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TWO CLASSICAL PERIODIC PROBLEMS ON TIME SCALES

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Abstract. We consider the generalization of two classical periodic problems to the context of time scales. On the one hand, we generalize a celebrated result by Castro for the forced pendulum equation. On the other hand, we extend a well-known result by Nirenberg to a resonant system of equations on time scales. Furthermore, the results are new even for classical difference equations.

1. Introduction

In recent years there has been an increasing interest in dynamic equations on time scales. The concept of time scales (also known as *measure chains*) was introduced in 1990 by Hilger [9] with the motivation of providing a unified approach to continuous and discrete calculus. Thus, the notion of a generalized derivative $y^{\Delta}(t)$ was introduced, where the domain of the function $y(t)$ is an arbitrary closed non-empty subset of $\mathbb{T} \subset \mathbb{R}$. If $\mathbb{T} = \mathbb{R}$ then the usual derivative is retrieved, that is $y^{\Delta}(t) = y'(t)$. On the other hand, if the time scale is taken to be $\mathbb{Z}$ then the generalized derivative reduces to the usual forward difference, that is $y^{\Delta}(t) = \Delta y(t)$.

The field of dynamic equations on time scales allows us to model hybrid processes where time may flow continuously in one part of the process (with the model leading to a differential equation) and then time may flow discretely in another part of the process (leading to a difference equation). Moreover, these types of stop-start hybrid processes occur naturally and for more on the current and future applications of dynamic equations on time scales the reader is referred to the cover story of New Scientist [16] or the monographs by Bohner and Peterson [3] and Bohner et al [4].

The field of dynamic equations on time scales is not only about unification. It is important to emphasize that by researching dynamic equations on time scales, new advances can be made into each of the theories of differential and difference equations in their own right. For example, once a result is proved in the general time scale setting, special cases of the new results may give new theorems for each of the theories of differential and difference equations.

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In this work, we consider a generalization of two classical resonant periodic problems to the context of time scales. On the one hand, we study the forced pendulum equation

$$y^{\Delta\Delta} + a\sin(y^{\sigma}) = p(t), \quad t \in [0,T]_{\mathbb{T}} \tag{1.1}$$

where a is a positive constant.

For the continuous case $\mathbb{T} = \mathbb{R}$, Castro proved in [5, Theorem A] that if $a \le (\frac{2\pi}{T})^2$ and $p_0 = p - c$ with $c = \overline{p} := \frac{1}{T}\int_0^T p(t)dt$, then there exist two real numbers $d(p_0)$ and $D(p_0)$ with

$$-a \le d(p_0) \le 0 \le D(p_0) \le a$$

such that equation (1.1) admits T-periodic solutions if and only if

$$d(p_0) \le c \le D(p_0).$$

A more general result has been obtained by Mawhin and Willem in [13], and by Fournier and Mawhin in [6], using topological methods.

Also, we investigate the existence of periodic solutions $y : [0,\sigma^2(T)]_{\mathbb{T}} \to \mathbb{R}^N$ to the following nonlinear system of second order differential equations on time scales

$$y^{\Delta\Delta} = f(t, y^{\sigma}), \quad t \in [0,T]_{\mathbb{T}}; \tag{1.2}$$

under Landesman-Lazer type conditions. We shall assume that the nonlinearity $f : [0,T]_{\mathbb{T}} \times \mathbb{R}^N \to \mathbb{R}^N$ is bounded and continuous although, unlike the pendulum equation, $f(t,z)$ will be typically a non-periodic function of z.

By investigating the general equation (1.2), special cases of our results give novel results for (classical) difference equations and also for non-classical difference equations, such as q-difference equations (used in physics). Thus this article not only makes a new contribution to time scales, it also provides new results for difference equations.

There exists a vast literature on Landesman-Lazer type conditions for resonant problems, starting at the pioneering work [10] for a second order elliptic (scalar) differential equation under Dirichlet conditions. For a survey on Landesman-Lazer conditions see e.g. [12]. In [14], Nirenberg extended the Landesman-Lazer conditions to a system of elliptic equations. Nirenberg's result can be adapted for a system of periodic ODE's in the following way:

Theorem 1.1. *Let $p \in C([0,T],\mathbb{R}^N)$ and let $g : \mathbb{R}^N \to \mathbb{R}^N$ be continuous and bounded. Further, assume that the radial limits $g_v := \lim_{r\to+\infty} g(rv)$ exist uniformly respect to $v \in S^{N-1}$, the unit sphere of $\mathbb{R}^N$. Then the problem*

$$y'' + g(y) = p(t)$$

has at least one T-periodic solution if the following conditions hold:

- *$g_v \ne \overline{p} := \frac{1}{T}\int_0^T p(t)dt$ for any $v \in S^{N-1}$.*
- *The degree of the mapping $\theta : S^{N-1} \to S^{N-1}$ given by*

$$\theta(v) = \frac{g_v - \overline{p}}{|g_v - \overline{p}|}$$

  is non-zero.

For completeness, let us introduce the essential terminology of time scales.

Definition 1.2. A time scale $\mathbb{T}$ is a non-empty, closed subset of $\mathbb{R}$, equipped with the topology induced from the standard topology on $\mathbb{R}$.

Definition 1.3. The forward (backward) jump operator $\sigma(t)$ at t for $t < \sup \mathbb{T}$ (respectively $\rho(t)$ at t for $t > \inf \mathbb{T}$) is given by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}, \quad (\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\},) \text{ for all } t \in \mathbb{T}.$$

Additionally $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$, if $\sup \mathbb{T} < \infty$, and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$, if $\inf \mathbb{T} > -\infty$. Furthermore, denote $\sigma^2(t) = \sigma(\sigma(t))$ and $y^\sigma(t) = y(\sigma(t))$.

Definition 1.4. If $\sigma(t) > t$ then the point t is called right-scattered; while if $\rho(t) < t$ then t is termed left-scattered. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$ then the point t is called right-dense; while if $t > \inf \mathbb{T}$ and $\rho(t) = t$ then we say t is left-dense.

If $\mathbb{T}$ has a left-scattered maximum at m then we define $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition 1.5. Fix $t \in \mathbb{T}^k$ and let $y : \mathbb{T} \to \mathbb{R}^n$. Then $y^\Delta(t)$ is the vector (if it exists) with the property that given $\epsilon > 0$ there is a neighborhood U of t such that, for all $s \in U$ and each $i = 1, \dots, n$

$$|[y_i(\sigma(t)) - y_i(s)] - y_i^\Delta(t)[\sigma(t) - s]| \le \epsilon |\sigma(t) - s|.$$

Here $y^\Delta(t)$ is termed the (delta) derivative of $y(t)$ at t.

Theorem 1.6 ([9]). *Assume that $y : \mathbb{T} \to \mathbb{R}^n$ and let $t \in \mathbb{T}^k$.*

(i) *If y is differentiable at t then y is continuous at t.*
(ii) *If y is continuous at t and t is right-scattered then y is differentiable at t and*
$$y^\Delta(t) = \frac{y(\sigma(t)) - y(t)}{\sigma(t) - t}.$$
(iii) *If y is differentiable and t is right-dense then*
$$y^\Delta(t) = \lim_{s \to t} \frac{y(t) - y(s)}{t - s}.$$
(iv) *If y is differentiable at t then $y(\sigma(t)) = y(t) + \mu(t) y^\Delta(t)$.*

Definition 1.7. The function y is said to be right-dense continuous, that is $y \in C_{rd}(\mathbb{T}; \mathbb{R}^n)$ if:

(a) y is continuous at every right-dense point $t \in \mathbb{T}$, and
(b) $\lim_{s \to t^-} y(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.

Proposition 1.8. *For any right-dense continuous function y there exists an anti-derivative; i.e., a differentiable function Y such that $Y^\Delta(t) = y(t)$. Moreover, Y is unique up to a constant term, and the time scale integral of y is thus defined by*

$$\int_a^t y(s) \Delta s = Y(t) - Y(a).$$

We shall use the standard notation for the different intervals in $\mathbb{T}$. For example, if $a, b \in \mathbb{R}$ with $a < b$, then the closed interval of numbers between a and b will be denoted by $[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \le t \le b\}$.

In this context, the periodic boundary conditions for problems (1.1) and (1.2) read:

$$y(0) = y(\sigma^2(T)), \quad y^\Delta(0) = y^\Delta(\sigma(T)). \tag{1.3}$$

The paper is organized as follows. In Section 2 we introduce some preliminary results concerning the Lebesgue integral on time scales, and the associated linear problem for (1.1) and (1.2).

In the third section, we study the periodic problem for equation (1.1). Following the ideas in [6], we generalize Castro's result for an equation on time scales.

Finally, in Section 4 we study an extension of the standard Landesman-Lazer conditions for system (1.2). We shall obtain a general result that extends Theorem 1.1 for a system of differential equations in time scales.

2. Preliminary results

Let us define a measure in the following way. For $a < b \in \mathbb{T}$, consider $\mathcal{A} \subset \mathcal{P}([a,b)_{\mathbb{T}})$ the completion of the Borel σ-algebra generated by the family

$$\{[x,y)_{\mathbb{T}} : a \le x < y \le b, x, y \in \mathbb{T}\}.$$

Hence, there is a unique σ-additive measure $\mu : \mathcal{A} \to \mathbb{R}^+$ defined over this basis as: $\mu([x,y)_{\mathbb{T}}) = y - x$.

In the following lemma we establish the equivalence between the Lebesgue integral with respect to μ and the Cauchy integral on time scales when the integrand is right-dense continuous. A more precise result is given in [7]; in particular, it is proved that any right-dense continuous function is Lebesgue integrable.

Lemma 2.1. *If $\varphi \in C_{rd}([a,b]_{\mathbb{T}})$, then*

$$\int_a^b \varphi(t)\Delta t = \int_{[a,b)} \varphi d\mu := \int_a^b \varphi \, d\mu.$$

Proof. For $t \in [a,b)_{\mathbb{T}}$, define $\phi(t) = \int_{[a,t)} \varphi \, d\mu$. If t is right-dense, take $s \in [a,b)_{\mathbb{T}} - \{t\}$. Assume for example that $s > t$, then:

$$\frac{\phi(s) - \phi(t)}{s - t} - \varphi(t) = \frac{1}{s-t} \int_{[t,s)} \varphi - \varphi(t) \, d\mu.$$

A similar equality holds for $s < t$, and by continuity of φ it is immediate to prove that $\frac{\phi(s)-\phi(t)}{s-t} - \varphi(t) \to 0$ as $s \to t$. Hence ϕ is Δ-differentiable at t and $\phi^\Delta(t) = \varphi(t)$.

If t is right-scattered, it is clear that ϕ is continuous at t, and

$$\phi(\sigma(t)) - \phi(t) = \int_{\{t\}} \varphi \, d\mu = \varphi(t)(\sigma(t) - t).$$

It follows that $\phi^\Delta(t) = \varphi(t)$. We conclude that ϕ is an antiderivative of φ, and the result holds. $\square$

Remark 2.2. It follows from the previous lemma that all the theorems for the Lebesgue integral theory such as dominated convergence or Fatou Lemma hold.

Lemma 2.3. *Let $\varphi \in C_{rd}([0,T]_{\mathbb{T}})$ and $s \in \mathbb{R}$. Then there exists a unique solution of the problem*

$$\begin{gathered} y^{\Delta\Delta}(t) = \varphi(t) \quad \text{in } [0,T]_{\mathbb{T}} \\ y(0) = y(\sigma^2(T)) = s. \end{gathered} \tag{2.1}$$

Furthermore,

$$y(t) = s + \int_0^{\sigma(T)} G(t,s)\varphi(s)\Delta s,$$

where the Green function G is given by

$$G(t,s) = \begin{cases} \frac{-t}{\sigma^2(T)}(\sigma^2(T) - \sigma(s)) & \text{if } t \le s \\ \frac{-\sigma(s)}{\sigma^2(T)}(\sigma^2(T) - t) & \text{if } t \ge \sigma(s). \end{cases} \tag{2.2}$$

Proof. By integration, it follows that

$$y(t) = s + \int_0^t \phi(r)\Delta r - \frac{t}{\sigma^2(T)} \int_0^{\sigma^2(T)} \phi(r)\Delta r,$$

where $\phi(r) = \int_0^r \varphi(s)\Delta s$. From the previous lemma, we have that

$$\int_0^r \varphi(s)\Delta s = \int_{[0,\sigma(T))} \varphi(s).\chi_{[0,r)}(s)d\mu = \int_{[0,\sigma(T))} \varphi(s).\chi_{(s,\sigma^2(T))}(r)d\mu.$$

If s is right-scattered then $\chi_{(s,\sigma^2(T))} = \chi_{[\sigma(s),\sigma^2(T))}$. On the other hand, if s is right-dense, then $\mu(\{s\}) = 0$ and we conclude that

$$\phi(r) = \int_{[0,\sigma(T))} \varphi(s).\chi_{[\sigma(s),\sigma^2(T))}(r)d\mu.$$

Hence,

$$y(t) = s + \int_0^{\sigma(T)} \varphi(s)\Big(\int_0^{\sigma^2(T)} \chi_{[\sigma(s),\sigma^2(T))}(r)\big[\chi_{[0,t)}(r) - \frac{t}{\sigma^2(T)}\big]\Delta r\Big)\Delta s,$$

and the result holds. □

3. The forced pendulum equation

In this section we extend Castro's result to the context of time scales. More precisely:

Theorem 3.1. *Assume that p_0 is rd-continuous, and that $\overline{p_0} = 0$, where*

$$\overline{p_0} := \frac{1}{\sigma(T)} \int_0^{\sigma(T)} p_0(t)\Delta t.$$

Then there exist two real numbers $d(p_0)$ and $D(p_0)$ with

$$-a \le d(p_0) \le D(p_0) \le a$$

such that problem (1.1-1.3) for $p = p_0 + c$ admits at least one solution if and only if $d(p_0) \le c \le D(p_0)$.

Remark 3.2. It may be noticed that no condition on a is assumed. Thus, Theorem 3.1 is indeed a generalization of a Mawhin-Willem result (see [13]).

Remark 3.3. In the continuous case $\mathbb{T} = \mathbb{R}$ a standard variational argument shows that $d(p_0) \le 0 \le D(p_0)$. The matter of extending this result to a general time scale was considered in [2].

Proof of Theorem 3.1. Let us introduce the function $P_0(t) = \int_0^{\sigma(T)} G(t,s)p_0(s)\Delta s$, where G is given by (2.2), and consider the following equivalent problem for $u = y - P_0$:

$$u^{\Delta\Delta} + a\sin(u^\sigma + P_0^\sigma) = c, \tag{3.1}$$

under periodic conditions. Define $c : C_{rd}([0,T]_{\mathbb{T}}) \to \mathbb{R}$ by

$$c(u) = \frac{a}{\sigma(T)} \int_0^{\sigma(T)} \sin(u^\sigma + P_0^\sigma)\Delta t$$

and consider the following integro-differential equation on time scales:

$$u^{\Delta\Delta} + a\sin(u^\sigma + P_0^\sigma) = c(u) \tag{3.2}$$

Claim: For each $r \in \mathbb{R}$ problem (3.2-1.3) admits at least one solution u such that $u(0) = r$.

Indeed, for $v \in C_{rd}([0,T]_{\mathbb{T}})$ let us define $u := T^r v$ as the unique solution of the linear problem

$$\begin{gathered} u^{\Delta\Delta} + a\sin(v^\sigma + P_0^\sigma)) = c(v) \\ u(0) = u(\sigma^2(T)) = r. \end{gathered}$$

From Lemma 2.3, we have that

$$T^r(v)(t) = r + \int_0^{\sigma(T)} G(t,s)(c(v) - a\sin(v^\sigma + P_0^\sigma))\Delta s,$$

and it follows from Arzelá-Ascoli Theorem that $T^r : C_{rd}([0,T]_{\mathbb{T}}) \to C_{rd}([0,T]_{\mathbb{T}})$ is compact. Furthermore, $\|T^r(v)\|_{C_{rd}([0,T]_{\mathbb{T}})} \le C$ for some constant C, and by Schauder Theorem T^r has a fixed point u. Integrating the equation, it follows that $u^\Delta(0) = u^\Delta(\sigma(T))$, and then u is a solution of (3.2-1.3).

Next, define the set

$$E = \{u : u \text{ solves (3.1-1.3) for some } c\}.$$

It is clear that $u \in E$ if and only if u is a solution of (3.2-1.3), with $c = c(u)$. Thus, E is nonempty, and it suffices to prove that $I(p_0) := c(E)$ is a compact interval.

From the periodicity of the equation it is immediate that $c(E) = c(E_{2\pi})$, where

$$E_{2\pi} = \{u \in E : u(0) \in [0, 2\pi]\}.$$

Let $\{u_n\}$ be a sequence in $E_{2\pi}$. Using the above Green representation, it follows that $\|u_n - u_n(0)\|_{C_{rd}([0,T]_{\mathbb{T}})} \le C$ for some constant C independent of n. Moreover, as $u_n - u_n(0) = T^0(u_n)$ and $u_n(0) \in [0, 2\pi]$, there exists a subsequence that converges to a function u for the C_{rd}-norm. By Lemma 2.1 and dominated convergence we obtain that $u = u(0) + T^0(u)$, and hence $u \in E_{2\pi}$. By continuity of the function c, compactness of $I(p_0)$ follows.

In order to see that $I(p_0)$ is connected, assume that $c_1, c_2 \in I(p_0)$, $c_1 < c_2$, and let $c \in (c_1, c_2)$. Choose $u_i \in E$ such that

$$u_i^{\Delta\Delta} + a\sin(u_i^\sigma + P_0^\sigma) = c_i.$$

As u_1 and u_2 are bounded, adding a multiple of 2π if necessary, we may assume that $u_1 \ge u_2$. Hence

$$u_1^{\Delta\Delta} + a\sin(u_1^\sigma + P_0^\sigma) \le c \le u_2^{\Delta\Delta} + a\sin(u_2^\sigma + P_0^\sigma).$$

It follows that (u_2, u_1) is an ordered couple of a lower and an upper solution of the problem $u^{\Delta\Delta} + a\sin(u^\sigma + P_0^\sigma) = c$, and the proof follows from Theorem 5 in [17]. $\square$

The following proposition gives some bounds for the numbers $d(p_0)$ and $D(p_0)$.

Proposition 3.4. *Let* $K = \sup_{t\in[0,\sigma(T)]_{\mathbb{T}}} \int_0^{\sigma(T)} |G(t,s)|\,\Delta s$, *and*

$$R(p_0) = \Big[\Big(\int_0^{\sigma(T)} \cos(P_0^\sigma)\,\Delta t\Big)^2 + \Big(\int_0^{\sigma(T)} \sin(P_0^\sigma)\,\Delta t\Big)^2\Big]^{1/2}.$$

Then

$$d(p_0) \le -a\Big(\frac{R(p_0)}{\sigma(T)} - 2aK\Big)$$

and

$$D(p_0) \geq a\Big(\frac{R(p_0)}{\sigma(T)} - 2aK\Big).$$

In particular, if $a < \frac{R(p_0)}{2K\sigma(T)}$, *then* $d(p_0) < 0 < D(p_0)$.

Proof. Let v be a solution of (3.2-1.3). A simple computation shows that

$$|c(v) - c(v(0))| \leq \frac{a}{\sigma(T)} \int_0^{\sigma(T)} |v^\sigma - v(0)|\ \Delta t \leq a\|v - v(0)\|_{C_{rd}([0,\sigma(T)])},$$

and

$$|v - v(0)| \leq \int_0^{\sigma(T)} |G(t,s)|.|c(v) - a\sin(v^\sigma + P_0^\sigma)|\ \Delta t \leq 2aK.$$

Thus, $|c(v) - c(v(0))| \leq 2a^2K$, and it follows that

$$d(p_0) = \inf_{v\in E} c(v) \leq \inf_{v\in E} c(v(0)) + 2a^2K = \inf_{x\in[0,2\pi]} c(x) + 2a^2K.$$

In the same way,

$$D(p_0) \geq \sup_{x\in[0,2\pi]} c(x) - 2a^2K.$$

For $x \in \mathbb{R}$,

$$\begin{aligned} c(x) &= \frac{a}{\sigma(T)} \int_0^{\sigma(T)} \sin(x + P_0^\sigma)\ \Delta t \\ &= \frac{a}{\sigma(T)}\Big(\sin x \int_0^{\sigma(T)} \cos(P_0^\sigma)\ \Delta t + \cos x \int_0^{\sigma(T)} \sin(P_0^\sigma)\ \Delta t\Big). \end{aligned}$$

Thus,

$$\sup_{x\in[0,2\pi]} c(x) = -\inf_{x\in[0,2\pi]} c(x) = \frac{a}{\sigma(T)} R(p_0),$$

and the proof is complete. $\square$

Remark 3.5. The smallness condition on a in the previous proposition may be improved by observing that:

$$|v - v(0)| \leq K_2\Big(\int_0^{\sigma(T)} [c(v) - a\sin(v^\sigma + P_0^\sigma)]^2\ \Delta t\Big)^{1/2},$$

where

$$K_2 = \sup_{t\in[0,\sigma(T)]_{\mathbb{T}}} \Big(\int_0^{\sigma(T)} G(t,s)^2\ \Delta s\Big)^{1/2}.$$

As

$$\begin{aligned} &\int_0^{\sigma(T)} [c(v) - a\sin(v^\sigma + P_0^\sigma)]^2\ \Delta t \\ &= -a\int_0^{\sigma(T)} \sin(v^\sigma + P_0^\sigma)[c(v) - a\sin(v^\sigma + P_0^\sigma)]\ \Delta t \\ &\leq a\sigma(T)^{1/2}\Big(\int_0^{\sigma(T)} [c(v) - a\sin(v^\sigma + P_0^\sigma)]^2\ \Delta t\Big)^{1/2}, \end{aligned}$$

it follows that $|v - v(0)| \le K_2 a\sigma(T)^{1/2}$. Hence,

$$d(p_0) \le -a\Big(\frac{R(p_0)}{\sigma(T)} - aK_2\sigma(T)^{1/2}\Big),$$
$$D(p_0) \ge a\Big(\frac{R(p_0)}{\sigma(T)} - aK_2\sigma(T)^{1/2}\Big),$$

and the smallness condition for a reads:

$$a < \frac{R(p_0)}{K_2\sigma(T)^{3/2}}.$$

For example, if $\mathbb{T} = \mathbb{R}$ then $K = T^2/8$, and $K_2 = \frac{T^{3/2}}{4\sqrt{3}}$, and thus $2K\sigma(T)^{1/2} > K_2\sigma(T)^{3/2}$.

Remark 3.6. It follows from the proof of Theorem 3.1 that E is infinite. However, the interval $I(p_0) = [d(p_0), D(p_0)]$ might reduce to a single point c_0; in this case the equation is called singular, and problem (1.1-1.3) with $p = p_0 + c_0$ admits infinitely many solutions.

The problem of finding p_0 for which (1.1-1.3) is singular, or proving that such a p_0 does not exist, is still open. For the standard case $\mathbb{T} = \mathbb{R}$, Ortega and Tarallo have proved in [15] that the following statements are equivalent:

(i) $I(p_0) = \{0\}$.
(ii) For any $r \in \mathbb{R}$ there exists a unique T-periodic solution u_r of (1.1-1.3) for $p = p_0$ such that $u_r(0) = r$.
(iii) There exists a continuous path $r \to u_r$ which satisfies
$$\lim_{r\to\pm\infty} u_r(t) = \pm\infty$$
uniformly in t.

When a is small, the following proposition gives a necessary condition for singularity.

Proposition 3.7. *Let $a < \frac{1}{K}$, where K is defined as before, and assume that $I(p_0) = \{c_0\}$. Then every solution of the problem*

$$u^{\Delta\Delta} + a\sin(u^\sigma) = p_0 + c_0$$
$$u(0) = u(\sigma^2(T))$$

also satisfies: $u^\Delta(0) = u^\Delta(\sigma(T))$.

Proof. For $s, c \in \mathbb{R}$ define $u_{s,c}$ as the unique solution of the problem

$$u^{\Delta\Delta} + a\sin(u^\sigma) = p_0 + c_0 + c$$
$$u(0) = u(\sigma^2(T)) = s.$$

We claim that the operator given by $(s, c) \to u_{s,c}$ is well defined and continuous. Indeed, if u and v are solutions of the previous problem, it follows that

$$(u - v)(t) = -a\int_0^{\sigma(T)} G(t, s)[\sin(u^\sigma(s)) - \sin(v^\sigma(s))]\Delta s,$$

and hence

$$\|u - v\|_{C_{rd}([0,\sigma(T)])} \le aK\|u - v\|_{C_{rd}([0,\sigma(T)])}.$$

As $aK < 1$, it follows that $u = v$. Moreover, if $c \to \hat{c}$ and $s \to \hat{s}$, then

$$(u_{s,c} - u_{\hat{s},\hat{c}})(t) = s - \hat{s} + a \int_0^{\sigma(T)} G(t,\xi)[c - \hat{c} - \sin(u^\sigma_{s,c}(\xi)) + \sin(u^\sigma_{\hat{s},\hat{c}}(\xi))]\Delta\xi.$$

Thus,

$$(1 - aK)\|u_{s,c} - u_{\hat{s},\hat{c}}\|_{C_{rd}([0,\sigma(T)])} \le |s - \hat{s}| + aK|c - \hat{c}|$$

and continuity follows. Next, define $\theta(s,c) = u^\Delta_{s,c}(\sigma(T)) - u^\Delta_{s,c}(0)$. By definition of $u_{s,c}$ it is clear that

$$\theta(s,c) = \int_0^{\sigma(T)} [p_0 + c_0 + c - a\sin(u^\sigma_{s,c})]\Delta t = c\sigma(T) - a\int_0^{\sigma(T)} \sin(u^\sigma_{s,c})\Delta t.$$

It follows that θ is continuous, and

$$\theta(s,a) \ge 0 \ge \theta(s,-a).$$

We conclude that for each s there exists a number $c(s)$ such that $\theta(s,c(s)) = 0$. As the problem is singular, we deduce that $c(s) = 0$, and it follows that $u_{s,0}$ also satisfies: $u^\Delta_{s,0}(\sigma(T)) - u^\Delta_{s,0}(0) = 0$. □

4. Landesman-Lazer conditions for a resonant system

In this section we shall give an existence result for problem (1.2-1.3), which may be regarded as an extension of Theorem 1.1.

Remark 4.1. A different existence result for (1.2-1.3) is given in [1] Theorem 3.3, assuming that f satisfies the Hartman-type condition (see [8]):

$$\langle f(t,z), z\rangle > 0 \quad \text{for } z \in \mathbb{R}^N \text{ with } |z| = R.$$

Our Landesman-Lazer type condition reads as follows.

Condition (F1): There exists a family $\{(U_j, w_j)\}_{j=1,\dots,K}$ where U_j is an open subset of S^{N-1} and $w_j \in S^{N-1}$, such that $\{U_j\}$ covers S^{N-1} and the limit

$$\limsup_{s\to+\infty} \langle f(t,su), w_j\rangle := \overline{f}_{u,j}(t) \tag{4.1}$$

exists uniformly for $u \in U_j$.

Remark 4.2. If condition (F1) holds, then a straightforward computation shows that the mapping $u \mapsto \overline{f}_{u,j}(t)$ is continuous in U_j for each fixed t.

Theorem 4.3. *Assume that f is bounded, and that condition (F1) holds. Then the periodic boundary value problem (1.2-1.3) admits at least one solution, provided that*

(1) *For each $u \in S^{N-1}$ there exists j such that $u \in U_j$ and*

$$\int_0^{\sigma(T)} \overline{f}_{u,j}(t)d\mu < 0,$$

where μ is the measure introduced in section 2.

(2) *There exists a constant R_0 such that $d_B(F, B_R, 0) \ne 0$ for any $R \ge R_0$, where d_B is the Brouwer degree, $B_R \subset \mathbb{R}^N$ denotes the open ball of radius R centered at 0, and $F : \mathbb{R}^N \to \mathbb{R}^N$ is defined by*

$$F(y) = \int_0^{\sigma(T)} f(t,y)\Delta t.$$

Remark 4.4. It follows from the proof below that $F(y) \neq 0$ for $y \in \mathbb{R}^N$ with $|y|$ large. Thus, the Brouwer degree in condition 2 is well defined.

Proof of Theorem 4.3. For $\lambda \in [0,1]$, let us define the compact operator $T_\lambda : C_{rd}([0,T]_{\mathbb{T}}) \to C_{rd}([0,T]_{\mathbb{T}})$ given by

$$T_\lambda y(t) = y(0) + \overline{f(\cdot, y^\sigma)} + \lambda \int_0^{\sigma(T)} G(t,s) f(s, y^\sigma(s)) \Delta s.$$

For $\lambda \neq 0$, if $y = T_\lambda y$ then evaluating at $t = 0$ it follows that $\overline{f(\cdot, y^\sigma)} = 0$. Moreover, $y(\sigma^2(T)) = y(0)$, and $y^{\Delta\Delta}(t) = \lambda f(t, y^\sigma)$. Integrating this last equation, we deduce that also $y^\Delta(0) = y^\Delta(\sigma(T))$.

We claim that the solutions of the equation $y = T_\lambda y$ are a priori bounded. Indeed, if $y_n = T_{\lambda_n} y_n$ with $\lambda_n \in (0,1]$ and $\|y\|_{C_{rd}([0,\sigma(T)])} \to \infty$, then

$$\|y_n - y_n(0)\|_{C_{rd}([0,\sigma(T)])} \leq K \|f\|_C,$$

and $y_n(0) \to \infty$. Let $z_n(t) = \frac{y_n(t)}{|y_n(t)|}$, then taking a subsequence if necessary we may assume that $z_n(t) \to u \in S^{N-1}$ as $n \to \infty$, uniformly in t. Thus, for some j we have by Fatou's Lemma that

$$0 = \int_0^{\sigma(T)} \langle f(t, y_n^\sigma), w_j \rangle d\mu < 0$$

for n large, a contradiction.

On the other hand, if $y = T_0 y$ then y is constant and $F(y) = 0$. As before, if we suppose that $F(y_n) = 0$ with $|y_n| \to \infty$, a contradiction yields. We conclude that if $\Omega = B_R(0) \subset C_{rd}([0,T]_{\mathbb{T}})$ with R large enough, then the Leray-Schauder degree $d_{LS}(I - T_\lambda, \Omega, 0)$ is well defined and $d_{LS}(I - T_1, \Omega, 0) = d_{LS}(I - T_0, \Omega, 0)$. Moreover, as $T_0 y = y(0) + \overline{f(\cdot, y^\sigma)} \in \mathbb{R}^N$ for any y, it follows that

$$d_{LS}(I - T_0, \Omega, 0) = d_B((I - T_0)|_{\mathbb{R}^N}, \Omega \cap \mathbb{R}^N, 0).$$

As $(I - T_0)|_{\mathbb{R}^N} = -\sigma(T) F$, this last degree is non-zero. We conclude that the equation $y = T_1 y$ admits a solution in Ω, which corresponds to a solution of (1.2-1.3). $\square$

Some examples are now provided to illustrate the main ideas of the paper.

Example 4.5. If $f(t,y) = p(t) - g(y)$ and $g_v := \lim_{r \to +\infty} g(rv)$ exist uniformly respect to $v \in S^{N-1}$, then for any $w \in S^{N-1}$ we have that $\langle f(t, sv), w \rangle \to \langle p - g_v, w \rangle$ uniformly in S^{N-1}. If $\overline{p} \neq g_v$, then for any $v_0 \in S^{N-1}$ there exists $w \in S^{N-1}$ such that $\langle \overline{p} - g_v, w \rangle < 0$ in a neighborhood of v_0. By compactness, (F1) and the first condition of Theorem 4.3 are fulfilled. Furthermore, if the degree of the mapping $\theta(v) = \frac{g_v - \overline{p}}{|g_v - \overline{p}|}$ is non-zero, it is immediate to see that $F(y) = \int_0^{\sigma(T)} p(t) - g(y) \Delta t = \sigma(T)(\overline{p} - g(y))$ satisfies: $d_B(F, B_R, 0) \neq 0$ when R is large. Thus, Theorem 1.1 can be regarded as a particular case of Theorem 4.3 for $\mathbb{T} = \mathbb{R}$.

Example 4.6. Let $f = (f_1, \dots, f_N)$ with $f_i(t,y) = \frac{\psi_i(t,y)}{|y|^2+1} + \xi_i(t) \arctan(y_i)$, where ψ_i is continuous such that $|\psi_i(t,y)| \leq A|y|^r + B$ for some $r < 2$, and ξ is rd-continuous. Then (1.2-1.3) admits at least one solution, provided that $\int_0^{\sigma(T)} \xi_i \Delta t \neq 0$ for $i = 1, \cdots, N$. Indeed, for $y \in \mathbb{R}^N$ with $y_i \neq 0$, set $k = \operatorname{sgn}\left(\operatorname{sgn}(y_i) \int_0^{\sigma(T)} \xi_i \Delta t\right)$

and $w_i = ke_i$. Then

$$\lim_{s\to+\infty}\langle f(t,sy),w_i\rangle = k\,\mathrm{sgn}(y_i)\frac{\pi}{2}\xi_i(t) := \overline{f}_{y,w_i}(t),$$

and

$$\int_0^{\sigma(T)} \overline{f}_{y,w_i}(t)d\mu = k\,\mathrm{sgn}(y_i)\int_0^{\sigma(T)} \xi_i\ \Delta t < 0.$$

Moreover, it is easy to see that if $|y_i| \gg 0$ then

$$F_i(y).F_i(-y) < 0.$$

Thus, the second condition in Theorem 4.3 is fulfilled.

References

[1] Amster, P.; Rogers, C.; Tisdell, C. C. *Existence of solutions to boundary value problems for dynamic systems on time scales*. J. Math. Anal. Appl. 308 (2005), no. 2, 565–577.

[2] P. Amster, P. De Npoli y C.C. Tisdell. International Journal of Difference Equations 2, No. 1 (2007).

[3] Bohner, Martin; Peterson, Allan. *Dynamic equations on time scales. An introduction with applications*. Birkhäuser Boston, Inc., Boston, MA, 2001.

[4] Bohner, Martin; Guseinov, Gusein; Peterson, Allan. *Introduction to the time scales calculus*. Advances in dynamic equations on time scales, 1–15, Birkhäuser Boston, Boston, MA, 2003.

[5] Castro, Alfonso. *Periodic solutions of the forced pendulum equation*. Differential equations (Proc. Eighth Fall Conf., Oklahoma State Univ., Stillwater, Okla., 1979), pp. 149–160, Academic Press, New York-London-Toronto, Ont., 1980.

[6] Fournier, G.; Mawhin, J. *On periodic solutions of forced pendulum-like equations*. J. Differential Equations 60 (1985), no. 3, 381–395.

[7] Guseinov, Gusein Sh. *Integration on time scales*. J. Math. Anal. Appl. 285 (2003), no. 1, 107–127.

[8] Hartman, P. *On boundary value problems for systems of ordinary nonlinear second order dierential equations*. Trans. Amer. Math. Soc. 96 (1960), 493–509.

[9] Hilger, Stefan. *Analysis on measure chains–a unified approach to continuous and discrete calculus*. Results Math. 18 (1990), no. 1-2, 18–56.

[10] Landesman, E. M.; Lazer, A. C. *Nonlinear perturbations of linear elliptic boundary value problems at resonance*. J. Math. Mech. 19 1969/1970 609–623.

[11] Mawhin, J. *Topological degree methods in nonlinear boundary value problems*. Expository lectures from the CBMS Regional Conference held at Harvey Mudd College, Claremont, Calif., June 9–15, 1977. CBMS Regional Conference Series in Mathematics, 40. American Mathematical Society, Providence, R.I., 1979.

[12] Mawhin J. *Landesman-Lazer conditions for boundary value problems: A nonlinear version of resonance*. Bol. de la Sociedad Española de Mat. Aplicada 16 (2000), 45–65.

[13] Mawhin, J.; Willem, M. *Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations*. Journal of Differential Equations, Volume 52, Issue 2, April 1984, Pages 264-287

[14] Nirenberg, L. *Generalized degree and nonlinear problems*. Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971), 1–9. Academic Press, New York, 1971.

[15] Ortega, Rafael; Tarallo, Massimo. *Degenerate equations of pendulum-type*. Commun. Contemp. Math. 2 (2000), no. 2, 127–149.

[16] Spedding, Vanessa. *Taming Nature's Numbers*, New Scientist, July 19, 2003, 28–31.

[17] Stehlík, Petr. *Periodic boundary value problems on time scales*. Adv. Difference Equ. 2005, no. 1, 81–92.

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