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VISCOSITY SOLUTIONS TO DEGENERATE DIFFUSION PROBLEMS

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Abstract. This paper concerns the weak solutions to a Cauchy problem in $\mathbb{R}^N$ for a degenerate nonlinear parabolic equation. We obtain the Hölder regularity of the weak solutions to this problem.

1. Introduction

We consider the Cauchy problem

$$\begin{gathered} u_t = \alpha_1 u^{\beta_1}\Delta u + \alpha_2 u^{\beta_2} w, \quad w = \frac{1}{2}|\nabla u|^2, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+ \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^N \end{gathered} \tag{1.1}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constants and u_0 is a bounded continuous and nonnegative function on R^N, denote $\Omega = \mathbb{R}^N \times \mathbb{R}^+$.

Problem (1.1) degenerates at the points where u vanishes. Therefore, in general, it has no classical solutions and we have to consider its weak solutions. The weak solution is defined as follows.

Definition 1.1. A function $u \in L^\infty(\Omega) \cap L^2_{\rm loc}([0,+\infty); H^1_{\rm loc}(\mathbb{R}^N))$ is called a weak solution of (1.1) if $u \geq 0$ a.e. in Ω and for all $T > 0$,

$$\int_{R^N} u_0\psi(0)dx + \int_{\mathbb{R}^N\times(0,T)} u\frac{\partial\psi}{\partial t} - \alpha_1\nabla u\cdot\nabla(u^{\beta_1}\psi) + \alpha_2 u^{\beta_2}|\nabla u|^2\psi\,dx\,dt = 0$$

for all $\psi \in C^{1,1}(\mathbb{R}^N \times [0,T])$ with the compact support in $\mathbb{R}^N \times [0,T)$.

Let $u_\epsilon(x,t) \geq 0$ be the classical solution of the problem

$$\begin{gathered} u_{\epsilon t} = \alpha_1 u_\epsilon^{\beta_1}\Delta u_\epsilon + \alpha_2 u_\epsilon^{\beta_2} w_\epsilon, \quad w_\epsilon = \frac{1}{2}|\nabla u_\epsilon|^2, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+ \\ u_\epsilon(x,0) = u_0(x) + \epsilon, \quad x \in \mathbb{R}^N \end{gathered} \tag{1.2}$$

By the maximum principle $u_\epsilon(x,t)$ is decreasing with respect to ϵ, thus

$$u(x,t) = \lim_{\epsilon\to 0} u_\epsilon(x,t)$$

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is well defined in $\bar{\Omega}$. The function u is a weak solution of (1.1). Because u_0 is bounded, using the maximum principle in problem (1.2), u_ϵ is bounded and $\{u_\epsilon\}_{\epsilon\to 0}$ is uniformly bounded.

Definition 1.2. The weak solution defined above is called a viscosity solution of (1.1).

As its special cases, Bertsh, Passo, Ughi and Lu had considered the equation $u_t = u\Delta u - \gamma|\nabla u|^2$ in [1-6]. When $\alpha_1 = m$, $\beta_1 = m-1$, $\alpha_2 = 2m(m-1)$, $\beta_2 = \beta_1 - 1$, problem (1.1) is the porous medium equation, the well known case.

2. Main Result

Theorem 2.1. *If $\alpha_1 > 0, \beta_2 = \beta_1 - 1$, there exists a constant s such that*

$$2\alpha_2\beta_1 - 2\alpha_2 - s\alpha_2 + 2s(s+1)\alpha_1 + N\alpha_1\beta_1^2 \le 0$$

and

$$|\nabla(u_0^{1+\frac{s}{2}})| \le M$$

for a nonnegative constant M. Then the viscosity solution u of (1.1) satisfies $|\nabla(u^{1+\frac{s}{2}})| \le M$ in $\bar{\Omega}$.

Proof. In the definition of the viscosity solution, we let $u_\epsilon > 0$ be the classical solution of (1.2). Then

$$u(x,t) = \lim_{\epsilon\to 0} u_\epsilon(x,t)$$

is the viscosity solution of (1.1). In the following we use the notation $u_{\epsilon,\cdot}$ to denote the derivative of function u_ϵ with respect to its independent variables. At first, we have

$$\begin{aligned}
w_{\epsilon,t} &= (\frac{1}{2}|\nabla u_\epsilon|^2)_t = \sum_{i=1}^N u_{\epsilon,x_i}(u_{\epsilon,x_i})_t \\
&= \sum_{i=1}^N u_{\epsilon,x_i}(\alpha_1 u_\epsilon^{\beta_1}\Delta u_\epsilon + \alpha_2 u_\epsilon^{\beta_2} w_{\epsilon,x_i} \\
&= \sum_{i=1}^N u_{\epsilon,x_i}(\alpha_1\beta_1 u_{\epsilon,x_i} u_\epsilon^{\beta_1-1}\Delta u_\epsilon + \alpha_1 u_\epsilon^{\beta_1}\Delta u_{\epsilon,x_i} \\
&\quad + \alpha_2\beta_2 u_{\epsilon,x_i} u_\epsilon^{\beta_2-1} w_\epsilon + \alpha_2 u_\epsilon^{\beta_2} w_{\epsilon,x_i}) \\
&= 2\alpha_1\beta_1 u_\epsilon^{\beta_1-1} w_\epsilon \Delta u_\epsilon + \alpha_1 u_\epsilon^{\beta_1}\sum_{i=1}^N u_{\epsilon,x_i}\Delta u_{\epsilon,x_i} \\
&\quad + 2\alpha_2\beta_2 u_\epsilon^{\beta_2-1} w_\epsilon^2 + \alpha_2 u_\epsilon^{\beta_2}\sum_{i=1}^N u_{\epsilon,x_i} w_{\epsilon,x_i} \\
&= 2\alpha_1\beta_1 u_\epsilon^{\beta_1-1} w_\epsilon \Delta u_\epsilon + \alpha_1 u_\epsilon^{\beta_1}\Delta w_\epsilon - \alpha_1 u_\epsilon^{\beta_1}\sum_{i,j=1}^N u_{\epsilon,x_ix_j}^2 \\
&\quad + 2\alpha_2\beta_2 u_\epsilon^{\beta_2-1} w_\epsilon^2 + \alpha_2 u_\epsilon^{\beta_2}\sum_{i=1}^N u_{\epsilon,x_i} w_{\epsilon,x_i}.
\end{aligned}$$

Let

$$z_\epsilon = u_\epsilon^s w_\epsilon, \tag{2.1}$$

then

$$\begin{aligned} z_{\epsilon,t} &= u_{\epsilon,t}^s w_\epsilon + u_\epsilon^s w_{\epsilon,t} \\ &= s\alpha_1 u_\epsilon^{s+\beta_1-1} w_\epsilon \Delta u_\epsilon + s\alpha_2 u_\epsilon^{s+\beta_2-1} w_\epsilon^2 + 2\alpha_1\beta_1 u_\epsilon^{s+\beta_1-1} w_\epsilon \Delta u_\epsilon + \alpha_1 u_\epsilon^{s+\beta_1} \Delta w_\epsilon \\ &\quad - \alpha_1 u_\epsilon^{s+\beta_1} \sum_{i,j=1}^N u_{\epsilon,x_i x_j}^2 + 2\alpha_2\beta_2 u_\epsilon^{s+\beta_2-1} w_\epsilon^2 + \alpha_2 u_\epsilon^{s+\beta_2} \sum_{i=1}^N u_{\epsilon,x_i} w_{\epsilon,x_i}. \end{aligned} \tag{2.2}$$

From (2.1) and (2.2),

$$\begin{aligned} z_{\epsilon,t} &= \alpha_1 u_\epsilon^{\beta_1} \Delta z_\epsilon + (\alpha_2 u_\epsilon^{\beta_2} - 2s\alpha_1 u_\epsilon^{\beta_1-1}) \sum_{i=1}^N u_{\epsilon,x_i} z_{\epsilon,x_i} \\ &\quad + [(2\alpha_2\beta_2 - s\alpha_2) u_\epsilon^{\beta_2-s-1} + 2s(s+1)\alpha_1 u_\epsilon^{\beta_1-s-2}] z_\epsilon^2 \\ &\quad + 2\alpha_1\beta_1 u_\epsilon^{\beta_1-1} z_\epsilon \Delta u_\epsilon - \alpha_1 u_\epsilon^{s+\beta_1} \sum_{i,j=1}^N u_{\epsilon,x_i x_j}^2. \end{aligned} \tag{2.3}$$

If $\beta_2 = \beta_1 - 1, \alpha_1 > 0$, then

$$\begin{aligned} z_{\epsilon,t} &= \alpha_1 u_\epsilon^{\beta_1} \Delta z_\epsilon + (\alpha_2 - 2s\alpha_1) u_\epsilon^{\beta_1-1} \sum_{i=1}^N u_{\epsilon,x_i} z_{\epsilon,x_i} \\ &\quad + [(2\alpha_2\beta_2 - s\alpha_2) + 2s(s+1)\alpha_1] u_\epsilon^{\beta_1-s-2} z_\epsilon^2 \\ &\quad + 2\alpha_1\beta_1 u_\epsilon^{\beta_1-1} z_\epsilon \Delta u_\epsilon - \alpha_1 u_\epsilon^{s+\beta_1} \sum_{i,j=1}^N u_{\epsilon,x_i x_j}^2. \end{aligned}$$

Since

$$\sum_{i,j=1}^N u_{\epsilon,x_i x_j}^2 \ge \frac{1}{N} (\Delta u_\epsilon)^2,$$

it follows that

$$\begin{aligned} z_{\epsilon,t} \le& \alpha_1 u_\epsilon^{\beta_1} \Delta z_\epsilon + (\alpha_2 - 2s\alpha_1) u_\epsilon^{\beta_1-1} \sum_{i=1}^N u_{\epsilon,x_i} z_{\epsilon,x_i} \\ &+ [(2\alpha_2\beta_1 - 2\alpha_2 - s\alpha_2) + 2s(s+1)\alpha_1] u_\epsilon^{\beta_1-s-2} z_\epsilon^2 \\ &+ 2\alpha_1\beta_1 u_\epsilon^{\beta_1-1} z_\epsilon \Delta u_\epsilon - \frac{\alpha_1}{N} u_\epsilon^{s+\beta_1} (\Delta u)_\epsilon^2 \\ =& \alpha_1 u_\epsilon^{\beta_1} \Delta z_\epsilon + (\alpha_2 - 2s\alpha_1) u_\epsilon^{\beta_1-1} \sum_{i=1}^N u_{\epsilon,x_i} z_{\epsilon,x_i} \\ &- \Big(\sqrt{\frac{\alpha_1}{N}} u_\epsilon^{\frac{s+\beta_1}{2}} \Delta u_\epsilon - \beta_1 \sqrt{N\alpha_1} u_\epsilon^{\frac{\beta_1-s-2}{2}} z_\epsilon\Big)^2 \\ &+ [(2\alpha_2\beta_1 - 2\alpha_2 - s\alpha_2) + 2s(s+1)\alpha_1 + N\alpha_1\beta_1^2] u_\epsilon^{\beta_1-s-2} z_\epsilon^2 \end{aligned} \tag{2.4}$$

By the condition

$$2\alpha_2\beta_1 - 2\alpha_2 - s\alpha_2 + 2s(s+1)\alpha_1 + N\alpha_1\beta_1^2 \le 0$$

and (2.4), we obtain

$$z_{\epsilon,t} \le \alpha_1 u_\epsilon^{\beta_1} \Delta z_\epsilon + (\alpha_2 - 2s\alpha_1) u_\epsilon^{\beta_1 - 1} \sum_{i=1}^{N} u_{\epsilon,x_i} z_{\epsilon,x_i}.$$

Using the maximum principle, we obtain

$$\|z_\epsilon\|_\infty \le \|z_0\|_\infty.$$

Because $z_\epsilon = u_\epsilon^s w_\epsilon = \frac{1}{2} u_\epsilon^s |\nabla u_\epsilon|^2$, thus

$$\|u_\epsilon^s |\nabla u_\epsilon|^2\|_\infty \le \|u_0^s |\nabla u_0|^2\|_\infty \le M + \epsilon.$$

Since $\nabla(u_\epsilon^{1+\frac{s}{2}})$ is continuous,

$$|\nabla(u_\epsilon^{1+\frac{s}{2}})| \le M + \epsilon. \tag{2.5}$$

Because $u(x,t) = \lim_{\epsilon\to 0} u_\epsilon(x,t)$, then

$$|\nabla(u^{1+\frac{s}{2}})| \le M. \tag{2.6}$$

□

Theorem 2.2. *Suppose α_1, α_2, β_1, β_2,u_0 are as in Theorem 2.1, if there exists a nonpositive constant $s \ne -2$ satisfying*

$$2\alpha_2\beta_1 - 2\alpha_2 - s\alpha_2 + 2s(s+1)\alpha_1 + N\alpha_1\beta_1^2 \le 0,$$

then the viscosity solution $u(x,t)$ of problem (1.1) is Lipschitz continuous in x and Hölder continuous in t with exponent $1/2$ in $\overline{\Omega}$.

Proof. Because $\{u_\epsilon\}_{\epsilon\to 0}$ is uniformly bounded, $u(x,t) = \lim_{\epsilon\to 0} u_\epsilon$, so there exists a constant M_1 such that $|u| < M_1$. By Theorem 2.1, $|\nabla(u^{1+\frac{s}{2}})| \le M$, then

$$|\nabla u| \le |1 + \frac{s}{2}|^{-1} M |u^{\frac{-s}{2}}| \le |1 + \frac{s}{2}|^{-1} M M_1^{\frac{-s}{2}}.$$

Therefore, u is Lipschitz continuous with respect to x. Hence, we get directly from [7] that u is Hölder continuous in t with exponent $1/2$ in $\overline{\Omega}$. □

3. Examples

Example 3.1. Consider the problem

$$\begin{gathered} u_t = u\Delta u - \gamma|\nabla u|^2, \quad (x,t) \in \Omega \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^N \end{gathered} \tag{3.1}$$

If $\gamma \ge \sqrt{N-1}$ ($N \ne 10$), and there are constants

$$\tau = \frac{3 - \sqrt{N-1}}{2},$$
$$s = \frac{1 - \gamma - 2\tau}{2\tau} + \frac{\sqrt{2\gamma^2 - 2N + 2 - [2\tau - (3-\gamma)]^2}}{2\tau}$$

satisfying $|\nabla u_0^{\tau(1+\frac{s}{2})}| \le M$, then the viscosity solution of (3.1) is Lipschitz continuous in x and Hölder continuous in t with exponent $1/2$ in $\overline{\Omega}$.

Proof. Set $v_\epsilon = u_\epsilon^\tau$. From problem (3.1), we obtain

$$\begin{aligned}
v_{\epsilon,t} &= \tau v\Delta u_\epsilon - \tau\gamma u_\epsilon^{\tau-1}|\nabla u_\epsilon|^2\\
&= \tau v_\epsilon \sum_{i=1}^N (\frac{1}{\tau} v_\epsilon^{\frac{1}{\tau}-1} v_{\epsilon,x_i})_{x_i} - \tau\gamma v_\epsilon^{\frac{\tau-1}{\tau}} |\frac{1}{\tau} v_\epsilon^{\frac{1}{\tau}-1}\nabla v_\epsilon|^2\\
&= v_\epsilon^{\frac{1}{\tau}}\Delta v_\epsilon + v_\epsilon \sum_{i=1}^N (\frac{1}{\tau}-1) v_\epsilon^{\frac{1}{\tau}-2} v_{\epsilon,x_i}^2 - \frac{\gamma}{\tau} v_{\epsilon,}^{\frac{1}{\tau}-1}|\nabla v_\epsilon|^2\\
&= v_\epsilon^{\frac{1}{\tau}}\Delta v_\epsilon + \frac{1-\gamma-\tau}{\tau} v_\epsilon^{\frac{1}{\tau}-1}|\nabla v_\epsilon|^2.
\end{aligned}$$

In problem (1.1), with $\alpha_1 = 1$, $\beta_1 = \frac{1}{\tau}$, $\alpha_2 = \frac{2-2\gamma-2\tau}{\tau}$, $\beta_2 = \beta_1 - 1$, we have

$$\begin{aligned}
&2\alpha_2\beta_1 - 2\alpha_2 - s\alpha_2 + 2s(s+1)\alpha_1 + N\alpha_1\beta_1^2\\
&= \frac{4(1-\gamma-\tau)}{\tau^2} - \frac{4(1-\gamma-\tau)}{\tau} - \frac{2s(1-\gamma-\tau)}{\tau} + 2s(s+1) + \frac{N}{\tau^2}\\
&= 2(s - \frac{1-\gamma-2\tau}{2\tau})^2 - \frac{(1-\gamma-2\tau)^2}{2\tau^2} + \frac{4(1-\gamma-\tau)}{\tau^2} - \frac{4(1-\gamma-\tau)}{\tau} + \frac{N}{\tau^2}\\
&= 2(s - \frac{1-\gamma-2\tau}{2\tau})^2 + \frac{1}{2\tau^2}[-\gamma^2 + (4\tau-6)\gamma + 4\tau^2 - 12\tau + 2N + 7]\\
&= 0.
\end{aligned}$$

From Theorem 2.1 we get $|\nabla(u^{\tau(1+\frac{s}{2})})| \le M$. Because $\tau(1+\frac{s}{2}) - 1 \le 0$, we have

$$|\nabla u| \le |\tau(1+\frac{s}{2})|^{-1} M |u^{-\tau(1+\frac{s}{2})+1}| \le M_2.$$

We get the Hölder continuity of u with respect to t from [7] directly. □

Remark 3.2. For the case $N = 10$, we take τ as a positive number, say δ, then similar to the above arguments we can get the result that when $\gamma \ge 2\delta - 3 + \sqrt{2(2\delta-3)^2 + 2N - 2}$ and $|\nabla u_0^{\delta(1+\frac{s}{2})}| \le M$, then u is Lipschitz continuous in x and Hölder continuous in t with exponent 1/2 in $\overline{\Omega}$. Since δ is any positive number which can be taken small enough so our conclusion for the parameter γ is that when $\gamma > \sqrt{N-1}$ the solution is Lipschitz continuous in x and Hölder continuous in t with exponent 1/2 in $\overline{\Omega}$. It is a improved result of the one in [9].

Remark 3.3. Let $\tau = 1$ in Example 3.1, we could get the result as $\gamma \ge \sqrt{2N} - 1$. It is the main result in [3].

Example 3.4. *The initial problem for the porous medium equation*

$$\begin{gathered}
u_t = \Delta(u^m), \quad (x,t) \in \Omega\\
u(x,0) = u_0(x), \quad x \in \mathbb{R}^N
\end{gathered} \tag{3.2}$$

If $m > 0$, $\frac{10+2N-\sqrt{16+2N}}{7+2N} \le m \le \frac{10+2N+\sqrt{16+2N}}{7+2N}$, *and* $|\nabla(u_0^{\frac{m+2}{4}})| \le M$, *then the viscosity solution* $u(x,t)$ *is Lipschitz continuous in* x *and Hölder continuous in* t *with exponent* 1/2 *in* $\overline{\Omega}$.

Proof. In Theorem 2.2, let $\alpha_1 = m$, $\beta_1 = m-1$, $\alpha_2 = 2m(m-1)$, $\beta_2 = \beta_1 - 1$, $s = \frac{m-2}{2}$. Then the result follows. □

Example 3.5. *Consider the initial-value problem for the singular equation*

$$\begin{aligned} u_t &= \Delta u + \frac{|\nabla u|^2}{u^m}, \quad (x,t) \in \Omega \\ u(x,0) &= u_0(x) \quad x \in \mathbb{R}^N \end{aligned} \tag{3.3}$$

where $m \geq 0$. If $|\nabla u_0| \leq M$, then $u(x,t)$ is Lipschitz continuous in x and Hölder continuous in t with exponent $1/2$ in $\overline{\Omega}$.

Proof. As the proof in Theorem 2.1, In problem (1.1), we take $\alpha_1 = 1$, $\beta_1 = 0$, $\alpha_2 = 2$, $\beta_2 = -m$. From (2.3),

$$z_{\epsilon,t} \leq \Delta z_\epsilon + (2u_\epsilon^{-m} - 2su_\epsilon^{-1}) \sum_{i=1}^N u_{\epsilon,x_i} z_{\epsilon,x_i} + [(-4m - 2s)u_\epsilon^{-m+1} + 2s(s+1)] z_\epsilon^2 u_\epsilon^{-s-2}.$$

Let $s = 0$, then

$$z_{\epsilon,t} \leq \Delta z_\epsilon + 2u_\epsilon^{-m} \sum_{i=1}^N u_{\epsilon,x_i} z_{\epsilon,x_i}.$$

Thus $\|z_\epsilon\|_\infty \leq \|z_0\|_\infty$ and so $|\nabla u| \leq M$. As in the proof in Theorem 2.2, u is Lipschitz continuous in x and Hölder continuous in t with exponent $1/2$ in $\overline{\Omega}$. $\square$

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