\tau) = -\alpha(\tau)r(t;\tau) + \|T(t)y_\alpha(\tau) - T(\tau)y_\alpha(\tau)\| + \|f(t) - f(\tau)\|$$
with $r(0;\tau) = \psi(0;\tau)$ exists. But it is clear that $r(\cdot;\tau)$ is defined up to $+\infty$. This proves boundedness for each fixed τ.

To prove the existence of a uniform bound assume that for any positive integer n there are $\tau_n > 0$ and $t_n \geq \tau_n$ such that $\psi(t_n;\tau_n) \geq n$. Without loss of generality we can assume that $\psi'(t_n;\tau_n) \geq 0$, for any n. Then from (3.8) we get

$$n\alpha(\tau_n) \leq \|T(t_n)y_\alpha(\tau_n) - T(\tau_n)y_\alpha(\tau_n)\| + \|f(t_n) - f(\tau_n)\|$$

and so

$$\begin{aligned} n \leq & \frac{\|T(t_n)y_\alpha(\tau_n) - Sy_\alpha(\tau_n)\|}{\alpha(t_n)} + \frac{\|f(t_n) - h\|}{\alpha(t_n)} \\ & + \frac{\|T(\tau_n)y_\alpha(\tau_n) - Sy_\alpha(\tau_n)\|}{\alpha(\tau_n)} + \frac{\|f(\tau_n) - h\|}{\alpha(\tau_n)}, \end{aligned}$$

contrary to the assumptions on T and f. The proof is complete. $\square$

Lemma 3.11. *Assume that S is a bounded operator and* (2.12)*,* (3.3) *keep in force. If the condition*

$$\sup_{t\geq 0} \int_\sigma^t [\alpha(s) - \alpha(t)] \exp\Big(- \int_s^t \alpha(u)du\Big) ds < +\infty, \tag{3.9}$$

holds, then any solution of the differential equation

$$x' + T(t)x(t) + \alpha(t)x(t) = f(t) \tag{3.10}$$

is defined on the whole interval $[0, +\infty)$*, it is bounded and uniformly continuous.*

It is easy to see that conditions (1.3) plus convexity of the function α imply (3.9). On the other hand there are functions satisfying (3.9) but not (1.3). For instance, $\alpha(t) := (t + 1)^{-1}$, $t \geq 0$ is such a function.

Proof of Lemma 3.11. Let x be a solution of (3.10). First we shall prove that x is bounded, and it has domain of definition the entire interval $[0, +\infty)$. Let τ be any positive real in the domain of the solution. Consider the solution $z(t;\tau)$ of the differential equation (3.6). Then we have

$$\begin{aligned} & x'(t) - z'(t;\tau) + [T(t)x(t) - T(t)z(t;\tau)] \\ & + \alpha(t)[x(t) - z(t;\tau)] + [\alpha(t) - \alpha(\tau)]z(t;\tau) = 0. \end{aligned}$$

Take any $t \in [0, \tau]$, multiply both sides with the quantity $x(t) - z(t;\tau)$ and use monotonicity of $T(t)$. Then we obtain

$$\frac{1}{2}\frac{d}{dt}\|x(t) - z(t;\tau)\|^2 \leq -\alpha(t)\|x(t) - z(t;\tau)\|^2 + [\alpha(t) - \alpha(\tau)]\|z(t;\tau)\|\|x(t) - z(t;\tau)\|.$$

Therefore the function $\phi(t;\tau) := \|x(t) - z(t;\tau)\|$ satisfies the differential inequality

$$\frac{d}{dt}\phi(t;\tau) \leq -\alpha(t)\phi(t;\tau) + [\alpha(t) - \alpha(\tau)]m, \tag{3.11}$$

where m is a bound (guaranteed from Lemma 3.10) of the set $\{z(t;\tau) : t \geq 0\}$. Integrate the differential inequality (3.11) from σ to $t(\leq \tau)$ to get

$$\phi(t;\tau) \leq \phi(\sigma;\tau) \exp\Big(- \int_\sigma^t \alpha(s)ds\Big) + m \int_\sigma^t [\alpha(s) - \alpha(\tau)] \exp\Big(- \int_s^t \alpha(u)du\Big) ds.$$

Thus it follows that

$$\begin{aligned}\phi(\tau;\tau) \le \phi(\sigma;\tau)\exp\Big(-\int_\sigma^\tau \alpha(s)ds\Big)\\ + m\int_\sigma^\tau [\alpha(s)-\alpha(\tau)]\exp\Big(-\int_s^\tau \alpha(u)du\Big)ds.\end{aligned}\tag{3.12}$$

Now, our claim is implied from (3.12) and the fact that the set $\{\phi(\sigma;\tau) : \tau \ge 0\}$ is bounded, because of Lemma 3.10.

To show uniform continuity it is, clearly, enough to show that $x'(t)$ is bounded. This follows from the fact that the set $\{x(t) : t \ge 0\}$ is bounded, the operator S maps bounded sets into bounded sets and it holds

$$\|x'(t)\| \le \|T(t)x(t) - Sx(t)\| + \|Sx(t)\| + \alpha(t)\|x(t)\| + \|f(t)\|.$$

□

4. The main semi-flow

In the sequel we shall assume that S is a bounded demicontinuous monotone operator.

We shall define the semi-flow for (S,h)-admissible equations and whose the (unique) limiting equation is of the form (1.4). To be more clear consider the product

$$\mathcal{E} := C(\mathbf{R}^+, H) \times C(\mathbf{R}^+, M) \times C(\mathbf{R}^+, \mathbf{R}^+) \times C(\mathbf{R}^+, H)$$

endowed with the product topology.

Let $\mathcal{U}$ be the subset of $\mathcal{E}$ with the following characteristic:

> *Property* (P): Given any point (x, T, α, f) of $\mathcal{U}$ the function x is a solution of the ordinary differential equation (1.2) with coefficients T, α, f and equation (1.2) is (S,h)-admissible.

Observe that the mapping

$$\pi(t,(x,T,\alpha,f)) := (p_H(t,x), p_{\mathcal{M}}(t,T), p_{\mathbb{R}}(t,\alpha), p_H(t,f))$$

leaves the set $\mathcal{U}$ invariant. Indeed, given $(x,T,\alpha,f) \in \mathcal{U}$ and any $t \ge 0$ we have

$$x'(t+s) + T(t+s)x(t+s) + \alpha(t+s)x(t+s) = f(t+s),$$

for all $s \ge 0$. Clearly, this shows the invariance.

Now we prove the following result:

Theorem 4.1. *Any trajectory in $\mathcal{U}$ is a closed subset of $\mathcal{E}$. Moreover, Property* (P) *holds for all points in the closure of any trajectory.*

Proof. Let $(p_H(t_n,x), p_{\mathcal{M}}(t_n,T), p_{\mathbb{R}}(t_n,\alpha), p_H(t_n,f))$ be a sequence of points in the trajectory of some (x,T,α,f) converging to a certain point (x_0,T_0,α_0,f_0). Then from the (S,h)-admissibility, it follows that equation

$$x'(t_n+s) + T(t_n+s)x(t_n+s) + \alpha(t_n+s)x(t_n+s) = f(t_n+s)$$

is equivalent to the integral equation

$$\begin{aligned}p_H(t_n,x)(t) = p_H(t_n,x)(0) - \int_0^t [p_{\mathcal{M}}(t_n,T)(s)p_H(t_n,x)(s)\\ + p_{\mathbb{R}}(t_n,\alpha)(s)p_H(t_n,x)(s) - p_H(t_n,f)(s)]ds,\end{aligned}\tag{4.1}$$

where the latter is meant with respect to the weak sense. By using the convergence of the original sequence, the demicontinuity of the operator $T(t)$ and applying the classical Lebesgue Dominated Theorem we conclude that the point $(x_0, T_0, \alpha_0, f_0)$ lies in $\mathcal{U}$. □

Next, we show that property (P) holds for any point $(x_0, T_0, \alpha_0, f_0)$ in the closure of the trajectory. If $(x_0, T_0, \alpha_0, f_0)$ lies in (a finite portion of) the trajectory, this fact follows from the previous arguments. So, assume that $(x_0, T_0, \alpha_0, f_0)$ lies in the ω-limit set of (x, T, α, f). It is clear that such a point is of the form $(\bar{x}, S, 0, h)$, where $\bar{x}$ is in the ω-limit set of x with respect to the semi-flow $p_H(\cdot,\cdot)$. Hence, there is a sequence of positive reals (t_n) converging to $+\infty$ and such that

$$p_H(t_n, x)(s) = x(t_n + s) \to \bar{x}(s),$$

uniformly for all s is compact intervals. By using (S, h)-admissibility we take (weak) limits in (4.1) and conclude that

$$\bar{x}(t) = \bar{x}(0) - \int_0^t (S\bar{x}(s) - h)ds, \tag{4.2}$$

in the weak sense, from which the result follows, since $S\bar{x}(s)$ is demicontinuous.

The previous arguments imply the following useful result.

Corollary 4.2. *The mapping π defines a semi-flow with phase-space $\mathcal{U}$. Moreover any (S, h)-admissible equation* (1.2) *has limiting equations consisting of equations of the form*

$$v'(t) + Sv(t) = h, \tag{4.3}$$

where v is a point of the ω-limit set of x with respect to the semi-flow $p_H(\cdot,\cdot)$.

Now we are ready to give our convergence result.

Theorem 4.3. *Assume that $S : H \to H$ is a demicontinuous bounded monotone operator and let* (1.2) *be a (S, h)-admissible ordinary differential equation. Assume, also, that α satisfies relation* (3.9) *and conditions* (2.12) *and* (3.3) *keep in force. Then any solution x of* (1.2) *is defined up to $+\infty$ and any full limiting function $\hat{x}$ of x satisfies*

$$\alpha(\hat{x}) \cup \omega(\hat{x}) \subseteq \mathcal{O}(S; h),$$

with respect to the flow $p_H(\cdot,\cdot)$. Moreover, if the set $\mathcal{O}(S; h)$ is a singleton, then any solution of (1.2) *converges to the unique point of $S^{-1}(h)$.*

Proof. Let x be a solution of equation (1.2). By Lemma 3.11, x is bounded and uniformly continuous. Hence, by using the (S, h)-admissibility the point x is compact with respect to the semi-flow $p_H(\cdot,\cdot)$. Take any full limiting function $\hat{x}$ of x. Then we have

$$\hat{x}'(t) + S\hat{x}(t) = h. \tag{4.4}$$

Let z be any point such that $Sz = h$. Hence from (4.4) we get

$$\hat{x}'(t) + S\hat{x}(t) - Sz = 0,$$

which implies that

$$\langle \hat{x}'(t), \hat{x}(t) - z\rangle + \langle S\hat{x}(t) - Sz, \hat{x}(t) - z\rangle = 0,$$

or

$$\frac{1}{2}\frac{d}{dt}\|\hat{x}(t) - z\|^2 + \langle S\hat{x}(t) - Sz, \hat{x}(t) - z\rangle = 0, \tag{4.5}$$

Since S is monotone and the solution $\hat{x}$ is bounded, from (4.5), we conclude that the function $\|\hat{x}(t)-z\|$ is nonnegative and decreasing, thus the limits

$$l_- := \lim_{t\to-\infty}\|\hat{x}(t)-z\|, \quad l_+ := \lim_{t\to+\infty}\|\hat{x}(t)-z\|$$

exist as real numbers. Integrate (4.5) from t to $t+\tau$ ($\tau\in\mathbf{R}$) and get

$$\begin{aligned}\frac{1}{2}\|\hat{x}(t+\tau)-z\|^2 &= \frac{1}{2}\|\hat{x}(t)-z\|^2 - \int_t^{t+\tau}\langle S\hat{x}(s)-Sz, \hat{x}(s)-z\rangle \\ &= \frac{1}{2}\|\hat{x}(t)-z\|^2 - \int_0^{\tau}\langle S\hat{x}(t+s)-Sz, \hat{x}(t+s)-z\rangle ds,\end{aligned} \tag{4.6}$$

Take any point u in the α-limit set of $\hat{x}$. Then

$$u(s) = \lim \hat{x}(t_n+s),$$

uniformly for all s in compact intervals of the real line and for some sequence (t_n) converging to $-\infty$. Then from (4.6) we obtain

$$\frac{1}{2}l_-^2 = \frac{1}{2}l_-^2 - \int_0^{\tau}\langle Su(s)-Sz, u(s)-z\rangle ds,$$

which, since τ is arbitrary, implies that

$$\langle Su(s)-Sz, u(s)-z\rangle = 0,$$

for all s. This shows that u lies in the orthogonality of S at h. Similarly we show that any function in the ω-limit set of $\hat{x}$ is an element of the set $\mathcal{O}(h;S)$. .

If the orthogonality set $\mathcal{O}(S;h)$ is a singleton, $\{w\}$, say, then, certainly, $S^{-1}(h) = \{w\}$. Therefore, from the previous arguments, we have

$$l_- = \lim_{t\to-\infty}\|\hat{x}(t)-z\| = \|w-z\| = \lim_{t\to+\infty}\|\hat{x}(t)-z\| = l_+.$$

Setting $\tau=-2t$ in (4.6) and taking the limits as t tends to $+\infty$ we get

$$\int_{-\infty}^{+\infty}\langle Su(s)-Sz, u(s)-z\rangle ds = 0.$$

Since the integrand is nonnegative, we conclude that

$$\langle Su(s)-Sz, u(s)-z\rangle = 0,$$

which implies that $u(s)$ lies in the orthogonality of S at h, hence it is a constant equal to w. The proof is complete. □

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The authors wish to correct some misprints and to clarify the uniform boundedness (in τ) of the function $r(t,\tau)$ considered in Lemma 3.10 of this article.

- Page 7: In (H1) add: $\sup_{t\geq 0} t\alpha(t) =: N < +\infty$ and in (2.12) add:
$$\sup_{t\geq 0}\frac{\|f(t)-h\|}{\alpha(t)} =: \delta < +\infty.$$
- Page 10, line -10: Instead of $\exp(\int_0^t \dots)$ write $\exp(-\int_0^t \dots)$
- Page 10, line -5: $\epsilon(B) := \sup_{z\in B}\sup_{t\geq 0}\frac{\|T(t)z-Sz\|}{\alpha(t)} < +\infty$ holds for any bounded $B\subset \mathcal{M}$.
- Page 11 line 10: Let Y be the range of y_α.
- Page 11: In equation (3.6) add the initial value: $z(0,\tau) = y_\alpha(\tau)$.
- Page 11, line 14: In Lemma 3.10 add the assumption:
$$P := \sup_{u\in Y}\sup_{t\geq 0}\int_0^t \big(\|T(s)u-Su\|+\|f(s)-h\|\big)e^{-(t-s)\alpha(t)}ds < +\infty.$$
- Page 11, line -2: This proves that $\psi(\cdot,\tau)$ is defined up to $+\infty$.
- Page 12, line 1: Replace the paragraph with the following: To prove the uniform boundedness, it is enough to prove the same fact for $r(t,\tau)$. To this end fix any $\tau\geq 0$. Integrating
$$\begin{aligned}&[r'(s;\tau)+\alpha(\tau)r(s;\tau)]e^{s\alpha(\tau)}\\ &= [\|T(s)y_\alpha(\tau)-T(\tau)y_\alpha(\tau)\|+\|f(s)-f(\tau)\|]e^{s\alpha(\tau)}\end{aligned}$$
on the interval $[0,\tau]$ we get $r(\tau;\tau)\leq P+\epsilon(Y)+\delta$. Now, if there is $s\leq\tau$ with $r(s;\tau) = R(\tau) := \sup_{t\geq 0} r(t;\tau)$, we obtain $0 = r(0;\tau)\leq r(s;\tau)\leq r(s;\tau)e^{s\alpha(\tau)}\leq r(\tau;\tau)e^N\leq (P+\epsilon(Y)+\delta)e^N$, while, if for all $t\leq\tau$ it happens $R(\tau) > r(t;\tau)$, then there is a sequence $t_n>\tau$ such that $r'(t_n;\tau)\geq 0$ and $r(t_n;\tau)\to R(\tau)$. Thus $\alpha(\tau)r(t_n;\tau)\leq \|T(t_n)y_\alpha(\tau)-T(\tau)y_\alpha(\tau)\|+\|f(t_n)-f(\tau)\|$ and therefore $R(\tau)\leq\max\{2\epsilon(Y)+2\delta,(P+\epsilon(Y)+\delta)e^N\}$. This completes the proof.
- Page 12, line -3: Replace the set $\{z(t;\tau): t\geq 0\}$ by $\{z(t;\tau): t\geq 0,\ \tau\geq 0\}$.
- Page 15, lines $-15, -16, -18$: Replace $u(s)$ with $\hat{x}(s)$.

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