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VERTICAL BLOW UPS OF CAPILLARY SURFACES IN $\mathbb{R}^3$, PART 1: CONVEX CORNERS

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Abstract. One technique which is useful in the calculus of variations is that of "blowing up". This technique can contribute to the understanding of the boundary behavior of solutions of boundary value problems, especially when they involve mean curvature and a contact angle boundary condition. Our goal in this note is to investigate the structure of "blown up" sets of the form $\mathcal{P}\times\mathbb{R}$ and $\mathcal{N}\times\mathbb{R}$ when $\mathcal{P},\mathcal{N}\subset\mathbb{R}^2$ and $\mathcal{P}$ (or $\mathcal{N}$) minimizes an appropriate functional; sets like $\mathcal{P}\times\mathbb{R}$ can be the limits of the blow ups of subgraphs of solutions of mean curvature problems, for example. In Part One, we investigate "blown up" sets when the domain has a convex corner. As an application, we illustrate the second author's proof of the Concus-Finn Conjecture by providing a simplified proof when the mean curvature is zero.

1. Introduction

Consider the nonparametric prescribed mean curvature problem with contact angle boundary data in the cylinder $\Omega\times\mathbb{R}$

$$Nf = H(x,f) \quad \text{for } x\in\Omega \tag{1.1}$$

$$Tf\cdot\nu = \cos\gamma \quad \text{on } \partial\Omega, \tag{1.2}$$

where $n\geq 2$, $\Omega\subset\mathbb{R}^n$ is bounded and open, $Tf=\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}$, $Nf=\nabla\cdot Tf$, ν is the exterior unit normal on $\partial\Omega$, $\gamma:\partial\Omega\to[0,\pi]$ and $f\in C^2(\Omega)$. Examples show that even if $\partial\Omega$ is smooth, a finitely valued solution f of (1.1)-(1.2) need not exist (e.g. [4]). If $\partial\Omega$ is locally Lipschitz, then one might consider formulating (1.1)-(1.2) as a variational problem. If $\partial\Omega$ is not smooth at $x_0\in\partial\Omega$ and a generalized solution f of (1.1)-(1.2) exists, the behavior of f near x_0 is often of great interest; when f is a variational solution of (1.1)-(1.2) and $H(\cdot,f(\cdot))\in L^\infty(\Omega)$, then the blow ups of f at x_0 will (usually) be minimal hypersurfaces and specific information about the behavior of these blow ups can contribute to an understanding of the behavior of f near x_0.

Suppose $f\in BV(\Omega)$ minimizes the functional $\mathcal{F}:BV(\Omega)\to\mathbb{R}$ given by

$$\mathcal{F}(g)=\int_\Omega\sqrt{1+|Dg|^2}\,dx+\int_\Omega\int_0^g H(x,t)\,dt\,dx-\int_{\partial\Omega}\cos(\gamma(x))g\,dx; \tag{1.3}$$

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that is, f is a variational solution of (1.1)-(1.2). Assume $\partial\Omega$ is locally Lipschitz, $O = (0, \dots, 0) \in \partial\Omega \subset \mathbb{R}^n$ and $H(\cdot, f(\cdot)) \in L^\infty(\Omega)$. Let $\{\epsilon_j\}$ and $\{z_j\}$ be real sequences with $\epsilon_j \downarrow 0$ as $j \to \infty$. For each integer $j \geq 1$, set $\Omega_j = \{x \in \mathbb{R}^n : \epsilon_j x \in \Omega\}$ and define $f_j \in BV(\Omega_j)$ by

$$f_j(x) = \frac{f(\epsilon_j x) - z_j}{\epsilon_j}; \tag{1.4}$$

notice that f_j minimizes

$$\mathcal{F}_j(g) = \int_{\Omega_j} \sqrt{1 + |Dg|^2}\, dx + \int_{\Omega_j} \int_0^g H_j(x, t)\, dt\, dx - \int_{\partial\Omega_j} \cos(\gamma_j(x)) g\, dx$$

and is a variational solution of $Ng = H_j(x, g)$ for $x \in \Omega_j$ and $Tg \cdot \nu_j = \cos\gamma_j$ on $\partial\Omega$, where $H_j(x, z) = \epsilon_j H(\epsilon_j x, \epsilon_j z)$, $\gamma_j(x) = \gamma(\epsilon_j x)$ and $\nu_j = \nu(\epsilon_j x)$. Notice that $H_j(\cdot, f_j(\cdot)) \to 0$ as $j \to \infty$. Suppose $\Omega_\infty = \lim_{j\to\infty} \Omega_j$, $\gamma_\infty = \lim_{j\to\infty} \gamma_j$ and $\nu_\infty = \lim_{j\to\infty} \nu_j$ exist. As in [9] (also [17, Lemma 1.2]), we can find a subsequence of $\{f_j\}$, which we continue to denote $\{f_j\}$, which converges locally to a function $f_\infty : \Omega \to [-\infty, \infty]$ in the sense that $\phi_{U_j} \to \phi_{U_\infty}$ in $L^1_{loc}(\Omega_\infty \times \mathbb{R})$ as $j \to \infty$, where $U_j = \{(x, t) \in \Omega_j \times \mathbb{R} : t < f_j(x)\}$ denotes the subgraph of f_j, $U_\infty = \{(x, t) \in \Omega_\infty \times \mathbb{R} : t < f_\infty(x)\}$ denotes the subgraph of f_∞ and ϕ_V denotes the characteristic function of a set V (e.g. [5], [6], [12]). Furthermore, f_∞ is a generalized solution of the functional

$$\mathcal{F}_\infty(g) = \int_{\Omega_\infty} \sqrt{1 + |Dg|^2}\, dx - \int_{\partial\Omega_\infty} \cos(\gamma_\infty) g\, dH_n \tag{1.5}$$

in the sense that for each compact subset K of $\mathbb{R}^{n+1}$ with finite perimeter, U_∞ minimizes the functional F_K defined on subsets of $\Omega_\infty \times \mathbb{R}$ by

$$F_K(V) = \int_{K\cap(\Omega_\infty\times\mathbb{R})} |D\phi_V| - \int_{K\cap(\partial\Omega_\infty\times\mathbb{R})} \cos(\gamma_\infty)\phi_V\ dH_n.$$

The sets

$$\mathcal{P} = \{x \in \Omega_\infty : f_\infty(x) = \infty\}, \tag{1.6}$$

$$\mathcal{N} = \{x \in \Omega_\infty : f_\infty(x) = -\infty\}, \tag{1.7}$$

have a special structure which is of principal interest to us. The set $\mathcal{P}$ minimizes the functional

$$\Phi(A) = \int_{\Omega_\infty} |D\phi_A| - \int_{\partial\Omega_\infty} \cos(\gamma_\infty)\phi_A\ dH_n. \tag{1.8}$$

and the set $\mathcal{N}$ minimizes the functional

$$\Psi(A) = \int_{\Omega_\infty} |D\phi_A| + \int_{\partial\Omega_\infty} \cos(\gamma_\infty)\phi_A\ dH_n \tag{1.9}$$

in the appropriate sense (e.g. [5], [16]). After modification on a set of measure zero, we may assume that ∂U_∞, $\partial\mathcal{P}$ and $\partial\mathcal{N}$ coincide with the essential boundaries of U_∞, $\mathcal{P}$ and $\mathcal{N}$ respectively (e.g. [5, Theorem 1.1]).

When the limiting contact angle γ_∞ satisfies certain conditions (depending on Ω_∞), the continuity at and the behavior near O of f are unknown. It would be valuable to understand the geometry of sets $\mathcal{P}$ and $\mathcal{N}$ in Ω_∞ which minimizes Φ and Ψ respectively when such conditions hold. Our goal is to determine the possible geometries of $\mathcal{P}$ and $\mathcal{N}$ when $n = 2$ and γ_∞ satisfies appropriate conditions and to illustrate the application of this knowledge by proving the Concus-Finn Conjecture in the special case that the prescribed mean curvature H is zero.

2. Statement of Results

Set $n = 2$; we will denote elements of $\mathbb{R}^2$ by (x, y) rather than by $x = (x_1, x_2)$ as in the previous section. Let Ω be an open subset of $\mathbb{R}^2$ with a corner at $O = (0, 0) \in \partial\Omega$ such that, for some $\delta_0 > 0$, $\partial\Omega$ is piecewise smooth in $B_{\delta_0}(O)$ and $\partial\Omega \cap B_{\delta_0}(O)$ consists of two C^1 arcs $\partial^+\Omega$ and $\partial^-\Omega$ whose tangent lines approach the lines $L^+ = \{\theta = \alpha\}$ and $L^- = \{\theta = -\alpha\}$, respectively, as the point O is approached. Let ν^+ and ν^- denote the exterior unit normals on $\partial^+\Omega$ and $\partial^-\Omega$ respectively. Here $\alpha \in [0, \pi]$, polar coordinates relative to O are denoted by r and θ and $B_\delta(O)$ is the ball in $\mathbb{R}^2$ of radius δ about O. Let $(x^+(s), y^+(s))$ be an arclength parametrization of $\partial^+\Omega$ and $(x^-(s), y^-(s))$ be an arclength parametrization of $\partial^-\Omega$, where $s = 0$ corresponds to the point O for both parametrizations. We will assume

$$\gamma_1 = \lim_{s\downarrow 0} \gamma(x^+(s), y^+(s)),$$
$$\gamma_2 = \lim_{s\downarrow 0} \gamma(x^-(s), y^-(s))$$

both exist and $\gamma_1, \gamma_2 \in (0, \pi)$. In this case,

$$\Omega_\infty = \{(r\cos\theta, r\sin\theta) : r > 0,\ -\alpha < \theta < \alpha\}, \tag{2.1}$$

$(\partial\Omega_\infty) \setminus \{O\} = \Sigma_1 \cup \Sigma_2$ with

$$\Sigma_j = \{(r\cos\theta, r\sin\theta) : r > 0, \theta = (-1)^{j+1}\alpha\}, \quad j = 1, 2,$$

and the limiting contact angle γ_∞ equals γ_1 on Σ_1 and γ_2 on Σ_2. A set $\mathcal{P} \subset \Omega_\infty$ minimizes Φ if and only if for each $T > 0$,

$$\Phi_T(\mathcal{P}) \le \Phi_T(\mathcal{P} \cup S), \quad \Phi_T(\mathcal{P}) \le \Phi_T(\mathcal{P} \setminus S) \quad \text{for every } S \subset \Omega_\infty^T\ ,$$

where $\Omega_\infty^T = \overline{B_T(O)} \cap \Omega_\infty$, $\Sigma_j^T = \overline{B_T(O)} \cap \Sigma_j$, $j = 1, 2$, and

$$\begin{aligned}\Phi_T(A) &= \int_{\Omega_\infty^T} |D\phi_A| - \cos(\gamma_1) \int_{\Sigma_1^T} \phi_A dH^1 - \cos(\gamma_2) \int_{\Sigma_2^T} \phi_A dH^1 \\ &= H^1(\Omega_\infty^T \cap \partial A) - \cos(\gamma_1) H^1(\Sigma_1^T \cap \partial A) - \cos(\gamma_2) H^1(\Sigma_2^T \cap \partial A)\ .\end{aligned}$$

A set $\mathcal{N} \subset \Omega_\infty$ minimizes Ψ if and only if for each $T > 0$,

$$\Psi_T(\mathcal{N}) \le \Psi_T(\mathcal{N} \cup S), \quad \Psi_T(\mathcal{N}) \le \Psi_T(\mathcal{N} \setminus S) \quad \text{for every} S \subset \Omega_\infty^T$$

where

$$\begin{aligned}\Psi_T(A) &= \int_{\Omega_\infty^T} |D\phi_A| + \cos(\gamma_1) \int_{\Sigma_1^T} \phi_A dH^1 + \cos(\gamma_2) \int_{\Sigma_2^T} \phi_A dH^1 \\ &= H^1(\Omega_\infty^T \cap \partial A) + \cos(\gamma_1) H^1(\Sigma_1^T \cap \partial A) + \cos(\gamma_2) H^1(\Sigma_2^T \cap \partial A).\end{aligned}$$

If $\mathcal{P}$ minimizes Φ, then after modification on a set of measure zero, we may assume $\partial\mathcal{P}$ coincides with the essential boundary of $\mathcal{P}$ (e.g. [5, Theorem 1.1]) and $\Omega_\infty \cap \partial\mathcal{P}$ consists of a union of rays. If $\mathcal{N}$ minimizes Ψ, then the same holds for $\partial\mathcal{N}$ and $\Omega_\infty \cap \partial\mathcal{N}$. We may also assume $\mathcal{P}$ and $\mathcal{N}$ are open.

It is convenient to introduce some notation. If A and B are points in $\mathbb{R}^2$, then OA denotes the open line segment with endpoints O and A and $\overline{OA}$ denotes the closed line segment with endpoints O and A. When the context is clear as in an arithmetic equality or inequality, OA will denote the length of the line segments

OA and $\overline{OA}$. If $A, B, C \in \overline{\Omega_\infty}$, then $\triangle ABC$ denotes the open triangular region in $\mathbb{R}^2$ with vertices A, B and C.

Theorem 2.1. *Suppose $\alpha \le \pi/2$ and $\mathcal{P} \subset \Omega_\infty$ minimizes Φ. Let (r,θ) be polar coordinates about O. Then one of the following holds:*

(i) $\mathcal{P} = \emptyset$ *or* $\mathcal{P} = \Omega_\infty$;

(ii) $\alpha < \frac{\pi}{2}$, $\gamma_1 + \gamma_2 - \pi = -2\alpha$ *and* $\mathcal{P} = \triangle AOB$, *where* $A \in \Sigma_1$, $B \in \Sigma_2$ *and angles* OAB *and* ABO *have measures* γ_1 *and* γ_2 *respectively;*

(iii) $\alpha < \frac{\pi}{2}$, $\gamma_1 + \gamma_2 - \pi = 2\alpha$ *and* $\mathcal{P} = \Omega_\infty \setminus \triangle AOB$, *where* $A \in \Sigma_1$, $B \in \Sigma_2$ *and angles* OAB *and* ABO *have measures* $\pi - \gamma_1$ *and* $\pi - \gamma_2$ *respectively;*

(iv) $\gamma_1 + \pi - \gamma_2 \le 2\alpha$ *and there exists* $A \in \Sigma_1$ *such that* $\partial\Omega_\infty \cap \partial\mathcal{P} = \Sigma_1 \setminus OA$, $\Omega_\infty \cap \partial\mathcal{P}$ *is the ray* L *in* Ω_∞ *starting at* A *and making an angle of measure* γ_1 *with* $\Sigma_1 \setminus OA$ *and* $\mathcal{P}$ *is the open sector between* $\Sigma_1 \setminus OA$ *and* L;

(v) $\gamma_2 + \pi - \gamma_1 \le 2\alpha$ *and there exists* $A \in \Sigma_1$ *such that* $\partial\Omega_\infty \cap \partial\mathcal{P} = \Sigma_2 \cup \overline{OA}$, $\Omega_\infty \cap \partial\mathcal{P}$ *is the ray* L *in* Ω_∞ *starting at* A *and making an angle of measure* γ_1 *with* OA *and* $\mathcal{P}$ *is the open connected region with boundary* $\Sigma_2 \cup \overline{OA} \cup L$;

(vi) $\gamma_1 + \pi - \gamma_2 \le 2\alpha$ *and there exists* $B \in \Sigma_2$ *such that* $\partial\Omega_\infty \cap \partial\mathcal{P} = \Sigma_1 \cup \overline{OB}$, $\Omega_\infty \cap \partial\mathcal{P}$ *is the ray* L *in* Ω_∞ *starting at* B *and making an angle of measure* γ_2 *with* OB *and* $\mathcal{P}$ *is the open connected region with boundary* $\Sigma_1 \cup \overline{OA} \cup L$;

(vii) $\gamma_2 + \pi - \gamma_1 \le 2\alpha$ *and there exists* $B \in \Sigma_2$ *such that* $\partial\Omega_\infty \cap \partial\mathcal{P} = \Sigma_2 \setminus OB$, $\Omega_\infty \cap \partial\mathcal{P}$ *is the ray* L *in* Ω_∞ *starting at* B *and making an angle of measure* γ_2 *with* $\Sigma_2 \setminus OB$ *and* $\mathcal{P}$ *is the open sector between* $\Sigma_2 \setminus OB$ *and* L;

(viii) $\gamma_1 + \pi - \gamma_2 \le 2\alpha$, $\partial\Omega_\infty \cap \partial\mathcal{P} = \Sigma_1 \cup \{O\}$, $\Omega_\infty \cap \partial\mathcal{P}$ *is a ray* $L = \{\theta = \beta\}$ *in* Ω_∞ *starting at* O *which makes an angle of measure greater than or equal to* γ_1 *with* Σ_1 *and an angle of measure greater than or equal to* $\pi - \gamma_2$ *with* Σ_2 *(i.e.* $\pi - \alpha - \gamma_2 \le \beta \le \alpha - \gamma_1$*) and* $\mathcal{P} = \{\beta < \theta < \alpha\}$; *or*

(ix) $\gamma_2 + \pi - \gamma_1 \le 2\alpha$, $\partial\Omega_\infty \cap \partial\mathcal{P} = \Sigma_2 \cup \{O\}$, $\Omega_\infty \cap \partial\mathcal{P}$ *is a ray* $L = \{\theta = \beta\}$ *in* Ω_∞ *starting at* O *which makes an angle of measure greater than or equal to* $\pi - \gamma_1$ *with* Σ_1 *and an angle of measure greater than or equal to* γ_2 *with* Σ_2 *(i.e.* $\gamma_2 - \alpha \le \beta \le \alpha + \gamma_1 - \pi$*) and* $\mathcal{P} = \{-\alpha < \theta < \beta\}$.

Theorem 2.2. *Suppose $\alpha \le \frac{\pi}{2}$ and $\mathcal{N} \subset \Omega_\infty$ minimizes Ψ. Let (r,θ) be polar coordinates about O. Then one of the following holds:*

(i) $\mathcal{N} = \emptyset$ *or* $\mathcal{N} = \Omega_\infty$;

(ii) $\alpha < \frac{\pi}{2}$, $\gamma_1 + \gamma_2 - \pi = -2\alpha$ *and* $\mathcal{N} = \Omega_\infty \setminus \triangle AOB$, *where* $A \in \Sigma_1$, $B \in \Sigma_2$ *and angles* OAB *and* ABO *have measures* γ_1 *and* γ_2 *respectively;*

(iii) $\alpha < \frac{\pi}{2}$, $\gamma_1 + \gamma_2 - \pi = 2\alpha$ *and* $\mathcal{N} = \triangle AOB$, *where* $A \in \Sigma_1$, $B \in \Sigma_2$ *and angles* OAB *and* ABO *have measures* $\pi - \gamma_1$ *and* $\pi - \gamma_2$ *respectively;*

(iv) $\gamma_1 + \pi - \gamma_2 \le 2\alpha$ *and there exists* $A \in \Sigma_1$ *such that* $\partial\Omega_\infty \cap \partial\mathcal{N} = \Sigma_2 \cup \overline{OA}$, $\Omega_\infty \cap \partial\mathcal{N}$ *is the ray* L *in* Ω_∞ *starting at* A *and making an angle of measure* $\pi - \gamma_1$ *with* OA *and* $\mathcal{N}$ *is the open connected region with boundary* $\Sigma_2 \cup \overline{OA} \cup L$;

(v) $\gamma_2 + \pi - \gamma_1 \le 2\alpha$ *and there exists* $A \in \Sigma_1$ *such that* $\partial\Omega_\infty \cap \partial\mathcal{N} = \Sigma_1 \setminus OA$, $\Omega_\infty \cap \partial\mathcal{N}$ *is the ray* L *in* Ω_∞ *starting at* A *and making an angle of measure* $\pi - \gamma_1$ *with* $\Sigma_1 \setminus OA$ *and* $\mathcal{N}$ *is the open sector between* $\Sigma_1 \setminus OA$ *and* L;

(vi) $\gamma_1 + \pi - \gamma_2 \le 2\alpha$ *and there exists* $B \in \Sigma_2$ *such that* $\partial\Omega_\infty \cap \partial\mathcal{N} = \Sigma_2 \setminus OB$, $\Omega_\infty \cap \partial\mathcal{N}$ *is the ray* L *in* Ω_∞ *starting at* B *and making an angle of measure* $\pi - \gamma_2$ *with* $\Sigma_2 \setminus OB$ *and* $\mathcal{N}$ *is the open sector between* $\Sigma_2 \setminus OB$ *and* L;

(vii) *$\gamma_2+\pi-\gamma_1 \le 2\alpha$ and there exists $B \in \Sigma_2$ such that $\partial\Omega_\infty \cap \partial\mathcal{N} = \Sigma_1 \cup \overline{OB}$, $\Omega_\infty \cap \partial\mathcal{N}$ is the ray L in Ω_∞ starting at B and making an angle of measure $\pi-\gamma_2$ with OB and $\mathcal{N}$ is the open connected region with boundary $\Sigma_1 \cup \overline{OA} \cup L$;*

(viii) *$\gamma_1+\pi-\gamma_2 \le 2\alpha$, $\partial\Omega_\infty \cap \partial\mathcal{N} = \Sigma_2 \cup \{O\}$, $\Omega_\infty \cap \partial\mathcal{N}$ is a ray $L = \{\theta = \beta\}$ in Ω_∞ starting at O which makes an angle of measure greater than or equal to γ_1 with Σ_1 and an angle of measure greater than or equal to $\pi-\gamma_2$ with Σ_2 (i.e. $\pi-\alpha-\gamma_2 \le \beta \le \alpha-\gamma_1$) and $\mathcal{P} = \{-\alpha < \theta < \beta\}$; or*

(ix) *$\gamma_2+\pi-\gamma_1 \le 2\alpha$, $\partial\Omega_\infty \cap \partial\mathcal{N} = \Sigma_1 \cup \{O\}$, $\Omega_\infty \cap \partial\mathcal{N}$ is a ray $L = \{\theta = \beta\}$ in Ω_∞ starting at O which makes an angle of measure greater than or equal to $\pi-\gamma_1$ with Σ_1 and an angle of measure greater than or equal to γ_2 with Σ_2 (i.e. $\gamma_2-\alpha \le \beta \le \alpha+\gamma_1-\pi$) and $\mathcal{P} = \{\beta < \theta < \alpha\}$.*

Figure 1 illustrates the geometry of $\mathcal{P}$ in Theorem 2.1 and $\mathcal{N}$ in Theorem 2.2. In each case, the shaded region illustrates the geometry of $\mathcal{P}$ and the unshaded region illustrates the geometry of $\mathcal{N}$.

3. Application to capillarity

Consider the stationary liquid-gas interface formed by an incompressible fluid in a vertical cylindrical tube with cross-section Ω. For simplicity, we assume that near $(0,0)$, $\partial\Omega$ has straight sides (as in [14]) and so we may assume

$$\Omega = \{(r\cos(\theta), r\sin(\theta)) : 0 < r < 1, -\alpha < \theta < \alpha\}. \tag{3.1}$$

In a microgravity environment or in a downward-oriented gravitational field, this interface will be a nonparametric surface $z = f(x,y)$ which is a solution of the boundary value problem (1.1)-(1.2) with $H(z) = \kappa z + \lambda$; that is,

$$Nf = \kappa f + \lambda \quad \text{in } \Omega \tag{3.2}$$

$$Tf\cdot\nu = \cos\gamma \quad \text{a.e. on } \partial\Omega \tag{3.3}$$

where $Tf = \nabla f/\sqrt{1+|\nabla f|^2}$, $Nf = \nabla\cdot Tf$, ν is the exterior unit normal on $\partial\Omega$, κ and λ are constants with $\kappa \ge 0$, $\gamma = \gamma(x,y) \in [0,\pi]$ is the angle at which the liquid-gas interface meets the vertical cylinder ([3]) and $\gamma_1, \gamma_1 \in (0,\pi)$ are as in §2.

Lemma 3.1. *Suppose $\alpha \le \frac{\pi}{2}$ and $\gamma_1+\pi-\gamma_2 < 2\alpha$. Let $f \in C^2(\Omega)\cap C^1(\overline{\Omega}\setminus\{O\})$ satisfy* (3.2) *and* (3.3) *and define*

$$\vec{n}(x,y) = \big(Tf(x,y), \frac{-1}{\sqrt{1+|\nabla f(x,y)|^2}}\big) \tag{3.4}$$

to be the (downward) unit normal to the graph of f at $(x,y,f(x,y))$. Let $\beta \in (-\alpha,\alpha)$ and let $\{(x_j,y_j)\}$ be a sequence in Ω satisfying $\lim_{j\to\infty}(x_j,y_j) = (0,0)$ and

$$\lim_{j\to\infty} \frac{(x_j,y_j)}{\sqrt{x_j^2+y_j^2}} = (\cos(\beta), \sin(\beta)). \tag{3.5}$$

(i) *If $\beta \in [-\alpha+\pi-\gamma_2, \alpha-\gamma_1]$, then $\lim_{j\to\infty}\vec{n}(x_j,y_j) = (-\sin(\beta), \cos(\beta), 0)$.*

(ii) *If $\beta \in (-\alpha, -\alpha+\pi-\gamma_2]$, then*
$\lim_{j\to\infty}\vec{n}(x_j,y_j) = (-\sin(-\alpha+\pi-\gamma_2), \cos(-\alpha+\pi-\gamma_2), 0)$.

(iii) *If $\beta \in [\alpha-\gamma_1, \alpha)$, $\lim_{j\to\infty}\vec{n}(x_j,y_j) = (-\sin(\alpha-\gamma_1), \cos(\alpha-\gamma_1), 0)$.*

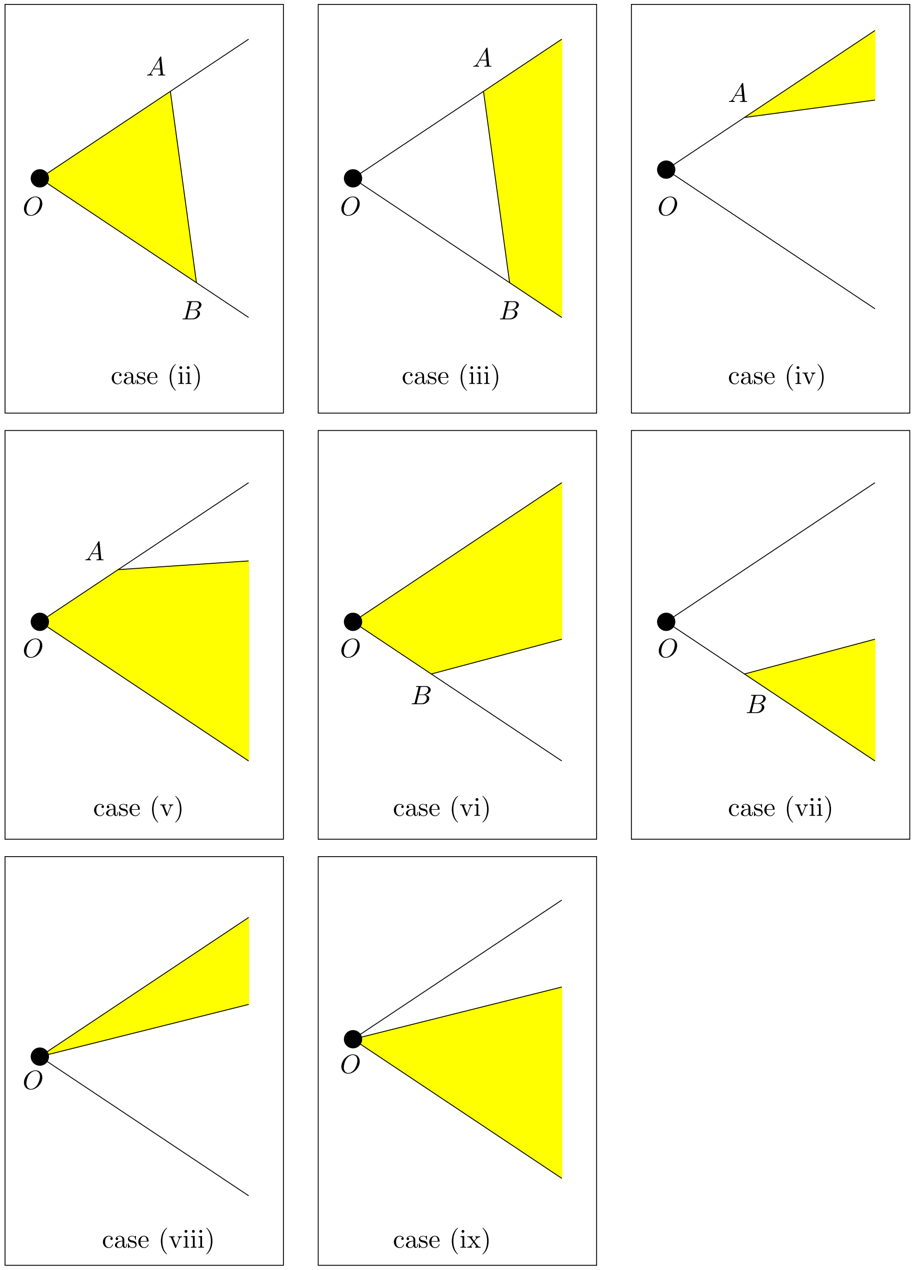

FIGURE 1. Illustration for Theorems 2.1 and 2.2

Proof. Suppose $\{(x_j, y_j) : j \in \mathbb{N}\}$ is a sequence in Ω converging to $(0,0)$ as $j \to \infty$ and satisfying (3.5); we may assume $x_j > 0$ for all $j \in \mathbb{N}$. For each $j \in \mathbb{N}$, define $f_j \in C^2(\Omega_j) \cap C^1(\overline{\Omega_j} \setminus \{O\})$ by (1.4) with $z_j = f(x_j, y_j)$ and $\epsilon_j = \sqrt{x_j^2 + y_j^2}$, so that

$$f_j(x,y) = \frac{f(\epsilon_j x, \epsilon_j y) - f(x_j, y_j)}{\epsilon_j}.$$

Let U_j be the subgraph of f_j as in §1 and $\vec{n}_j$ be the downward unit normal to the graph of f_j; that is,

$$\vec{n}_j(x,y) = \big(Tf_j(x,y), \frac{-1}{\sqrt{1+|\nabla f_j(x,y)|^2}}\big), \quad (x,y)\in\Omega_j.$$

As in §1, there exists a subsequence of $\{(x_j,y_j)\}$, still denoted $\{(x_j,y_j)\}$, and a generalized solution $f_\infty : \Omega_\infty \to [-\infty,\infty]$ of (1.5) such that f_j converges to f_∞ in the sense that $\phi_{U_j} \to \phi_{U_\infty}$ in $L^1_{loc}(\Omega_\infty \times \mathbb{R})$ as $j\to\infty$, where Ω_∞ is given in (2.1). Let $\mathcal{P}$ and $\mathcal{N}$ be given by (1.6) and (1.7) respectively. Notice that $f_j(x_j/\epsilon_j, y_j/\epsilon_j) = 0$ for all $j\in\mathbb{N}$ and so $f_\infty(\cos(\beta),\sin(\beta)) = 0$. Using "density lower bounds" (e.g. [18, Lemma 3.1]), we see that $(\cos(\beta),\sin(\beta),0) \in \partial U_\infty$ and $(\cos(\beta),\sin(\beta)) \in \Omega_\infty \setminus (\mathcal{P}\cup\mathcal{N})$; hence $\mathcal{P}\neq\Omega_\infty$ and $\mathcal{N}\neq\Omega_\infty$. Since $\gamma_1+\pi-\gamma_2<2\alpha$, either $\mathcal{P}=\emptyset$ or one of cases (iv), (vi) or (viii) of Theorem 2.1 holds and either $\mathcal{N}=\emptyset$ or one of cases (iv), (vi) or (viii) of Theorem 2.2 holds.

We claim that $(\Omega_\infty\times\mathbb{R})\cap\partial U_\infty$ is the portion of a plane Π in $\Omega_\infty\times\mathbb{R}$, with $(\cos(\beta),\sin(\beta),0)\in\Pi$. There are two possibilities to consider. The first is that there exists $\epsilon>0$ such that $B_\epsilon(\cos(\beta),\sin(\beta))\cap(\mathcal{P}\cup\mathcal{N})=\emptyset$. The second is that $(\cos(\beta),\sin(\beta))\in\partial\mathcal{P}\cup\partial\mathcal{N}$.

Suppose the first case holds and let $\mathcal{G}=\Omega_\infty\setminus(\overline{\mathcal{P}}\cup\overline{\mathcal{N}})$. Notice that f_∞ is a classical (i.e. $C^2(\mathcal{G})$) solution of the minimal surface equation in $\mathcal{G}$ (e.g. [5], [11]). Let $h(x,y)=f(x,y)-Rf(\beta)$, where $Rf(\beta)=\lim_{r\downarrow 0} f(r\cos(\beta), r\sin(\beta))$ ([9]), and define $h_j:\Omega_j\to\mathbb{R}$ by

$$h_j(x,y) = \frac{f(\epsilon_j x,\epsilon_j y)-Rf(\beta)}{\epsilon_j} = \frac{h(\epsilon_j x,\epsilon_j y)}{\epsilon_j} \quad \text{for } j\in\mathbb{N}.$$

Set $c_j = \frac{f(x_j,y_j)-Rf(\beta)}{\epsilon_j}$ and observe that $h_j(x,y)=f_j(x,y)+c_j$. Hence

$$\nabla h_j(x,y) = \nabla f_j(x,y), \quad (x,y)\in\mathcal{G}.$$

From [10, Theorem 3], we see that

$$\lim_{j\to\infty}\nabla f_j(x,y) = \nabla f_\infty(x,y) \quad \text{for } (x,y)\in\mathcal{G} \tag{3.6}$$

and so $\lim_{j\to\infty}\nabla h_j(x,y)=\nabla f_\infty(x,y)$ for $(x,y)\in\mathcal{G}$. Set $E=\{(x,y,z)\in\Omega\times\mathbb{R} : z<h(x,y)\}$ and $V_j=\{(x,y,z)\in\Omega_j\times\mathbb{R} : z<h_j(x,y)\}$, $j\in\mathbb{N}$. Notice that $(r\cos(\beta), r\sin(\beta), h(r\cos(\beta), r\sin(\beta)))\in\partial E$ for $r>0$ and so $(0,0,0)\in\partial E$ since $(r\cos(\beta), r\sin(\beta), h(r\cos(\beta), r\sin(\beta)))\to(0,0,0)$ as $r\downarrow 0$. By [14, Theorem 4.5], $E_t=\{(x,y,z) : t(x,y,z)\in E\}$ converges to a (solid) minimal cone C (with vertex at $(0,0,0)$) in $\mathbb{R}^3$ in the sense that $\phi_{E_t}\to\phi_C$ in $L^1_{loc}(\mathbb{R}^3)$ as $r\downarrow 0$. Now $(\Omega_\infty\times\mathbb{R})\cap\partial C$ is a minimal surface which is a cone with vertex at $(0,0,0)$ and hence is a portion of a plane. Notice that $V_j=E_{\epsilon_j}$ and so h_j converges to h_∞, where h_∞ denote the generalized solution with subgraph ϕ_C, in the sense that $\phi_{V_j}\to\phi_C$ in $L^1_{loc}(\mathbb{R}^3)$ as $j\to\infty$. By [10, Theorem 3] and (3.6), we see that

$$\lim_{j\to\infty}\nabla h_j(x,y) = \nabla h_\infty(x,y) \quad \text{for } (x,y)\in\mathcal{G},$$

∂C is a nonvertical plane, $\nabla f_\infty = \nabla h_\infty$ is a constant vector and $(\Omega_\infty \times \mathbb{R}) \cap \partial U_\infty$ is the portion of a nonvertical plane Π in $\Omega_\infty \times \mathbb{R}$. Since $\vec{n}_j(x_j/\epsilon_j, y_j/\epsilon_j) = \vec{n}(x_j, y_j)$, we see that $\lim_{j\to\infty} \vec{n}(x_j, y_j) = (\nabla h_\infty, -1)/\sqrt{|\nabla h_\infty|^2 + 1}$.

Suppose the second case holds. We may assume $(\cos(\beta), \sin(\beta)) \in \partial\mathcal{P}$; then $(\cos(\beta), \sin(\beta), 0)$ is a point on $\partial\mathcal{P} \times \mathbb{R}$ and, in a neighborhood of $(\cos(\beta), \sin(\beta), 0)$, ∂U_∞ is a real analytic surface. Since $\mathcal{P} \times \mathbb{R} \subset U_\infty$, this real analytic surface contains a portion Λ of $\partial\mathcal{P} \times \mathbb{R}$ with $(\cos(\beta), \sin(\beta), 0) \in \Lambda$. This implies that the component of $\partial U_\infty \cap (\Omega_\infty \times \mathbb{R})$ which contains $(\cos(\beta), \sin(\beta), 0)$ is $(\partial\mathcal{P} \cap \Omega_\infty) \times \mathbb{R}$. Since $\vec{n}_j(x_j/\epsilon_j, y_j/\epsilon_j) = \vec{n}(x_j, y_j)$, we see from [10, Theorem 3] that

$$\lim_{j\to\infty} \vec{n}(x_j, y_j) = \xi = (\xi_1, \xi_2, 0), \tag{3.7}$$

where ξ is the normal to $\partial\mathcal{P} \times \mathbb{R}$ at $(\cos(\beta), \sin(\beta), 0)$ which points into $\mathcal{P} \times \mathbb{R}$.

If we let $\xi = (\xi_1, \xi_2, \xi_3)$ denote the unit normal to Π at $(\cos(\beta), \sin(\beta), 0)$ pointing into U_∞, then it is easy to see (as Concus and Finn observed in [2]) that ξ_3 cannot be less than 0 (i.e. no plane can meet $\Sigma_1 \times \mathbb{R}$ in angle γ_1 and $\Sigma_2 \times \mathbb{R}$ in angle γ_2) and so $\xi_3 = 0$. Hence $U_\infty = \mathcal{P} \times \mathbb{R}$, where $\mathcal{P}$ is given in one of (iv), (vi) or (viii) of Theorem 2.1 and $\mathcal{N} = \Omega_\infty \setminus \overline{\mathcal{P}}$.

Suppose $\beta \in [-\alpha + \pi - \gamma_2, \alpha - \gamma_1]$ holds. From Theorem 2.1, we see that case (viii) must hold. Since $\partial\mathcal{P}$ is a line going through $(0, 0)$ and $(\cos(\beta), \sin(\beta))$, we have $\xi = (-\sin(\beta), \cos(\beta), 0)$ and (i.) of Lemma 3.1 follows from (3.7).

Suppose $\beta \in (-\alpha, -\alpha + \pi - \gamma_2]$ holds. Then case (vi) of Theorem 2.1 holds, $\xi = (-\sin(-\alpha + \pi - \gamma_2), \cos(-\alpha + \pi - \gamma_2), 0)$ and (ii.) of Lemma 3.1 is established.

Finally, suppose $\beta \in [\alpha - \gamma_1, \alpha)$. Then case (iv) of Theorem 2.1 holds, $\xi = (-\sin(\alpha - \gamma_1), \cos(\alpha - \gamma_1), 0)$ and (iii.) of Lemma 3.1 is established. $\square$

Remark 3.2. In [13] and [14], Shi assumes Ω is a wedge domain (i.e. (3.1), although with $\frac{\pi}{2} < \alpha < \pi$); for simplicity, we consider (convex) wedge domains. However, as noted in [13] and [14], the assumption that $\partial\Omega$ has straight sides near O can be relaxed. She also assumes $\gamma = \gamma_1$ on $\{\theta = \alpha\}$ and $\gamma = \gamma_2$ on $\{\theta = -\alpha\}$. Neither of these assumptions has a significant impact on the critical monotonicity estimates (i.e. (8) and page 339, [14]) which imply that the limit C of the $\{V_j\}$ is a (solid, minimal) cone with vertex at $(0, 0, 0)$. Also, the hypothesis of [14, Theorem 4.5] only requires f to satisfy (1.1) (with $n = 2$ and H bounded near O) rather than (3.2). Thus, we may assume in this section that Ω and γ are as described in §2 and f satisfies (1.1) and (3.3).

Using (v), (vii) and (ix) of Theorems 2.1 and 2.2 and the techniques used in the proof of Lemma 3.1, we see that the following holds.

Lemma 3.3. *Suppose $\alpha \le \frac{\pi}{2}$ and $\gamma_2 + \pi - \gamma_1 < 2\alpha$. Let $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\})$ satisfy* (3.2) *and* (3.3) *and define $\vec{n}(x, y)$ by (3.4). Let $\beta \in (-\alpha, \alpha)$ and let $\{(x_j, y_j)\}$ be a sequence in Ω satisfying $\lim_{j\to\infty}(x_j, y_j) = (0, 0)$ and (3.5).*

(i) *If $\beta \in [-\alpha + \gamma_2, \alpha + \gamma_1 - \pi]$, then* $\lim_{j\to\infty} \vec{n}(x_j, y_j) = (\sin(\beta), -\cos(\beta), 0)$.

(ii) *If $\beta \in (-\alpha, -\alpha + \gamma_2]$, then* $\lim_{j\to\infty} \vec{n}(x_j, y_j) = (\sin(-\alpha + \gamma_2), -\cos(-\alpha + \gamma_2), 0)$.

(iii) *If $\beta \in [\alpha + \gamma_1 - \pi, \alpha)$, then*

$$\lim_{j\to\infty} \vec{n}(x_j, y_j) = (\sin(\alpha + \gamma_1 - \pi), -\cos(\alpha + \gamma_1 - \pi), 0).$$

The Concus-Finn Conjecture is the conjecture that if $\kappa \geq 0$, $0 < \alpha < \frac{\pi}{2}$ and $\pi - |\gamma_1 - \gamma_2| < 2\alpha$, then every solution of (3.2)-(3.3) must be discontinuous at $O = (0,0)$ (see, for example, [8, 13] and [14]). The second author proved the Concus-Finn Conjecture in [8]. Here we wish to illustrate the applicability of Theorems 2.1 and 2.2 to calculus of variations problems in $\mathbb{R}^3$ by providing a proof of the Concus-Finn Conjecture in the special case that $\kappa = 0$ and $\lambda = 0$. (We note that while searching for a proof of the conjecture, this case was the first which the second author considered and its proof provided the roadmap for the proof in the general case.)

Theorem 3.4 (Concus-Finn Conjecture with $\kappa = \lambda = 0$)**.** *Suppose* $0 < \alpha < \frac{\pi}{2}$ *and* $\kappa = \lambda = 0$ *in* (3.2)*. Suppose further that* $\pi - |\gamma_1 - \gamma_2| < 2\alpha$*. Then every solution of* (3.2)-(3.3) *must be discontinuous at* $O = (0,0)$*.*

Proof. We may suppose $\gamma_1 + \pi - \gamma_2 < 2\alpha$, $\kappa = \lambda = 0$ in (3.2), $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\})$ satisfies (3.2) and (3.3) and f is continuous at O. The graph of f, $G = \{(x, y, f(x, y)) : (x, y) \in \Omega\}$, may be represented in isothermal coordinates. It follows from the arguments in [7] and [9] that there exists $X : \overline{B} \to \mathbb{R}^3$, $X(u, v) = (x(u, v), y(u, v), z(u, v))$, (with $B = \{(u, v) : u^2 + v^2 < 1\}$) such that

- X is a homeomorphism of $\overline{B}$ and $\overline{G}$ and a diffeomorphism of B and G,
- $X_u \cdot X_v = 0$ and $|X_u| = |X_v|$ on B,
- $\triangle X(u, v) = (0, 0, 0)$ for $(u, v) \in B$.

Since f is continuous at O, we know, in addition, $K(u, v) = (x(u, v), y(u, v))$ is a homeomorphism of $\overline{B}$ and $\overline{\Omega}$; if f were discontinuous at O, then there would be a closed, nontrivial interval $I \subset \partial B$ such that $K(u, v) = (0, 0)$ for each $(u, v) \in I$ [7, 9]. We will assume $X(1, 0) = (0, 0, f(0, 0))$.

Consider the Gauss map $\vec{n} \circ K$ of G; in particular, recall that since G has zero mean curvature, the stereographic projection (from the north pole) of the Gauss map, which we denote $g : B \to \mathbb{R}^2$, is an analytic function of $u + iv$ when we consider B and $\mathbb{R}^2$ as being in the complex plane. Notice that $g : B \to B$.

Let $m_\theta : [0, \delta] \to B$ be defined by $K(m_\theta(t)) = t(\cos(\theta), \sin(\theta))$ for each $\theta \in (-\alpha, \alpha)$. Let θ_1 and θ_2 satisfy $-\alpha + \pi - \gamma_2 < \theta_1 < \theta_2 < \alpha - \gamma_1$. From Lemma 3.1, we see that

$$\lim_{t \downarrow 0} g(m_\theta(t)) = -\sin(\theta) + i\cos(\theta) \quad \text{for each } \theta \in [\theta_1, \theta_2]$$

and, in particular, $|\lim_{t\downarrow 0} g(m_\theta(t))| = 1$ for $\theta \in (\theta_1, \theta_2)$.

On the other hand, the Phragmen-Lindelof Theorem for (bounded) analytic functions applied to g (when restricted to a suitable subdomain of B and composed with a conformal map) implies that $\lim_{t\downarrow 0} g(m_\theta(t))$ lies on the line segment joining $-\sin(\theta_1) + i\cos(\theta_1)$ and $-\sin(\theta_2) + i\cos(\theta_2)$ and hence $|\lim_{t\downarrow 0} g(m_\theta(t))| < 1$ for $\theta \in (\theta_1, \theta_2)$ (e.g. [1]). This contradiction implies that our assumption that f was continuous at O was incorrect. □

4. Auxiliary Lemmas

In this section, let $\mathcal{P}$ denote a minimizer of Φ and $\mathcal{N}$ denote a minimizer of Ψ. We do not assume $\alpha \leq \frac{\pi}{2}$ since the results here will also be used in Part Two, where $\frac{\pi}{2} < \alpha < \pi$. After modification on a set of measure zero, we may assume

(i) each component of $\Omega_\infty \cap \partial\mathcal{P}$ (and of $\Omega_\infty \cap \partial\mathcal{N}$) is a connected component of the intersection of Ω_∞ and a line;

(ii) $\Omega_\infty \cap \partial\mathcal{P}$ consists of the union of rays (and lines) in Ω_∞ which do not intersect in Ω_∞ and

(iii) $\Omega_\infty \cap \partial\mathcal{N}$ consists of the union of rays (and lines) in Ω_∞ which do not intersect in Ω_∞.

These items follow since the sets $\partial\mathcal{P} \times \mathbb{R}$ and $\partial\mathcal{N} \times \mathbb{R}$ are ruled minimal surfaces in $\mathbb{R}^3$ which are area minimizing with respect to compact perturbations (e.g. [16]).

Lemma 4.1. *For each $r, T \in \mathbb{R}$ with $0 < r < T$, there are only finitely many components of $\Omega_\infty \cap \partial\mathcal{P}$ (or of $\Omega_\infty \cap \partial\mathcal{N}$) which intersect $\Omega_\infty^T \setminus \Omega_\infty^r$.*

Proof. Suppose $\{M_n : n \in \mathbb{N}\}$ is a countably infinite collection of distinct components of $\Omega_\infty \cap \partial\mathcal{P}$, $0 < r < T$ and $M_n \cap \Omega_\infty^T \setminus \Omega_\infty^r \neq \emptyset$ for each $n \in \mathbb{N}$. Let $D \in \overline{\Omega_\infty^T \setminus \Omega_\infty^r}$ be an accumulation point of $\{M_n\}$. If $D \in \Omega_\infty$, pick $\epsilon > 0$ such that $B_{2\epsilon}(D) \subset \Omega_\infty$. If $D \in \Sigma_1$, pick $\epsilon > 0$ such that $B_{2\epsilon}(D) \cap \overline{\Sigma_2} = \emptyset$. If $D \in \Sigma_2$, pick $\epsilon > 0$ such that $B_{2\epsilon}(D) \cap \overline{\Sigma_1} = \emptyset$. Since D is an accumulation point of $\{M_n\}$, there is a subsequence $\{M_{n_k}\}$ of $\{M_n\}$ such that $M_{n_k} \cap B_\epsilon(D) \neq \emptyset$ for each $k \in \mathbb{N}$. Hence $H^1(\Omega_\infty \cap M_{n_k}) \geq \epsilon$ for each k. Since $H^1(\Sigma_1^T \cup \Sigma_2^T) < \infty$. we see that $\Phi_T(\mathcal{P}) = \infty$, a contradiction. The argument for $\Omega_\infty \cap \partial\mathcal{N}$ is similar. □

Remark 4.2. We will later show in Lemma 4.18 that for each $T > 0$, $\Omega_\infty \cap \partial\mathcal{P}$ and $\Omega_\infty \cap \partial\mathcal{N}$ each have only finitely many components which intersect Ω_∞^T. We note that if $\alpha \geq \pi/2$ or if an infinite number of components of $\Omega_\infty \cap \partial\mathcal{P}$ (or of $\Omega_\infty \cap \partial\mathcal{N}$) contain O as an endpoint, then the proof of Lemma 4.1 already yields this result. However, if $\alpha < \pi/2$, the fact that $\Phi_T(\mathcal{P}) < \infty$ (or $\Psi_T(\mathcal{N}) < \infty$) does not suffice to exclude an infinite number of components of $\Omega_\infty \cap \partial\mathcal{P}$ (or of $\Omega_\infty \cap \partial\mathcal{N}$) from intersecting Ω_∞^T and this is illustrated in the following example.

Example 4.3. Let $0 < \alpha < \pi/2$ and set $S = \cup_{n=1}^\infty (2^{-2n-1}, 2^{-2n}) \times \mathbb{R}$ and $U = \Omega_\infty \cap S$. Notice that $\Omega_\infty \cap \partial U = \cup_{n=2}^\infty M_n$, where

$$M_n = \{2^{-n}\} \times (-2^{-n}\tan(\alpha), 2^{-n}\tan(\alpha)),$$

and O is an accumulation point of the sequence $\{M_n : n \in \mathbb{N}\}$. If $T > 1$, then $H^1(\Omega_\infty^T \cap \partial U) = \tan(\alpha)$ and $\Phi_T(U) < \infty$ and $\Psi_T(U) < \infty$ for any choice of γ_1 and γ_2.

Lemma 4.4. *If M_1 and M_2 are two distinct components of $\Omega_\infty \cap \partial\mathcal{P}$ (or of $\Omega_\infty \cap \partial\mathcal{N}$) and $\overline{M_1} \cap \overline{M_2} = \{A\}$, then $A = O$ and the angle between M_1 and M_2 has measure greater than or equal to π.*

Proof. Suppose two components of $\Omega_\infty \cap \partial\mathcal{P}$ intersect at a point $A \in \partial\Omega_\infty$ and, if $A = O$, then they meet in an angle of measure less than π. Then either one component $\mathcal{P}_1$ of $\mathcal{P}$ is a convex wedge with vertex A or one component $\mathcal{N}_1$ of $\Omega_\infty \setminus \mathcal{P}$ is a convex wedge with vertex A. Suppose the first case holds and let

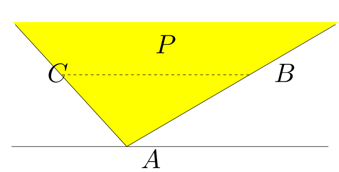

FIGURE 2. Convex wedge 1

I_1 and I_2 be the components of $\Omega_\infty \cap \partial\mathcal{P}_1$; then $\overline{I_1} \cap \overline{I_2} = \{A\}$. Let $B \in I_1$, $C \in I_2$ and $T > \max\{OA, OB, OC\}$ (see Figure 2). Since $\mathcal{P}$ minimizes Φ, $\phi_T(\mathcal{P}) \le \phi_T(\mathcal{P} \setminus \triangle ABC)$ and so

$$AC + AB \le BC,$$

which is a contradiction. Thus the first case cannot hold.

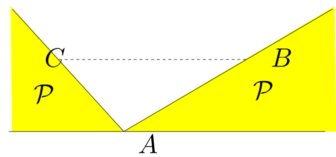

FIGURE 3. Convex wedge 2

Suppose the second case holds and let I_1 and I_2 be the components of $\Omega_\infty \cap \partial\mathcal{N}_1$ and $B \in I_1$, $C \in I_2$ (see Figure 3). Let T be as above. Since $\mathcal{P}$ minimizes Φ, $\phi_T(\mathcal{P}) \le \phi_T(\mathcal{P} \cup \triangle ABC)$ and so $AC + AB \le BC$, which is a contradiction. Thus the second case cannot hold. The argument for components of $\Omega_\infty \cap \partial\mathcal{N}$ is essentially the same. □

Remark 4.5. We may state Lemma 4.4 informally as follows:

> Two components of $\Omega_\infty \cap \partial\mathcal{P}$ (or of $\Omega_\infty \cap \partial\mathcal{N}$) cannot meet on $\Sigma_1 \cup \Sigma_2$. Two components of $\Omega_\infty \cap \partial\mathcal{P}$ (or of $\Omega_\infty \cap \partial\mathcal{N}$) which meet at O meet in a angle of measure greater than or equal to π.

Lemma 4.6. *Suppose $A \in \overline{\Sigma_1} \cap \partial\mathcal{P} \cap \partial(\Omega_\infty \setminus \mathcal{P})$ and $B \in \Sigma_1 \setminus \{A\}$ with $AB \subset \partial\mathcal{P}$. Let $\mathcal{P}_1$ denote the connected component of $\mathcal{P}$ whose closure contains AB. Then the measure of the angle $\mathcal{P}_1$ makes at A is greater than or equal to γ_1.*

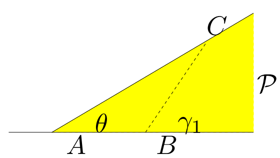

FIGURE 4. $\theta < \gamma_1$

Proof. From Lemma 4.4 and (i)-(iii), we see that only one component of $\mathcal{P}$ contains AB in its closure; let us denote this component by $\mathcal{P}_1$. Denote by θ the measure of the angle at A formed by $\partial\mathcal{P}_1$ and assume $\theta < \gamma_1$. Let C be the point on $\Omega_\infty \cap \partial\mathcal{P}_1$ for which the angle CBA has measure $\pi - \gamma_1$ (see Figure 4). Since $\mathcal{P}$ minimizes Φ, $\phi_T(\mathcal{P}) \le \phi_T(\mathcal{P} \setminus \triangle ABC)$ for T large. Hence $AC - \cos(\gamma_1)AB \le BC$. Now $AC = (\sin(\gamma_1)/\sin(\gamma_1 - \theta))AB$ and $BC = (\sin(\theta)/\sin(\gamma_1 - \theta))AB$ and so $\sin(\gamma_1) - \cos(\gamma_1)\sin(\gamma_1 - \theta) \le \sin(\theta)$. A short calculation shows that this implies $1 \le \cos(\gamma_1 - \theta)$, which contradicts the assumption that $\theta < \gamma_1$. □

Lemma 4.7. *Suppose $A \in \overline{\Sigma_2} \cap \partial\mathcal{P} \cap \partial(\Omega_\infty \setminus \mathcal{P})$ and $B \in \Sigma_2 \setminus \{A\}$ with $AB \subset \partial\mathcal{P}$. Let $\mathcal{P}_1$ denote the connected component of $\mathcal{P}$ whose closure contains AB. Then the measure of the angle $\mathcal{P}_1$ makes at A is greater than or equal to γ_2.*

Lemma 4.8. *Suppose $A \in \Sigma_1 \cap \partial\mathcal{P} \cap \partial(\Omega_\infty \setminus \mathcal{P})$. Then A lies in the closure of exactly one connected component $\mathcal{P}_1$ of $\mathcal{P}$ and $\partial\mathcal{P}_1$ makes an angle of measure exactly γ_1 at A.*

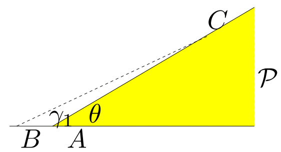

Figure 5. $\theta > \gamma_1$

Proof. Since $A \neq O$, Lemma 4.4 implies that only one component $\mathcal{P}_1$ of $\mathcal{P}$ contains A in its closure. Let θ be the measure of the angle $\partial\mathcal{P}_1$ makes at A and assume $\theta > \gamma_1$. Let $B \in \Sigma_1$ and $C \in \Omega_\infty \cap \partial\mathcal{P}_1$ such that $B \notin \overline{\mathcal{P}_1}$ and angle CBA has measure γ_1 (see Figure 5). Now $\phi_T(\mathcal{P}) \leq \phi_T(\mathcal{P} \cup \triangle ABC)$ for large T and so $AC \leq BC - \cos(\gamma_1)AB$. Now $AC = (\sin(\gamma_1)/\sin(\theta-\gamma_1))AB$ and $BC = (\sin(\theta)/\sin(\theta-\gamma_1))AB$ and so

$$\sin(\gamma_1) \leq \sin(\theta) - \cos(\gamma_1)\sin(\theta-\gamma_1).$$

A short calculation shows that this implies $1 \leq \cos(\theta-\gamma_1)$, which contradicts the assumption that $\theta > \gamma_1$. □

Lemma 4.9. *Suppose $O \in \partial\mathcal{P} \cap \partial(\Omega_\infty \setminus \mathcal{P})$, $A \in \Omega_\infty \cap \partial\mathcal{P}$, $B \in \Sigma_1$, $\triangle AOB \subset \Omega_\infty \setminus \mathcal{P}$ and θ_1 is the measure of the angle AOB. Then $\theta_1 \geq \pi - \gamma_1$.*

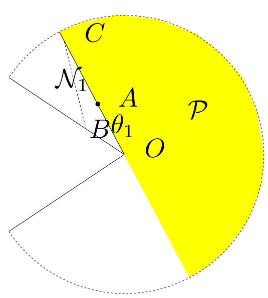

Figure 6. $O \in \partial\mathcal{P} \cap \partial(\Omega_\infty \setminus \mathcal{P})$

Proof. Assume $\theta_1 < \pi - \gamma_1$. Let $\mathcal{N}_1$ be the component of $\Omega_\infty \setminus \mathcal{P}$ which contains $\triangle AOB$. Pick $C \in \Omega_\infty \cap \partial\mathcal{N}_1$ such that angle CBO has measure γ_1 (see Figure 6). Now $\phi_T(\mathcal{P}) \leq \phi_T(\mathcal{P} \cup \triangle CBO)$ for large T and so $OC \leq BC - \cos(\gamma_1)OB$. Since $OC = (\sin(\gamma_1)/\sin(\theta_1+\gamma_1))OB$ and $BC = (\sin(\theta_1)/\sin(\theta_1+\gamma_1))OB$, we see that

$$\sin(\gamma_1) \leq \sin(\theta_1) - \cos(\gamma_1)\sin(\theta_1+\gamma_1).$$

A short calculation shows that this implies $1 \leq \cos(\pi - \theta_1 - \gamma_1)$, which contradicts the assumption that $\theta_1 < \pi - \gamma_1$. □

Lemma 4.10. *Suppose $A \in \Sigma_2 \cap \partial\mathcal{P} \cap \partial(\Omega_\infty \setminus \mathcal{P})$. Then A lies in the closure of exactly one connected component $\mathcal{P}_1$ of $\mathcal{P}$ and $\partial\mathcal{P}_1$ makes an angle of measure exactly γ_2 at A.*

Lemma 4.11. *Suppose $O \in \partial\mathcal{P} \cap \partial(\Omega_\infty \setminus \mathcal{P})$, $A \in \Omega_\infty \cap \partial\mathcal{P}$, $B \in \Sigma_2$, $\triangle AOB \subset \Omega_\infty \setminus \mathcal{P}$ and θ_2 is the measure of the angle AOB. Then $\theta_2 \geq \pi - \gamma_2$.*

Lemma 4.12. *Suppose $A \in \overline{\Sigma_1} \cap \partial\mathcal{N} \cap \partial(\Omega_\infty \setminus \mathcal{N})$ and $B \in \Sigma_1 \setminus \{A\}$ with $AB \subset \partial\mathcal{N}$. Let $\mathcal{N}_1$ denote the connected component of $\mathcal{N}$ whose closure contains AB. Then the measure of the angle $\mathcal{N}_1$ makes at A is greater than or equal to $\pi - \gamma_1$.*

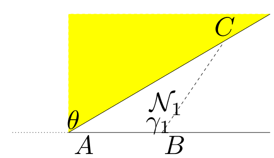

FIGURE 7. $\pi - \theta < \pi - \gamma_1$

Proof. From Lemma 4.4 and (i)-(iii), we see that only one component of $\mathcal{N}$ contains AB in its closure; let us denote this component by $\mathcal{N}_1$. Denote by $\pi - \theta$ the measure of the angle at A formed by $\partial\mathcal{N}_1$ and assume $\theta > \gamma_1$. Let C be the point on $\Omega_\infty \cap \partial\mathcal{N}_1$ for which the angle CBA has measure γ_1 (see Figure 7). Since $\mathcal{N}$ minimizes Ψ, $\psi_T(\mathcal{N}) \leq \psi_T(\mathcal{N} \setminus \triangle ABC)$ for T large. Hence $AC + \cos(\gamma_1)AB \leq BC$. Now $AC = (\sin(\gamma_1)/\sin(\theta - \gamma_1))AB$ and $BC = (\sin(\theta)/\sin(\theta - \gamma_1)AB$ and so $\sin(\gamma_1) + \cos(\gamma_1)\sin(\theta - \gamma_1) \leq \sin(\theta)$. A short calculation shows that this implies $1 \leq \cos(\theta - \gamma_1)$, which contradicts the assumption that $\theta > \gamma_1$. Thus $\theta \leq \gamma_1$ and so the measure $\pi - \theta$ of the angle which $\mathcal{N}_1$ makes at A is greater than or equal to $\pi - \gamma_1$. □

Lemma 4.13. *Suppose $A \in \overline{\Sigma_2} \cap \partial\mathcal{N} \cap \partial(\Omega_\infty \setminus \mathcal{N})$ and $B \in \Sigma_2 \setminus \{A\}$ with $AB \subset \partial\mathcal{N}$. Let $\mathcal{N}_1$ denote the connected component of $\mathcal{N}$ whose closure contains AB. Then the measure of the angle $\mathcal{N}_1$ makes at A is greater than or equal to $\pi - \gamma_2$.*

Lemma 4.14. *Suppose $A \in \Sigma_1 \cap \partial\mathcal{N} \cap \partial(\Omega_\infty \setminus \mathcal{N})$. Then A lies in the closure of exactly one connected component $\mathcal{N}_1$ of $\mathcal{N}$ and $\partial\mathcal{N}_1$ makes an angle of measure exactly $\pi - \gamma_1$ at A.*

Lemma 4.15. *Suppose $A \in \Sigma_2 \cap \partial\mathcal{N} \cap \partial(\Omega_\infty \setminus \mathcal{N})$. Then A lies in the closure of exactly one connected component $\mathcal{N}_1$ of $\mathcal{N}$ and $\partial\mathcal{N}_1$ makes an angle of measure exactly $\pi - \gamma_2$ at A.*

Lemma 4.16. *Suppose $O \in \partial\mathcal{N} \cap \partial(\Omega_\infty \setminus \mathcal{N})$, $A \in \Omega_\infty \cap \partial\mathcal{N}$, $B \in \Sigma_1$, $\triangle AOB \subset \Omega_\infty \setminus \mathcal{N}$ and θ_1 is the measure of the angle AOB. Then $\theta_1 \geq \gamma_1$.*

Lemma 4.17. *Suppose $O \in \partial\mathcal{N} \cap \partial(\Omega_\infty \setminus \mathcal{N})$, $A \in \Omega_\infty \cap \partial\mathcal{N}$, $B \in \Sigma_2$, $\triangle AOB \subset \Omega_\infty \setminus \mathcal{N}$ and θ_2 is the measure of the angle AOB. Then $\theta_2 \geq \gamma_2$.*

The proofs of Lemmas 4.7, 4.10, 4.11 and 4.13 follow those of Lemmas 4.6, 4.8, 4.9 and 4.12 respectively. The proofs of Lemmas 4.14 and 4.15 are similar to that of Lemma 4.8 while the proofs of Lemmas 4.16 and 4.17 are similar to that of Lemma 4.9.

Lemma 4.18. *For each $T > 0$, only finitely many components of $\Omega_\infty \cap \partial\mathcal{P}$ (or of $\Omega_\infty \cap \partial\mathcal{N}$) intersect Ω_∞^T.*

Proof. Example 4.3 illustrates the only case with which we need to deal. Suppose $0 < \alpha < \pi/2$, $T > 0$ and $\Omega_\infty^T \cap \partial\mathcal{P} = \cup_{n=1}^\infty M_n$, where $M_n = A_nB_n$, $A_n \in \Sigma_1$, $B_n \in \Sigma_2$, $OA_{n+1} < OA_n$ and $OB_{n+1} < OB_n$ for each $n \in \mathbb{N}$ and $A_n \to O$, $B_n \to O$ as $n \to \infty$. Let $\mathcal{P}_1$ be a component of $\mathcal{P}$ whose boundary contains M_3. Then either $\partial\mathcal{P}_1 = M_2 \cup M_3 \cup \overline{A_2A_3} \cup \overline{B_2B_3}$ or $\partial\mathcal{P}_1 = M_3 \cup M_4 \cup \overline{A_3A_4} \cup \overline{B_3B_4}$. Let us assume the first case holds. Notice then by Lemmas 4.8 and 4.9 that $\partial\mathcal{P}$ makes angles of measure γ_1 at A_2 and A_3 and angles of measure γ_2 at B_2 and B_3. Hence $2\gamma_1 + 2\gamma_2 = 2\pi$ or $\gamma_1 + \gamma_2 = \pi$; this implies Σ_1 and Σ_2 are parallel, a contradiction. A similar argument applied to $\Omega_\infty \cap \partial\mathcal{N}$ shows that $2(\pi - \gamma_1) + 2(\pi - \gamma_2) = 2\pi$ or $\gamma_1 + \gamma_2 = \pi$ again. □

Lemma 4.19. *Suppose $\alpha \geq \frac{\pi}{2}$. Then:*

(a) *Two components of $\Omega_\infty \cap \partial\mathcal{P}$ for which the closure of one (or both) of the components is disjoint from $\partial\Omega_\infty$ cannot be parallel.*
(b) *Two components of $\Omega_\infty \cap \partial\mathcal{N}$ for which the closure of one (or both) of the components is disjoint from $\partial\Omega_\infty$ cannot be parallel.*

Proof. Suppose M_1 and M_2 are distinct components of $\Omega_\infty \cap \partial\mathcal{P}$ such that M_1 is a line or ray in Ω_∞, M_2 is a line in Ω_∞ and M_1 and M_2 are parallel. We may assume that either

(i) M_1 and M_2 lie on the boundary of a component $\mathcal{P}_1$ of $\mathcal{P}$ or
(ii) M_1 and M_2 lie on the boundary of a component $\mathcal{N}_1$ of $\Omega_\infty \setminus \mathcal{P}$.

Let $A \in M_1$ and $B \in M_2$ be fixed with AB orthogonal to M_1 and M_2. Now pick $C \in M_1$ and $D \in M_2$ such that AB and CD are parallel and $AB = CD < AC = BD$. Fix $T > \max\{OA, OB, OC, OD\}$. Assuming (i.), the minimality of P implies $\Phi_T(\mathcal{P}) \leq \Phi_T(\mathcal{P} \setminus ACDB)$ or $AC+BD \leq AB+CD$. However $AC+BD > AB+CD$ and we have a contradiction. Assuming (ii.), the minimality of P implies $\Phi_T(\mathcal{P}) \leq \Phi_T(\mathcal{P} \cup ACDB)$ or $AC + BD \leq AB + CD$. However $AC + BD > AB + CD$ and we have a contradiction. This proves the Lemma when M_1 and M_2 are distinct components of $\partial\mathcal{P}$. The proof when M_1 and M_2 are distinct components of $\partial\mathcal{N}$ is similar. □

Lemma 4.20. *Let $\Omega_1 = \{0 < \theta < \alpha\}$ and $\Omega_2 = \{-\alpha < \theta < 0\}$. Then no component of $\Omega_1 \cap \partial\mathcal{P}$ nor of $\Omega_1 \cap \partial\mathcal{N}$ can be parallel to Σ_1 and no component of $\Omega_2 \cap \partial\mathcal{P}$ nor of $\Omega_2 \cap \partial\mathcal{N}$ can be parallel to Σ_2.*

Proof. Suppose L is a component of $\Omega_\infty \cap \partial\mathcal{P}$ such that $L_1 = L \cap \Omega_\infty \neq \emptyset$ and L_1 is parallel to Σ_1. From (i)-(iii) of §4, we know that L is either a ray starting at a point on Σ_2 (if $\alpha < \frac{\pi}{2}$) or a line in Ω_∞ (if $\alpha \geq \frac{\pi}{2}$) and, in either case, L_1 is a ray in Ω_1 starting at a point $A = (a, 0)$. Let $B \in \Sigma_1$ and set $C(= C(B)) = B + (a, 0) \in L_1$. Select $T > \max\{OA, OB, OC\}$. Let Δ denote the open region with boundary $OACB$. Now the open region $U \subset \Omega_1$ with boundary $\Sigma_1 \cup L_1 \cup OA$ is a component of $\Omega_1 \cap \mathcal{P}$ or a component of $\Omega_1 \cap \mathcal{N}$.

Suppose first that U is a component of $\Omega_1 \cap \mathcal{P}$. Since

$$\Phi_T(\mathcal{P}) \leq \Phi_T(\mathcal{P} \setminus \Delta),$$

we obtain

$$AC - \cos(\gamma_1)\, OB \leq BC + OA.$$

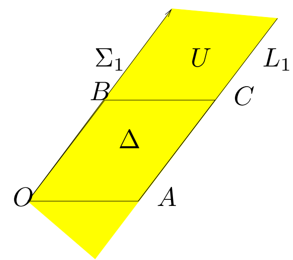

FIGURE 8. $U \subset \mathcal{P}$

Since $BC = OA$ is fixed (independent of B), $AC = OB$ and $|\cos(\gamma_1)| < 1$, we obtain

$$OB \le \frac{2OA}{1-\cos(\gamma_1)}$$

for all $B \in \Sigma_1$, which is impossible for sufficiently large OB.

Suppose second that U is a component of $\Omega_1 \setminus \mathcal{P}$. Since

$$\Phi_T(\mathcal{P}) \le \Phi_T(\mathcal{P} \cup \Delta),$$

we obtain $AC \le BC + OA - \cos(\gamma_1)\ OB$. Since $BC = OA$ is fixed (independent of B), $AC = OB$ and $|\cos(\gamma_1)| < 1$, we obtain

$$OB \le \frac{2OA}{1+\cos(\gamma_1)}$$

for all $B \in \Sigma_1$, which is impossible for sufficiently large OB.

The case in which L is a component of $\Omega_\infty \cap \partial\mathcal{N}$ and $L_1 = L \cap \Omega_\infty$ is parallel to Σ_1 follows by a similar argument. The cases in which L is a component of $\Omega_\infty \cap \partial\mathcal{P}$ or of $\Omega_\infty \cap \partial\mathcal{N}$ such that $L_2 = L \cap \Omega_\infty$ is parallel to Σ_2 follow using similar comparison arguments. □

5. Proofs for convex corners: Theorems 2.1, 2.2

In this section, we assume $\alpha \le \frac{\pi}{2}$ and let $\mathcal{P}$ and $\mathcal{N}$ denote minimizers of Φ and Ψ respectively. Notice that the only situations in which differences between the geometry when $\alpha = \frac{\pi}{2}$ and $\alpha < \frac{\pi}{2}$ occur first in cases (ii) and (iii), where we assume $\alpha < \frac{\pi}{2}$, second when $\alpha = \frac{\pi}{2}$ and components of $\partial\mathcal{P}$ or $\partial\mathcal{N}$ might be distinct parallel lines in Ω_∞, a possibility eliminated by Lemma 4.19, and third when $\alpha = \frac{\pi}{2}$ and $\partial\mathcal{P}$ (or $\partial\mathcal{N}$) contains a single line parallel to $\partial\Omega_\infty$. This last possibility cannot occur as a simple comparison argument shows. Thus we may assume $\alpha < \frac{\pi}{2}$ subsequently. The arguments used here are similar to those of Tam ([15]). The conclusions of Theorems 2.1 and 2.2 follow from the eight claims proven here.

Claim (1). Every component of $\Omega_\infty \cap \partial\mathcal{P}$ is unbounded and every component of $\Omega_\infty \cap \partial\mathcal{N}$ is unbounded unless $|\gamma_1 + \gamma_2 - \pi| = 2\alpha$.

Suppose that $\gamma_1 + \gamma_2 - \pi \ne -2\alpha$ and $M = AB$ is a bounded component of $\Omega_\infty \cap \partial\mathcal{P}$, where $A \in \Sigma_1$ and $B \in \Sigma_2$. Notice that no component of $\Omega_\infty \cap \partial\mathcal{P}$ can have O in its closure since otherwise $\Omega_\infty \cap \partial\mathcal{P}$ would contain a ray in Ω_∞ with O as one endpoint and this ray would intersect M, in contradiction to (i)-(iii). Hence $O \in \overline{\mathcal{P}} \setminus \overline{\Omega_\infty \setminus \mathcal{P}}$ or $O \in \overline{\Omega_\infty \setminus \mathcal{P}} \setminus \overline{\mathcal{P}}$.

In the first case, there exist $C \in \Sigma_1$ and $D \in \Sigma_2$ such that $\triangle COD$ is a component of $\mathcal{P}$ and $\triangle COD \subset \triangle AOB$. From Lemmas 4.8 and 4.9, we see that $\triangle COD$ makes angles of measure γ_1 at C, γ_2 at D and 2α at O. Therefore $\gamma_1 + \gamma_2 + 2\alpha = \pi$ or $\gamma_1 + \gamma_2 - \pi = -2\alpha$, a contradiction.

In the second case, there exist $C \in \Sigma_1$ and $D \in \Sigma_2$ such that $\triangle COD$ is a component of $\Omega_\infty \setminus \mathcal{P}$. From Lemmas 4.8 and 4.9, we see that $\triangle COD$ makes angles of measure $\pi - \gamma_1$ at C, $\pi - \gamma_2$ at D and 2α at O. Therefore $\pi - \gamma_1 + \pi - \gamma_2 + 2\alpha = \pi$ or $\gamma_1 + \gamma_2 - \pi = 2\alpha$, a contradiction. The argument for $\Omega_\infty \setminus \mathcal{N}$ is similar.

Claim (2). $\Omega_\infty \cap \partial\mathcal{P}$ and $\Omega_\infty \cap \partial\mathcal{N}$ have at most one component.

Suppose M_1 and M_2 are distinct components of $\Omega_\infty \cap \partial\mathcal{P}$ such that M_1 and M_2 lie on the boundary of a component $\mathcal{P}_1$ of $\mathcal{P}$ or a component $\mathcal{N}_1$ of $\Omega_\infty \setminus \mathcal{P}$. Now Lemma 4.4 implies M_1 and M_2 are either parallel or the lines L_1 and L_2 on which they respectively lie intersect at a point E outside of $\overline{\Omega_\infty}$. We will consider the various cases individually and show that each leads to a contradiction.

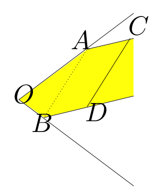

FIGURE 9. Case (a)

(a) M_1 and M_2 are parallel, $M_1 \cup M_2 \subset \partial\mathcal{P}_1$, $L_1 \cap \Sigma_1 \neq \emptyset$ and $L_2 \cap \Sigma_2 \neq \emptyset$. Let $A \in L_1 \cap \Sigma_1$ and $B \in L_2 \cap \Sigma_2$. We will pick $C \in M_1$ and $D \in M_2$ such that AB and CD are parallel (see Figure 9); we let $T > \max\{OA, OB, OC, OD\}$. Now $\Phi_T(\mathcal{P}) \leq \Phi_T(\mathcal{P} \setminus OACDB)$, where $OACDB$ denotes the open polygon with vertices O, A, C, D and B. Then

$$AC + BD \leq \cos(\gamma_1)OA + \cos(\gamma_2)OB + CD.$$

Since the lengths OA, OB and CD are fixed, this inequality is false when AC and BD are sufficiently large.

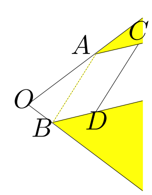

FIGURE 10. Case (b)

(b) M_1 and M_2 are parallel, $M_1 \cup M_2 \subset \partial\mathcal{N}_1$, $L_1 \cap \Sigma_1 \neq \emptyset$ and $L_2 \cap \Sigma_2 \neq \emptyset$. Let $A \in L_1 \cap \Sigma_1$ and $B \in L_2 \cap \Sigma_2$. We will pick $C \in M_1$ and $D \in M_2$ such that AB and CD are parallel (see Figure 10); we let $T > \max\{OA, OB, OC, OD\}$. Now $\Phi_T(\mathcal{P}) \leq \Phi_T(\mathcal{P} \cup OACDB)$, where $OACDB$ denotes the open polygon with vertices O, A, C, D and B. Then

$$AC + BD \leq CD - \cos(\gamma_1)OA - \cos(\gamma_2)OB.$$

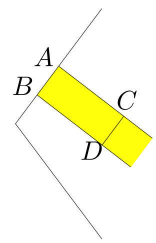

FIGURE 11. Case (c)

Since the lengths OA, OB and CD are fixed, this inequality is false when AC and BD are sufficiently large.

(c) M_1 and M_2 are parallel, $M_1 \cup M_2 \subset \partial\mathcal{P}_1$, $L_1 \cap \Sigma_1 \neq \emptyset$ and $L_2 \cap \overline{\Sigma_1} \neq \emptyset$. Let $A \in L_1 \cap \Sigma_1$ and $B \in L_2 \cap \overline{\Sigma_1}$. We will pick $C \in M_1$ and $D \in M_2$ such that AB and CD are parallel (see Figure 11); we let $T > \max\{OA, OB, OC, OD\}$. Now $\Phi_T(\mathcal{P}) \leq \Phi_T(\mathcal{P} \setminus ACDB)$, where $ACDB$ denotes the open polygon with vertices A, C, D and B. Then

$$AC + BD \leq \cos(\gamma_1)AB + CD.$$

Since the lengths AB and CD are fixed, this inequality is false when AC and BD are sufficiently large.

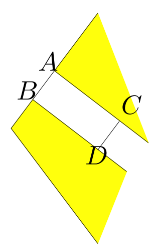

FIGURE 12. Case (d)

(d) M_1 and M_2 are parallel, $M_1 \cup M_2 \subset \partial\mathcal{N}_1$, $L_1 \cap \Sigma_1 \neq \emptyset$ and $L_2 \cap \overline{\Sigma_1} \neq \emptyset$. Let $A \in L_1 \cap \Sigma_1$ and $B \in L_2 \cap \overline{\Sigma_1}$. We will pick $C \in M_1$ and $D \in M_2$ such that AB and CD are parallel (see Figure 12); we let $T > \max\{OA, OB, OC, OD\}$. Now $\Phi_T(\mathcal{P}) \leq \Phi_T(\mathcal{P} \cup ACDB)$, where $ACDB$ denotes the open polygon with vertices A, C, D and B. Then

$$AC + BD \leq CD - \cos(\gamma_1)AB.$$

Since the lengths AB and CD are fixed, this inequality is false when AC and BD are sufficiently large.

(e) M_1 and M_2 are parallel, $M_1 \cup M_2 \subset \partial\mathcal{P}_1$, $L_1 \cap \overline{\Sigma_2} \neq \emptyset$ and $L_2 \cap \Sigma_2 \neq \emptyset$. Let $A \in L_1 \cap \overline{\Sigma_2}$ and $B \in L_2 \cap \Sigma_2$. We will pick $C \in M_1$ and $D \in M_2$ such that AB and CD are parallel (see Figure 13); we let $T > \max\{OA, OB, OC, OD\}$. Now

$$\Phi_T(\mathcal{P}) \leq \Phi_T(\mathcal{P} \setminus ACDB),$$

where $ACDB$ denotes the open polygon with vertices A, C, D and B. Then

$$AC + BD \leq \cos(\gamma_2)AB + CD.$$

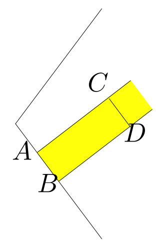

FIGURE 13. Case (e)

Since the lengths AB and CD are fixed, this inequality is false when AC and BD are sufficiently large.

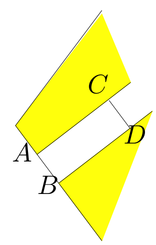

FIGURE 14. Case (f)

(f) M_1 and M_2 are parallel, $M_1 \cup M_2 \subset \partial\mathcal{N}_1$, $L_1 \cap \overline{\Sigma_2} \neq \emptyset$ and $L_2 \cap \Sigma_2 \neq \emptyset$. Let $A \in L_1 \cap \overline{\Sigma_2}$ and $B \in L_2 \cap \Sigma_2$. We will pick $C \in M_1$ and $D \in M_2$ such that AB and CD are parallel (see Figure 14); we let $T > \max\{OA, OB, OC, OD\}$. Now $\Phi_T(\mathcal{P}) \leq \Phi_T(\mathcal{P} \cup ACDB)$, where $ACDB$ denotes the open polygon with vertices A, C, D and B. Then

$$AC + BD \leq CD - \cos(\gamma_2)AB.$$

This inequality is false when AC and BD are sufficiently large.

(g) M_1 and M_2 are not parallel, $M_1 \cup M_2 \subset \partial\mathcal{P}_1$, $L_1 \cap \Sigma_1 \neq \emptyset$ and $L_2 \cap \overline{\Sigma_1} \neq \emptyset$. Let $A \in L_1 \cap \Sigma_1$ and $B \in L_2 \cap \overline{\Sigma_1}$. We will pick $C \in M_1$ and $D \in M_2$ such that AB and CD are parallel (see Figure 15); we let $T > \max\{OA, OB, OC, OD\}$. Now $\Phi_T(\mathcal{P}) \leq \Phi_T(\mathcal{P} \setminus ACDB)$, where $ACDB$ denotes the open polygon with vertices A, C, D and B. Then

$$AC + BD - \cos(\gamma_1)AB \leq CD.$$

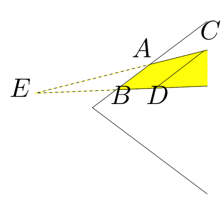

FIGURE 15. Case (g)

Now $AC = EC - EA$ and $BD = ED - EB$ and we have

$$EC + ED \le CD + EA + EB + \cos(\gamma_1)AB.$$

Since the lengths EA, EB and AB are fixed, this inequality is false when EC and ED are sufficiently large.

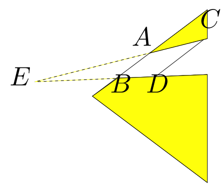

FIGURE 16. Case (h)

(h) M_1 and M_2 are not parallel, $M_1 \cup M_2 \subset \partial\mathcal{N}_1$, $L_1 \cap \Sigma_1 \neq \emptyset$ and $L_2 \cap \overline{\Sigma_1} \neq \emptyset$. Let $A \in L_1 \cap \Sigma_1$ and $B \in L_2 \cap \overline{\Sigma_1}$. We will pick $C \in M_1$ and $D \in M_2$ such that AB and CD are parallel (see Figure 16); we let $T > \max\{OA, OB, OC, OD\}$. Now $\Phi_T(\mathcal{P}) \le \Phi_T(\mathcal{P} \cup ACDB)$, where $ACDB$ denotes the open polygon with vertices A, C, D and B. Then

$$AC + BD \le CD - \cos(\gamma_1)AB.$$

Now $AC = EC - EA$ and $BD = ED - EB$ and we have

$$EC + ED \le CD + EA + EB - \cos(\gamma_1)AB.$$

This inequality is false when EC and ED are sufficiently large.

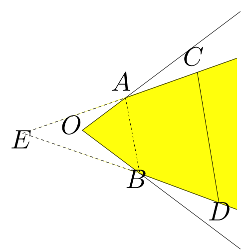

FIGURE 17. Case (i)

(i) M_1 and M_2 are not parallel, $M_1 \cup M_2 \subset \partial\mathcal{P}_1$, $L_1 \cap \Sigma_1 \neq \emptyset$ and $L_2 \cap \Sigma_2 \neq \emptyset$ and Let $A \in L_1 \cap \Sigma_1$ and $B \in L_2 \cap \Sigma_2$. We will pick $C \in M_1$ and $D \in M_2$ such that AB and CD are parallel (see Figure 17); we let $T > \max\{OA, OB, OC, OD\}$. Now $\Phi_T(\mathcal{P}) \le \Phi_T(\mathcal{P} \setminus OACDB)$, where $OACDB$ denotes the open polygon with vertices O, A, C, D and B. $AC + BD - \cos(\gamma_1)OA - \cos(\gamma_2)OB \le CD$, $AC = EC - EA$ and $BD = ED - EB$ implies

$$EC + ED \le CD + EA + EB + \cos(\gamma_1)OA + \cos(\gamma_2)OB.$$

Since the lengths EA, EB, OA and OB are fixed, this inequality is false when EC and ED are sufficiently large.

(j) M_1 and M_2 are not parallel, $M_1 \cup M_2 \subset \partial\mathcal{N}_1$, $L_1 \cap \Sigma_1 \neq \emptyset$ and $L_2 \cap \Sigma_2 \neq \emptyset$. Let $A \in L_1 \cap \Sigma_1$ and $B \in L_2 \cap \Sigma_2$. We will pick $C \in M_1$ and $D \in M_2$ such that

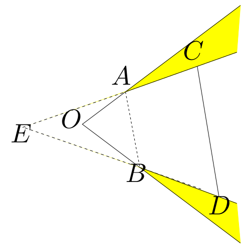

Figure 18. Case (j)

AB and CD are parallel (see Figure 18); we let $T > \max\{OA, OB, OC, OD\}$. Now $\Phi_T(\mathcal{P}) \le \Phi_T(\mathcal{P} \cup OACDB)$, where $OACDB$ denotes the open polygon with vertices O, A, C, D and B. $AC + BD \le CD - \cos(\gamma_1)OA - \cos(\gamma_2)OB$, $AC = EC - EA$ and $BD = ED - EB$ implies

$$EC + ED \le CD + EA + EB - \cos(\gamma_1)OA - \cos(\gamma_2)OB.$$

This inequality is false when EC and ED are sufficiently large.

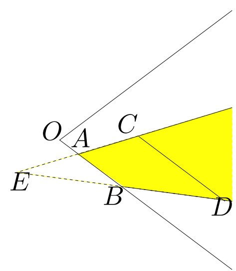

Figure 19. Case (k)

(k) M_1 and M_2 are not parallel, $M_1 \cup M_2 \subset \partial\mathcal{P}_1$, $L_1 \cap \overline{\Sigma_2} \neq \emptyset$ and $L_2 \cap \Sigma_2 \neq \emptyset$. Let $A \in L_1 \cap \overline{\Sigma_2}$ and $B \in L_2 \cap \Sigma_2$. We will pick $C \in M_1$ and $D \in M_2$ such that AB and CD are parallel (see Figure 19); we let $T > \max\{OA, OB, OC, OD\}$. Now $\Phi_T(\mathcal{P}) \le \Phi_T(\mathcal{P} \setminus ACDB)$, where $ACDB$ denotes the open polygon with vertices A, C, D and B. Then

$$AC + BD - \cos(\gamma_2)AB \le CD.$$

Now $AC = EC - EA$ and $BD = ED - EB$ and we have

$$EC + ED \le CD + EA + EB + \cos(\gamma_2)AB.$$

Since the lengths EA, EB and AB are fixed, this inequality is false when EC and ED are sufficiently large.

(l) M_1 and M_2 are not parallel, $M_1 \cup M_2 \subset \partial\mathcal{N}_1$, $L_1 \cap \overline{\Sigma_2} \neq \emptyset$ and $L_2 \cap \Sigma_2 \neq \emptyset$. Let $A \in L_1 \cap \overline{\Sigma_2}$ and $B \in L_2 \cap \Sigma_2$. We will pick $C \in M_1$ and $D \in M_2$ such that AB and CD are parallel (see Figure 20); we let $T > \max\{OA, OB, OC, OD\}$. Now $\Phi_T(\mathcal{P}) \le \Phi_T(\mathcal{P} \cup ACDB)$, where $ACDB$ denotes the open polygon with vertices A, C, D and B. Then

$$AC + BD \le CD - \cos(\gamma_2)AB.$$

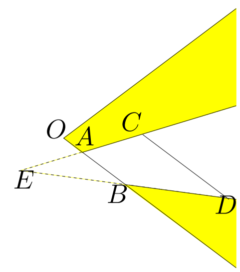

FIGURE 20. Case (l)

Now $AC = EC - EA$ and $BD = ED - EB$ and we have

$$EC + ED \leq CD + EA + EB - \cos(\gamma_2)AB.$$

This inequality is false when EC and ED are sufficiently large. This proves Claim (2.) for $\Omega_\infty \cap \partial\mathcal{P}$. The proof for $\Omega_\infty \cap \partial\mathcal{N}$ is essentially the same.

Claim (3). Suppose $\Omega_\infty \cap \partial\mathcal{P}$ has $A \in \Sigma_1$ as an endpoint. Then either

(i) $\gamma_1 + \gamma_2 - \pi = -2\alpha$ and $\mathcal{P} = \triangle AOB$, where $B \in \Sigma_2$ and angles OAB and ABO have measures γ_1 and γ_2 respectively,

(ii) $\gamma_1 + \gamma_2 - \pi = 2\alpha$ and $\mathcal{P} = \Omega_\infty \setminus \triangle AOB$, where $B \in \Sigma_2$ and angles OAB and ABO have measures $\pi - \gamma_1$ and $\pi - \gamma_2$ respectively, or

(iii) $\Omega_\infty \cap \partial\mathcal{P}$ is a ray with endpoint A which makes angles of measure γ_1 and $\pi - \gamma_1$ with Σ_1 and $\mathcal{P}$ is the component of $\Omega_\infty \setminus \partial\mathcal{P}$ which forms an angle of measure γ_1 at A.

If $|\gamma_1 + \gamma_2 - \pi| \neq 2\alpha$, Claim (1) implies $\Omega_\infty \cap \partial\mathcal{P}$ is an infinite ray with endpoint A. Even when $|\gamma_1 + \gamma_2 - \pi| = 2\alpha$, $\Omega_\infty \cap \partial\mathcal{P}$ might be an infinite ray with endpoint A. We shall suppose this is the case. Then (iii) follows from Lemma 4.8.

Claim (4). Suppose $\Omega_\infty \cap \partial\mathcal{P}$ has $B \in \Sigma_2$ as an endpoint. Then either

(i) $\gamma_1 + \gamma_2 - \pi = -2\alpha$ and $\mathcal{P} = \triangle AOB$, where $A \in \Sigma_1$ and angles OAB and ABO have measures γ_1 and γ_2 respectively,

(ii) $\gamma_1 + \gamma_2 - \pi = 2\alpha$ and $\mathcal{P} = \Omega_\infty \setminus \triangle AOB$, where $A \in \Sigma_1$ and angles OAB and ABO have measures $\pi - \gamma_1$ and $\pi - \gamma_2$ respectively, or

(iii) $\Omega_\infty \cap \partial\mathcal{P}$ is a ray with endpoint B which makes angles of measure γ_2 and $\pi - \gamma_2$ with Σ_2 and $\mathcal{P}$ is the component of $\Omega_\infty \setminus \partial\mathcal{P}$ which forms an angle of measure γ_2 at B.

Let us suppose $\Omega_\infty \cap \partial\mathcal{P}$ is an infinite ray with endpoint B. Then our claim follows from Lemma 4.10.

Claim (5). Suppose $\Omega_\infty \cap \partial\mathcal{P}$ has $O = (0,0)$ as an endpoint. Then $\Omega_\infty \cap \partial\mathcal{P}$ is a ray with endpoint O which makes angles of measure θ_1 and θ_2 with Σ_1 and Σ_2 respectively (and $\theta_1 + \theta_2 = 2\alpha$) such that

(i) $\theta_1 \geq \gamma_1$ and $\theta_2 \geq \pi - \gamma_2$ if $\mathcal{P}$ is the open (infinite) sector whose boundary contains Σ_1 and

(ii) $\theta_1 \geq \pi - \gamma_1$ and $\theta_2 \geq \gamma_2$ if $\mathcal{P}$ is the open (infinite) sector whose boundary contains Σ_2.

The proof of our claim follows from Lemmas 4.6 and 4.11 when $\mathcal{P}$ is the open (infinite) sector whose boundary contains Σ_1 and from Lemmas 4.9 and 4.7 when $\mathcal{P}$ is the open (infinite) sector whose boundary contains Σ_2.

Claim (6). Suppose $\Omega_\infty \cap \partial\mathcal{N}$ has $A \in \Sigma_1$ as an endpoint. Then either

(i) $\gamma_1 + \gamma_2 - \pi = 2\alpha$ and $\mathcal{N} = \triangle AOB$, where $B \in \Sigma_2$ and angles OAB and ABO have measures $\pi - \gamma_1$ and $\pi - \gamma_2$ respectively,

(ii) $\gamma_1 + \gamma_2 - \pi = -2\alpha$ and $\mathcal{N} = \Omega_\infty \setminus \triangle AOB$, where $B \in \Sigma_2$ and angles OAB and ABO have measures γ_1 and γ_2 respectively, or

(iii) $\Omega_\infty \cap \partial\mathcal{N}$ is a ray with endpoint A which makes angles of measure γ_1 and $\pi - \gamma_1$ with Σ_1 and $\mathcal{N}$ is the component of $\Omega_\infty \setminus \partial\mathcal{N}$ which forms an angle of measure $\pi - \gamma_1$ at A.

Let us suppose $\Omega_\infty \cap \partial\mathcal{N}$ is an infinite ray with endpoint A. Then our claim follows from Lemma 4.14.

Claim (7). Suppose $\Omega_\infty \cap \partial\mathcal{N}$ has $B \in \Sigma_2$ as an endpoint. Then either

(i) $\gamma_1 + \gamma_2 - \pi = 2\alpha$ and $\mathcal{N} = \triangle AOB$, where $A \in \Sigma_1$ and angles OAB and ABO have measures $\pi - \gamma_1$ and $\pi - \gamma_2$ respectively,

(ii) $\gamma_1 + \gamma_2 - \pi = -2\alpha$ and $\mathcal{N} = \Omega_\infty \setminus \triangle AOB$, where $A \in \Sigma_1$ and angles OAB and ABO have measures γ_1 and γ_2 respectively, or

(iii) $\Omega_\infty \cap \partial\mathcal{N}$ is a ray with endpoint B which makes angles of measure γ_2 and $\pi - \gamma_2$ with Σ_2 and $\mathcal{N}$ is the component of $\Omega_\infty \setminus \partial\mathcal{N}$ which forms an angle of measure $\pi - \gamma_2$ at B.

Let us suppose $\Omega_\infty \cap \partial\mathcal{N}$ is an infinite ray with endpoint B. Then our claim follows from Lemma 4.15.

Claim (8). Suppose $\Omega_\infty \cap \partial\mathcal{N}$ has $O = (0,0)$ as an endpoint. Then $\Omega_\infty \cap \partial\mathcal{N}$ is a ray with endpoint O which makes angles of measure θ_1 and θ_2 with Σ_1 and Σ_2 respectively (and $\theta_1 + \theta_2 = 2\alpha$) such that

(i) $\theta_1 \geq \pi - \gamma_1$ and $\theta_2 \geq \gamma_2$ if $\mathcal{N}$ is the open (infinite) sector whose boundary contains Σ_1 and

(ii) $\theta_1 \geq \gamma_1$ and $\theta_2 \geq \pi - \gamma_2$ if $\mathcal{N}$ is the open (infinite) sector whose boundary contains Σ_2.

The proof of our claim follows from Lemmas 4.12 and 4.17 when $\mathcal{N}$ is the open (infinite) sector whose boundary contains Σ_1 and from Lemmas 4.16 and 4.13 when $\mathcal{N}$ is the open (infinite) sector whose boundary contains Σ_2.

The conclusions of Theorem 2.1 follow from the results in §4 and the Claims proven above except for the restrictions on γ_1 and γ_2 in (iv)-(vii). For example, one consequence of (v) is that if $\gamma_2 + \pi - \gamma_1 > 2\alpha$, then the conclusion of (v), illustrated in Figure 1 (v), cannot hold. To see this, assume the inequality above and the conclusion of (v) both hold. Set $\theta = 2\alpha + \gamma_1 - \pi$ and $\beta = \gamma_1 + \pi - \gamma_2 - 2\alpha > 0$. Let D be the point of intersection of the lines on which rays L and Σ_2 lie; notice that $D \notin \overline{\Omega_\infty}$. Fix $B \in L$ and let C be the point on Σ_2 determined by the condition that angle ABC has measure β. Then angle OCB has measure $\pi - \gamma_2$. Let $\triangle$ be the open subset of Ω_∞ whose boundary is the quadrilateral $ABCO$. For $T > \max\{OA, OB, OC\}$, $\Phi_T(\mathcal{P}) \leq \Phi_T(\mathcal{P} \setminus \triangle)$ or

$$AB \leq BC + \cos(\gamma_1)OA + \cos(\gamma_2)OC.$$

Notice that $AB = DB - DA$, $OC = DC - DO$, $DC = \big(\sin(\beta)/\sin(\gamma_2)\big)DB$ and $BC = \big(\sin(\theta)/\sin(\gamma_2)\big)DB$. Set $m = DA + \cos(\gamma_1)OA - \cos(\gamma_2)DO$. Then

$$DB \le \frac{\sin(\theta)}{\sin(\gamma_2)}DB + \cos(\gamma_2)\frac{\sin(\beta)}{\sin(\gamma_2)}DB + m$$

or

$$\sin(\gamma_2) \le \sin(\theta) + \cos(\gamma_2)\sin(\beta) + \frac{m\sin(\gamma_2)}{DB}.$$

Since $\beta = \gamma_2 - \theta$, we obtain

$$\sin(\gamma_2) \le \sin(\theta) - \cos^2(\gamma_2)\sin(\theta) + \cos(\gamma_2)\sin(\gamma_2)\cos(\theta) + \frac{m\sin(\gamma_2)}{DB}$$

or, after dividing by $\sin(\gamma_2)$ and simplifying,

$$1 \le \cos(\gamma_2 - \theta) + \frac{m}{DB}. \tag{5.1}$$

Now $0 < \gamma_2 - \theta < 2\pi$ and so $\cos(\gamma_2 - \theta) < 1$. Therefore, for DB sufficiently large, $\cos(\gamma_2 - \theta) + \frac{m}{DB} < 1$, in contradiction to (5.1).

The other restrictions on γ_1 and γ_2 in ***(iv) - (vii)*** of Theorem 2.1 follow using similar arguments. The conclusions of Theorem 2.2 follow from the results in §4 and the Claims proven previously in this section except that the restrictions on γ_1 and γ_2 in ***(iv) - (vii)*** follow as above.

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