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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO NONLINEAR PARABOLIC EQUATIONS WITH VARIABLE VISCOSITY AND GEOMETRIC TERMS

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This article is dedicated to the memory of Professor S. L. Yadava

Abstract. In this paper we study the asymptotic behaviour of solutions of certain nonlinear parabolic equations with variable viscosity and geometric terms. We generalize the results on the large time behaviour and vanishing viscosity limits obtained earlier for planar Burgers equation by Hopf [7], Lighthill [20] and others. For several classes of systems of equations we derive explicit solution for initial value problem with different types of initial conditions and study large time behaviour of the solutions and its asymptotic form. We derive the simple hump solutions and N-wave solutions as its asymptotes depending on the conditions on the data and derive L^p decay estimates for solutions and show that they depend on the variable viscosity coefficient and geometric terms. We also analyse the small viscosity limit of these solutions.

1. Introduction

One of the well studied nonlinear partial differential equation which describes the interplay between nonlinearity and diffusion is the Burgers equation
$$u_t + uu_x = \frac{\nu}{2} u_{xx}. \tag{1.1}$$
It was introduced by Burgers [2] as a simple model for fluid flow. Hopf [7] and Cole [5] showed that it has a remarkable feature that its solution with initial conditions of the form
$$u(x,0) = u_0(x) \tag{1.2}$$
can be explicitly written down. Starting with the pioneering work of Hopf [7] properties of the solutions for initial value problem was studied by many authors with regards to large time behaviour, vanishing viscosity limits etc. A table of solutions is contained in Benton and Platzman [1]. Initial boundary value problems for (1.1) were studied later, see Joseph [8], Calogero and De Lillo [3, 4] and Joseph and Sachdev [13, 14] and the references there.

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A more general class of partial differential equation which models physical phenomenon balancing nonlinear convection, geometrical expansion and variable viscosity effects which is studied extensively is the generalized Burgers equation

$$u_t + u^n u_x + \beta(t)u = \alpha(t)u_{xx} \tag{1.3}$$

where n is a positive integer and the viscosity coefficient $\alpha(t) > 0$ and the geometric coefficient $\beta(t)$ are smooth functions for $t > 0$. For the derivation of the equation(1.3) for specific physical situations and analysis of simple hump and N-wave solutions see Lighthill [20], Leibovich and Seebass [19], Crighton and Scott [6], Lee -Bapty and Crighton [18], Sachdev [24] and Sachdev and his Collaborators [25, 26, 27, 28, 29] and the references there.

The aim of this paper is to understand the effect of variable viscosity and geometric effect on the large time behaviour of solutions. First we consider the scalar equation (1.3), with a given initial data and get decay estimates for the solution. Then we study special nonlinear systems of partial differential equations containing variable viscosity and geometrical expansion terms which can be linearized using a generalized Hopf-Cole transformation and explicitly construct exact solutions for the initial value problem and study its large time behaviour. We show that the asymptotic form depends on the variable viscosity coefficient and the geometrical term.

The paper is organized in the following way. In the second section we study asymptotic behaviour of solutions of the scalar parabolic equations (1.3) in $-\infty < x < \infty$ $t > 0$, with initial condition

$$u(x, 0) = u_0(x). \tag{1.4}$$

This equation cannot be solved explicitly except for $n = 1$ and α and β are related by $\beta = -\frac{\alpha'}{\alpha}$. However following the analysis of Zingano [32, 33] we will show that the L^1 norm and L^2 norm with respect to x of the solution of (1.3) and (1.4) decays at a rate depending on α and β.

In the third section, we consider a system of n equations for n unknown variables $u_1, u_2, \dots, u_n$ in $-\infty < x < \infty$, $t > 0$,

$$(u_j)_t + (\sum_{k=1}^{n} c_k u_k)(u_j)_x - \frac{\nu'(t)}{\nu(t)} u_j = \frac{\nu(t)}{2}(u_j)_{xx}, \tag{1.5}$$

with initial condition

$$u_j(x, 0) = u_{0j}(x). \tag{1.6}$$

When $n = 1$, $c_1 = 1$ and $\nu(t) = \nu > 0$, a constant independent of t, (1.5) is standard Burgers equation (1.1). A special case of (1.5), $n = 2$, $c_1 = 1$, $c_2 = 0$ $\nu(t) = \nu$ a positive constant, was used to construct solution to a model in the study of pressure less gas by passing to ν goes to 0, see Joseph [9], Joseph and Vasudeva Murthy [15]. For $\nu(t) = \nu$, a constant, large time behavior was studied in [12] and vanishing viscosity limit in [16]. We generalize these results to variable $\nu(t)$. We solve the initial value problem explicitly and show that the limiting asymptotic form depends on $\int_0^\infty \nu(s)ds$ is finite or infinite.

In the fourth section we study

$$\begin{aligned} u_t + (\frac{u^2}{2})_x - \frac{\nu'(t)}{\nu(t)}u &= \frac{\nu(t)}{2}u_{xx}, \\ v_t + (uv)_x - \frac{\nu'(t)}{\nu(t)}v &= \frac{\nu(t)}{2}v_{xx}, \\ w_t + (\frac{v^2}{2} + uw)_x - \frac{\nu'(t)}{\nu(t)}w &= \frac{\nu(t)}{2}w_{xx}, \end{aligned} \tag{1.7}$$

in $-\infty < x < \infty$, $t > 0$, supplemented with an initial condition at $t = 0$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \tag{1.8}$$

When $\nu(t) = \nu$ is a positive constant, this system was considered earlier by Joseph [10] and Shelkovich [31] to construct solution to the corresponding inviscid case

$$u_t + (\frac{u^2}{2})_x = 0, \quad v_t + (uv)_x = 0, \quad w_t + (\frac{v^2}{2} + uw)_x = 0. \tag{1.9}$$

Joseph [10] observed that the system (1.9) does not have a solution in the class of bounded Borel measures even for Riemann type initial data and hence he used the special case of the system (1.7) namely $\nu(t) = \nu > 0$, a constant and constructed solutions for general initial data in the class of generalized functions of Colembeau. Shelkovich [31] constructed solutions of (1.7) with $\nu(t) = \nu > 0$ a constant for Riemann initial data and showed that the solution contains derivative of δ measure in the passage to the limit. We get explicit formula for (1.7) and (1.8) with $\nu(t) > 0$ and study large time behaviour and and show that the asymptotic form depends on $\int_0^\infty \nu(s)ds$ is finite or infinite.

In the fifth section we consider vector equivalent of Burgers equation with an additional geometrical term and variable viscosity namely

$$U_t + U.\nabla U - \frac{\nu'(t)}{\nu(t)}U = \frac{\nu(t)}{2}\Delta U. \tag{1.10}$$

The special case $\nu(t)$ a positive constant was treated first by Nerney et al [22] and by Joseph and Sachdev [14]. We generalize their method and get explicit solutions of the equation (1.10) with initial conditions of the form

$$U(x,0) = \nabla_x \phi_0(x) \tag{1.11}$$

and study the large time behaviour and small viscosity limits.

2. L^2 decay estimates for solutions of (1.3)

In this section, we consider initial value problem for the scalar equation

$$u_t + u^n u_x + \beta(t)u = \alpha(t)u_{xx}, \tag{2.1}$$

in $-\infty < x < \infty, t > 0$ with initial condition at $t = 0$

$$u(x,0) = u_0(x). \tag{2.2}$$

In the description of asymptotic behaviour of solutions of (2.1) and (2.2), the following functions of t, naturally appear:

$$\tau(t) = \int_0^t \alpha(s)ds, \eta(t) = \int_0^t \beta(s)ds, \gamma(\tau) = \int_0^\tau \frac{\beta(t(s)}{\alpha(t(s))}ds \tag{2.3}$$

To see this consider the linear part of (2.1) namely

$$u_t + \beta(t)u = \alpha(t)u_{xx}, \tag{2.4}$$

Dividing this equation through out by $\alpha(t)$ and using the definition of τ the equation (2.4) becomes

$$u_\tau + \frac{\beta(t(\tau))}{\alpha(t(\tau))}u = u_{xx}, \tag{2.5}$$

Set $v(x,\tau) = e^{\gamma(\tau)}u(x,\tau)$, (2.5) and (2.2) reduces to

$$v_\tau = v_{xx}, v(x,0) = u_0(x). \tag{2.6}$$

Solving (2.6) we get the explicit formula

$$v(x,\tau) = \frac{1}{(4\pi\tau)^{1/2}}\int_{-\infty}^{\infty} u_0(y)e^{-\frac{(x-y)^2}{4\tau}}dy.$$

This gives the formula for the solution of the linear problem (2.4) with initial data (2.2), namely

$$u(x,t) = \frac{e^{-\eta(t)}}{(4\pi\tau(t))^{1/2}}\int_{-\infty}^{\infty} u_0(y)e^{-\frac{(x-y)^2}{4\tau(t)}}dy. \tag{2.7}$$

since

$$\gamma(\tau(t)) = \int_0^{\tau(t)} \frac{\beta(t(s))}{\alpha(t(s))}ds = \int_0^t \beta(y)dy = \eta(t).$$

By Young's inequality for convolutions, the L^p norm of u with respect to x decays at the rate $\tau(t)^{-\frac{1}{2}(1-1/p)}e^{-\eta(t)}$. We will show that for any integer $n \geq 1$ the L^1 and L^2 norm of the solution of (2.1) and (2.2) with respect to x decays as t goes to infinity. More precisely we shall prove

Theorem 2.1. *Assume that $\beta(t) \geq 0$, and $\alpha(t) > 0$ for $t \geq 0$ and initial data $u_0 \in L^1(R^1) \cap L^\infty(R^1)$. Then there exits a smooth solution $u(x,t)$ for* (2.1) *and* (2.2) *satisfying the following decay estimates:*

$$\|u(x,t)\|_{L^1(dx)} \leq e^{-\eta(t)}\|u_0\|_{L^1(dx)} \tag{2.8}$$

$$\|u(x,t)\|_2 \leq C(1+\tau(t))^{-1/2}e^{-\frac{\eta(t)}{2}}.\Big(\int_0^t (1+\tau(s))^{-1/2}e^{-\eta(s)}\alpha(s)ds\Big)^{1/2}. \tag{2.9}$$

Further if $\|\frac{du_0}{dx}\|_{L^1(dx)} < \infty$, then

$$\|u_x(x,t)\|_{L^1(dx)} \leq e^{-\eta(t)}\|\frac{du_0}{dx}\|_{L^1(dx)} \tag{2.10}$$

Proof. Existence of smooth solutions to (2.1) and (2.2) follows in a standard way using fixed point arguments. Further we get for each fixed $t > 0$, $u(x,t)$ is bounded and integrable with respect to x and satisfy the estimate

$$\|u(.,t)\|_{L^\infty(dx)} \leq \|u_0\|_{L^\infty(dx)} \tag{2.11}$$

which follows from the maximum principle.

To prove decay estimates in L^1 norm,in a standard way we write the L^1- norm as a sum of integrals on intervals where u has the same sign. For fixed $t > 0$, let $y_i(t), i \in Z$ are the points where $u(x,t)$ as a function of x change its sign with $y_i(t) < y_{i+1}(t)$ and the index is chosen such that on $(y_0(t), y_1(t))$, $u(x,t) > 0$. Thus

$u(x,t)$ is positive on $(y_{2k}(t), y_{2k+1}(t))$ and negative on $(y_{2k+1}(t), y_{2k+2}(t))$. Hence $u_x(y_{2k+1}(t),t) \le 0$ and $u_x(y_{2k}(t),t) \ge 0$. Hence

$$\|u(x,t)\|_{L^1(dx)} = \sum_{i=-\infty}^{\infty} (-1)^i \int_{y_i(t)}^{y_{i+1}(t)} u(x,t)dx$$

and differentiating it with respect to t and using $u(y_i(t),t) = 0$ and the differential equation (2.1), we get

$$\begin{aligned}
\frac{d}{dt}\|u(x,t)\|_{L^1(dx)} &= \sum_{i=-\infty}^{\infty} (-1)^i \int_{y_i(t)}^{y_{i+1}(t)} u_t(x,t)dx \\
&= \sum_{i=-\infty}^{\infty} (-1)^i \int_{y_i(t)}^{y_{i+1}(t)} (-u^n u_x - \beta(t)u + \alpha(t)u_{xx})dx, \\
&= -\beta(t)\|u(.,t)\|_{L^1(dx)} + \alpha(t) \sum_{i=-\infty}^{\infty} (-i)^i (u_x(y_{i+1}(t)) - u_x(y_i(t))) \\
&\le -\beta(t)\|u(.,t)\|_{L^1}(dx)
\end{aligned}$$

Integrating this from 0 to t we get the estimate (2.8) follows.

The estimate (2.10) can be derived by differentiating the equation (2.1) with respect to x and following the same procedure for $v = u_x$. The details are omitted.

To get L^2 decay estimates we multiply the (2.1) by $u(x,t)$ and integrate with respect to x

$$\frac{d}{dt}\int_{R^1} u^2(x,t)dx + \beta(t)\int_{R^1} u^2(x,t)dx = -\alpha(t)\int_R {u_x}^2 dx, \tag{2.12}$$

$$\frac{d}{d\tau}\int_{R^1} u^2(x,t)dx + \frac{\beta(t(\tau))}{\alpha(t(\tau))}\int_{R^1} u^2(x,t(\tau)dx = -\int_R {u_x}^2 dx \tag{2.13}$$

Using (2.3) we can write (2.13) in the form

$$\frac{d}{d\tau}\int_{R^1} e^{\gamma(\tau)}u^2(x,t)dx + \int_{R^1} e^{\gamma(\tau)}{u_x}^2 dx = 0. \tag{2.14}$$

Multiply (2.14) by $(1+\tau)$ and integrate from 0 to τ_0, we get

$$\begin{aligned}
&(1+\tau_0)e^{\gamma(\tau_0)}\int_R u^2(x,t)dx + \int_0^{\tau_0}(1+\tau)e^{\gamma(\tau)}\int_R {u_x}^2 dxdt \\
&= \int_R u_0(x)^2 dx + \int_0^{\tau_0} e^{\gamma(\tau)}\int_R u^2(x,t)dxd\tau
\end{aligned} \tag{2.15}$$

Now

$$\|u(x,t(\tau))\|^2_{L^2(dx)} \le \|u(x,t(\tau))\|_{L^\infty}(dx).\|u(x,t(\tau))\|_{L^1(dx)} \tag{2.16}$$

and

$$\begin{aligned}
u^2(x,t) &= \int_{-\infty}^x (u(x,t)^2)_x dx = 2\int_{-\infty}^x u(x,t)u_x(x,t)dx \\
&\le 2\|u(x,t)\|_{L^2(dx)}\|u_x(x,t)\|_{L^2(dx)}
\end{aligned} \tag{2.17}$$

from which we get

$$\|u(x,t)\|_{L^\infty(dx)} \le 2.\|u(x,t)\|^{1/2}_{L^2(dx)}\|u_x(x,t)\|^{1/2}_{L^2(dx)} \tag{2.18}$$

Using (2.18) in (2.16) we have

$$\|u(x,t)\|^2_{L^2(dx)} \le 2.\|u(x,t)\|^{1/2}_{L^2(dx)}\|u_x(x,t)\|^{1/2}_{L^2(dx)}\|u(x,t)\|_{L^1(dx)}. \tag{2.19}$$

From (2.19), we get

$$\|u(x,t)\|^2_{L^2(dx)} \leq 4.\|u_x(x,t)\|^{2/3}_{L^2(dx)}\|u(x,t)\|^{4/3}_{L^1(dx)} \tag{2.20}$$

Now

$$\begin{aligned}
&\int_0^{\tau_0} e^{\gamma(\tau)}\|u(x,t(\tau))\|^2_{L^2(dx)}d\tau \\
&\leq 4\int_0^{\tau_0}(e^{\gamma(\tau)}\|u_x(x,t(\tau))\|^{2/3}_{L^2(dx)}\|u(x,t(\tau))\|^{4/3}_{L^1(dx)})d\tau \\
&= 4\int_0^{\tau_0}((1+\tau)^{1/3}e^{\frac{1}{3}\gamma(\tau)}\|u_x(x,t(\tau))\|^{2/3}_{L^2(dx)}(1+\tau)^{-1/3}e^{\frac{2}{3}\gamma(\tau)}\|u(x,t(\tau))\|^{4/3}_{L^1(dx)})d\tau \\
&\leq 4(\int_0^{\tau_0}((1+\tau)e^{\gamma(\tau)}\|u_x(x,t(\tau))\|^2_{L^2(dx)}d\tau)^{1/3} \\
&\quad\times(\int_0^{\tau_0}(1+\tau)^{-1/2}e^{\gamma(\tau)}\|u(x,t(\tau))\|^2_{L^1(dx)}d\tau)^{2/3}
\end{aligned} \tag{2.21}$$

and from (2.8), we get

$$\begin{aligned}
&\int_0^{\tau_0}(1+\tau)^{-\frac{1}{2}}e^{\gamma(\tau)}\|u(x,t(\tau))\|^2_{L^1(dx)}d\tau \\
&\leq \|u_0\|^2_{L^1(dx)}\int_0^{\tau_0}[(1+\tau)^{-1/2}e^{\gamma(\tau)}e^{-2\gamma(\tau)}]d\tau \\
&= \|u_0\|^2_{L^1(dx)}\int_0^{\tau_0}(1+\tau)^{-1/2}e^{-\gamma(\tau)}d\tau.
\end{aligned} \tag{2.22}$$

Let $F(\tau)$ be defined by

$$F(\tau) = (1+\tau)e^{\gamma(\tau)}\int_R u^2(x,t)dx + \int_0^\tau(1+s)e^{\gamma(s)}\int_R {u_x}^2(x,s)dxds. \tag{2.23}$$

and

$$g(\tau) = \int_0^{\tau_0}(1+\tau)^{-1/2}e^{-\gamma(\tau)}d\tau \tag{2.24}$$

Using (2.21)-(2.24) in (2.15), we get

$$F(\tau) \leq 8(\|u_0\|^2_{L^2(dx)} + \|u_0\|^{1/3}_{L^1(dx)}g(\tau)^{2/3}F(\tau)^{1/3}). \tag{2.25}$$

Since $\|u_0\|^2_{L^2} \leq \|u_0\|_{L^1}.\|u_0\|_{L^\infty}$, from (2.25), it follows that, for some constant C depending only on $\|u_0\|_{L^1}$ and $\|u_0\|_{L^\infty}$,

$$F(\tau) \leq Cg(\tau) \tag{2.26}$$

Substituting $F(\tau)$ from (2.23) $g(\tau)$ from (2.24) in (2.26) we get

$$\begin{aligned}
&(1+\tau)e^{\gamma(\tau)}\|u(x,t(\tau))\|^2_{L^2(dx)} + \int_0^\tau(1+s)e^{\gamma(s)}\|u_x(x,t(s))\|^2_{L^2(dx)}ds \\
&\leq C\int_0^\tau(1+s)^{-1/2}e^{-\gamma(s)}ds.
\end{aligned} \tag{2.27}$$

From (2.27), we get

$$\|u(x,t)\|^2_{L^2(dx)} \leq (1+\tau(t))^{-1}e^{-\gamma(\tau(t))}\int_0^{\tau(t)}(1+s)^{-1/2}e^{-\gamma(s)}ds. \tag{2.28}$$

Now observe that making a change of variable, $t(s) = y$, we get

$$\gamma(\tau(t)) = \int_0^{\tau(t)} \frac{\beta(t(s))}{\alpha(t(s))} ds = \int_0^t \beta(y)dy = \eta(t). \tag{2.29}$$

From (2.27) - (2.29) we get

$$\|u(x,t)\|^2_{L^2(dx)} \le (1+\tau(t))^{-1} e^{-\eta(t)} \int_0^t (1+\tau(s))^{-1/2} e^{-\eta(s)} \alpha(s) ds. \tag{2.30}$$

From (2.30) the estimate (2.9) follows. □

We remark that since β is non-negative, $\alpha > 0$ and $d\tau(s) = \alpha(s)ds$, it follows that

$$\int_0^t (1+\tau(s))^{-1/2} e^{-\eta(s)} \alpha(s) ds \le \int_0^t (1+\tau(s))^{-1/2} \alpha(s) ds \le 2(1+\tau(t))^{1/2}$$

and hence from (2.30) we get the estimate

$$\|u(x,t)\|_{L^2(dx)} \le (1+\tau(t))^{-1/4} e^{-\frac{\eta(t)}{2}}. \tag{2.31}$$

For the special case $\beta = 0$, the estimate (2.31) becomes

$$\|u(x,t)\|_{L^2(dx)} \le (1+\tau(t))^{-1/4}.$$

and this agrees with the L^2 decay results of Zingano [32, 33] for the Burgers equation ($n = 1$, α, a constant, $\tau(t) = \alpha t$) and $\beta = 0$) and for certain systems of equations with $\beta = 0$ and diffusion term is in conservation form $(B(u)u_x)_x$, with $B(u)$ positive definite matrix.

3. Explicit solution of (1.5) and its asymptotic behaviour

In this section we consider the initial value problem for the system for u_j, $j = 1, 2, \dots n$, in a domain $-\infty < x < \infty$, $t > 0$

$$(u_j)_t + \Big(\sum_{k=1}^n c_k u_k\Big)(u_j)_x - \frac{\nu'(t)}{\nu(t)} u_j = \frac{\nu(t)}{2} (u_j)_{xx}, \tag{3.1}$$

with initial conditions at $t = 0$

$$u_j(x,0) = u_{0j}(x). \tag{3.2}$$

Here we assume that $\nu(t) > 0$ for $t \ge 0$ and two times continuously differentiable and $u_{0j}, j = 1, 2, \dots n$ are measurable functions. We use a generalised Hopf-Cole transformation to linearize the system of equations in (3.1) and solve it in terms of initial data (3.2). Through out this section we use

$$\tau(t) = \int_0^t \nu(s)ds, \sigma_0(x) = \sum_1^n c_j u_{0j}(x), w_0(x) = \int_0^x \sigma_0(z)dz \tag{3.3}$$

We shall prove the following result.

Theorem 3.1. *(a). Under the assumptions, $u_{0j} \in L^p(R^1)$ for some $1 \le p \le \infty$ the functions $u_j, j = 1, 2, \dots n$ given by*

$$u_j(x,t) = \frac{\nu(t)}{\nu(0)} \frac{\int_{-\infty}^{\infty} u_{0j}(y) e^{-[\frac{w_0(y)}{\nu(0)} + \frac{(x-y)^2}{2\tau(t)}]} dy}{\int_{-\infty}^{\infty} e^{-[\frac{w_0(y)}{\nu(0)} + \frac{(x-y)^2}{2\tau(t)}]} dy}. \tag{3.4}$$

are twice continuously differentiable in the variables (x,t) and is exact solution of the initial value problem for (3.1) and (3.2).
(b). Assume that $u_{0j}, j=1,2,\dots,n$ is such that $u_{0j}(\infty)$,$u_{0j}(-\infty)$ exists and there is a cancellation in $\sigma_0=\sum_1^n c_k u_{0k}$, so that σ_0 is integrable. If $\tau(t)$ goes to infinity as t goes to infinity, then

$$\frac{\nu(0)}{\nu(t)}u_j(x,t) \approx \frac{u_{0j}(\infty)e^{\frac{-w_0(\infty)}{\nu(0)}}\int_{-\infty}^{\xi} e^{-\frac{y^2}{2}}dy + u_{0j}(-\infty)e^{\frac{-w_0(-\infty)}{\nu(0)}}\int_{\xi}^{\infty} e^{-\frac{y^2}{2}}dy}{e^{\frac{-w_0(\infty)}{\nu(0)}}\int_{-\infty}^{\xi} e^{-\frac{y^2}{2}}dy + e^{\frac{-w_0(-\infty)}{\nu(0)}}\int_{\xi}^{\infty} e^{-\frac{y^2}{2}}dy} \tag{3.5}$$

uniformly in the variable $\xi = x/\sqrt{\tau}(t)$ lying on bounded subsets. If $\tau(t)$ goes to a finite value $\tau(\infty)$, as t goes to infinity then

$$\lim_{t\to\infty}\frac{\nu(0)}{\nu(t)}u_j(x,t) = \frac{\int_{-\infty}^{\infty}u_{0j}(y)e^{-[\frac{w_0(y)}{\nu(0)}+\frac{(x-y)^2}{2\tau(\infty)}]}dy}{\int_{-\infty}^{\infty}e^{-[\frac{w_0(y)}{\nu(0)}+\frac{(x-y)^2}{2\tau(\infty)}]}dy}. \tag{3.6}$$

(c). If $u_{0j}\in L^1(R^1)$ then we have the following estimates

$$\|u_j(x,t)\|_p = O(1)\tau^{\frac{-1}{2}(1-1/p)}. \tag{3.7}$$

(d).If $\nu(t)=\epsilon\nu_0(t), \epsilon>0, \tau_0(t)=\int_0^t \nu_0(s)ds$ and $u_j^\epsilon, j=1,2,\dots,n$, the corresponding solution given by (3.4), then for each fixed $t>0$ except for a countable number of x, the limit

$$\lim_{\epsilon\to 0}u_j^\epsilon(x,t) = \frac{\nu_0(t)}{\nu_0(0)}u_{0j}(y(x,t)) \tag{3.8}$$

exists, where $y(x,t)$ is the minimizer of

$$\min_{-\infty<y<\infty}[\frac{w_0(y)}{\nu_0(0)}+\frac{(x-y)^2}{2\tau_0(t)}] \tag{3.9}$$

Proof. To prove the result first we introduce a new unknown variable

$$\sigma = \sum_{k=1}^n c_k u_k. \tag{3.10}$$

It follows that, the equation (3.1) can be written as

$$(u_j)_t + \sigma(u_j)_x - \frac{\nu'(t)}{\nu(t)}u_j = \frac{\nu(t)}{2}(u_j)_{xx}. \tag{3.11}$$

Now multiplying this equation by c_j and summing from 1 to n we get σ is the solution to

$$\begin{gathered}\sigma_t + \frac{1}{2}(\sigma^2)_x - \frac{\nu'(t)}{\nu(t)}\sigma = \frac{\nu(t)}{2}\sigma_{xx}\\ \sigma(x,0)=\sigma_0(x).\end{gathered} \tag{3.12}$$

So to solve the initial value problem (3.1) and (3.2) first we solve (3.12) and then we solve the linear system (3.11) with initial conditions

$$u_j(x,0) = u_{0j}(x), \tag{3.13}$$

for $j=1,2,\dots,n$. To solve (3.11) and (3.12) we observe that if $w(x,t)$ is a solution of

$$w_t + \frac{(w_x)^2}{2} - \frac{\nu'(t)}{\nu(t)}w = \frac{\nu(t)}{2}w_{xx} \tag{3.14}$$

with initial condition

$$w(x,0) = w_0(x) \tag{3.15}$$

then

$$\sigma(x,t) = w_x(x,t) \tag{3.16}$$

is a solution of (3.12). Here w_0 is defined by (3.3). To get explicit solution, we use a modified form of the Hopf- Cole transformation by introducing new unknown variables $v, v_j, j = 1, 2, \dots n$, in the following way

$$v = e^{-\frac{w}{\nu(t)}}, v_j = \frac{u_j}{\nu(t)} e^{-\frac{w}{\nu(t)}}, j = 1, 2, 3, \dots, n. \tag{3.17}$$

An easy calculation shows that

$$\begin{gathered}
v_t - \frac{\nu(t)}{2} v_{xx} = -\frac{1}{\nu(t)}[w_t + \frac{(w_x)^2}{2} - \frac{\nu'(t)}{\nu(t)} w - \frac{\nu(t)}{2} w_{xx}] e^{-\frac{w}{\nu(t)}}, \\
(v_j)_t - \frac{\nu(t)}{2}(v_j)_{xx} = \frac{1}{\nu(t)}[(u_j)_t + \sigma(u_j)_x - \frac{\nu'(t)}{\nu(t)} u_j - \frac{\nu(t)}{2} w_{xx}] e^{-\frac{w}{\nu(t)}} \\
-\frac{1}{\nu(t)^2}[w_t + \frac{(w_x)^2}{2} - \frac{\nu'(t)}{\nu(t)} w - \frac{\nu(t)}{2} w_{xx}] u_j e^{-\frac{w}{\nu(t)}}.
\end{gathered} \tag{3.18}$$

From (3.10) - (3.18), it follows that $v, v_j, j = 1, 2, \dots, n$ are solutions of

$$\begin{gathered}
v_t = \frac{\nu(t)}{2} v_{xx}, v(x,0) = e^{-\frac{w_0(x)}{\nu(0)}}, \\
(v_j)_t = \frac{\nu(t)}{2}(v_j)_{xx}, v_j(x,0) = \frac{u_{0j}(x)}{\nu(0)} e^{-\frac{w_0(x)}{\nu(0)}},
\end{gathered} \tag{3.19}$$

if and only if w is a solution of (3.14) and (3.15) and $u_j, j = 1, 2, 3, \dots n$ are solutions of (3.11) and (3.13). Solving (3.19) explicitly we get

$$\begin{gathered}
v(x,t) = \frac{1}{(2\pi\tau(t))^{1/2}} \int_{R^1} e^{-[\frac{w_0(y)}{\nu(0)} + \frac{(x-y)^2}{2\tau(t)}]} dy, \\
v_j(x,t) = \frac{1}{\nu(0)(2\pi\tau(t))^{1/2}} \int_{R^1} u_{0j}(y) e^{-[\frac{w_0(y)}{\nu(0)} + \frac{(x-y)^2}{2\tau(t)}]} dy.
\end{gathered} \tag{3.20}$$

From (3.16) and (3.17) we get

$$\sigma(x,t) = w_x(x,t) = -\nu(t)\frac{v_x}{v}, u_j(x,t) = \nu(t)\frac{v_j}{v}, j = 1, 2, \dots, n. \tag{3.21}$$

and substituting the formulas of v and v_j from (3.20) in (3.21) we get the explicit formula (3.4) for the solution of (3.1) and (3.2).

We show that integrals defined in (3.4) is well defined and $u_j(x,t)$ is twice continuously differentiable, when $u_{0j} \in L^p$. This follows from the fact that $w_0(y)$ grows at most linearly at infinity as shown below.

For $1 < p < \infty$ using Holder's inequality we get

$$|w_0(y)| \leq \int_0^y |\sigma(z)| \leq |y|^{\frac{p-1}{p}} \sum_{j=1}^n |c_j| \|u_{0j}\|_p.$$

Hence for each fixed $\{(x,t), x \in R^1, t > 0\}$ and k a non-negative integer, the map $y \to y^k e^{-[\frac{w_0(y)}{\nu(0)} + \frac{(x-y)^2}{2\tau(t)}]}$ is in $L^{p/p-1}$. So for any k non negative integer, the product $u_{0j}(y).y^k e^{-[\frac{w_0(y)}{\nu(0)} + \frac{(x-y)^2}{2\tau(t)}]}$ is integrable with respect to y by the Holder's inequality.

Hence the differentiation under the integral sign is justified and the smoothness of u_j in (x,t) follows. The cases for $p=1$ and $p=\infty$ also follows as for $p=1$,

$$\|w_0\|_{L^\infty} \le \sum_{j=1}^{n} |c_j| \|u_{0j}\|_1$$

and for $p=\infty$

$$|w_0(y)| \le \sum_{j=1}^{n} |c_j| \|u_{0j}\|_\infty |y|.$$

Next we prove part (b) of the theorem. First we take the case $\tau(\infty)=\infty$. We rewrite the solution $u_j(x,t), j=1,2,\dots,n$ in a convenient way.

Setting $\xi = x/\sqrt{\tau(t)}$, and then making a change of variable $z = \frac{\sqrt{\tau(t)}\xi - y}{\sqrt{\tau(t)}}$ and renaming z as y, we get

$$u_j(x,t) = \frac{\nu(t)}{\nu(0)} \frac{\int_{-\infty}^{\infty} u_{0j}(\sqrt{\tau(t)}(\xi-y)) e^{-[\frac{w_0(\sqrt{\tau(t)}(\xi-y))}{\nu(0)} + y^2/2]} dy}{\int_{-\infty}^{\infty} e^{-[\frac{w_0(\sqrt{\tau(t)}(\xi-y))}{\nu(0)} + y^2/2]} dy}. \tag{3.22}$$

Now we fix $\delta > 0$ and split the integrals appearing in the explicit solution into three parts and study each of these integrals as t tends to infinity. We have under the assumptions of the theorem on $w_0(x)$ and $u_{0j}(x)$, as t tends to infinity:

$$\int_{-\infty}^{\xi-\delta} u_{0j}(\sqrt{\tau(t)}(\xi-y) e^{-[\frac{w_0(\sqrt{\tau(t)}(\xi-y)}{\nu(0)} + y^2/2]} dy \approx e^{-\frac{w_0(+\infty)}{\nu(0)}} u_{0j}(\infty) \int_{-\infty}^{\xi-\delta} e^{-y^2/2} dy.$$

$$\int_{\xi+\delta}^{\infty} u_{0j}(\sqrt{\tau(t)}(\xi-y) e^{-[\frac{w_0(\sqrt{\tau(t)}(\xi-y)}{\nu(0)} + y^2/2]} dy \approx e^{-\frac{w_0(-\infty)}{\nu(0)}} u_{0j}(-\infty) \int_{\xi+\delta}^{\infty} e^{-y^2/2} dy,$$

$$\limsup_{t\to\infty} \Big| \int_{\xi-\delta}^{\xi+\delta} u_{0j}(\sqrt{\tau(t)}(\xi-y) e^{-[\frac{w_0(\sqrt{\tau(t)}(\xi-y)}{\nu(0)} + y^2/2]} dy \Big| = O(\delta).$$

Now first let t tends to infinity in these integrals and add and then δ tends to 0, we get

$$\int_{-\infty}^{\infty} u_{0j}(\sqrt{\tau(t)}(\xi-y)) e^{-[w_0(\sqrt{\tau(t)}(\xi-y)\nu + y^2/2]} dy \approx$$

$$e^{-\frac{w_0(+\infty)}{\nu(0)}} u_{0j}(\infty) \int_{-\infty}^{\xi} e^{-y^2/2} dy + e^{\frac{-w_0(-\infty)}{\nu(0)}} u_{0j}(-\infty) \int_{\xi}^{\infty} e^{-y^2/2} dy. \tag{3.23}$$

Similarly

$$\int_{-\infty}^{\infty} e^{-[\frac{w_0(\sqrt{\tau(t)}(\xi-y)}{\nu(0)} + y^2/2]} dy \approx e^{\frac{-w_0(+\infty)}{\nu(0)}} \int_{-\infty}^{\xi} e^{-y^2/2} dy + e^{\frac{-w_0(-\infty)}{\nu(0)}} \int_{\xi}^{\infty} e^{-y^2/2} dy. \tag{3.24}$$

We observe that due to our assumption on $w_0(x)$, this limit in (3.24) is positive and hence from (3.22), (3.23) and (3.24) we get the asymptotic form (3.5). When $\tau(\infty) < \infty$, then (3.6) follows immediately from the explicit formula (3.4).

Now we shall prove (c). Since $u_{0j} \in L^1(R^1)$,

$$|w_0(x)| = \Big| \int_0^x \sum_{0}^{n} c_j u_{0j}(y) dy \Big| \le \sum_{0}^{n} |c_j| \|u_{0j}\|_{L^1} = c < \infty$$

Hence from (3.20), we have

$$v(x,t) \geq e^{-\frac{c}{\nu(0)}}, |v_j(x,t)| \leq \frac{e^{\frac{c}{\nu(0)}}}{\nu(0)(2\pi\tau(t))^{1/2}} \int_{R^1} |u_{0j}(y)| e^{-\frac{(x-y)^2}{2\tau(t)}]} dy.$$

Using these estimates in (3.21), we get

$$|u_j(x,t)| \leq \frac{\nu(t) e^{\frac{2c}{\nu(0)}}}{\nu(0)(2\pi\tau(t))^{1/2}} \int_{R^1} |u_{0j}(y)| e^{-\frac{(x-y)^2}{2\tau(t)}]} dy.$$

The right hand side is a function of t times the convolution of heat kernel and $|u_{0j}|$. By Young's inequality we get the estimate (3.7)

Lastly we consider the vanishing diffusion limit. Let $\nu(t) = \epsilon\nu_0(t)$ where $\nu_0(t) > 0$, for $t \geq 0$. We denote the solution $u(x,t)$ of (3.1) and (3.2) given by (3.4) by u^ϵ. Following the analysis of Hopf [7] and Lax [17], we have, for each $t > 0$, except for a countable points of x, (3.9) has a unique minimum $y(x,t)$and at those points $\lim_{\epsilon\to 0} u^\epsilon$ has limit given by the formula (3.8). The proof of the theorem is complete. □

It is easy to see from the present analysis that initial boundary value problem for (3.1) in $\{(x,t), x > 0, t > 0\}$ with initial condition

$$u_j(x,0) = u_{0j}(x), \quad x > 0$$

and boundary condition

$$u_j(0,t) = 0, \quad t > 0$$

has an explicit formula given by

$$u_j(x,t) = \frac{\nu(t)}{\nu(0)} \frac{\int_0^\infty u_{0j}(y)\{e^{-\frac{(x-y)^2}{2\tau(t)}} - e^{-\frac{(x-y)^2}{2\tau(t)}}\} e^{-\frac{w_0(y)}{\nu(0)}} dy}{\int_0^\infty \{e^{-\frac{(x-y)^2}{2\tau(t)}} + e^{-\frac{(x-y)^2}{2\tau(t)}}\} e^{-\frac{w_0(y)}{\nu(0)}} dy}$$

and its asymptotic behaviour easily follows. We have the following result:
If $\tau(t)$ goes to infinity as t goes to infinity, then

$$\lim_{t\to\infty} \frac{\nu(0)}{\nu(t)} u_j(x,t) = 0, \quad j = 1,2,\dots n.$$

If $\tau(t)$ goes to a finite value $\tau(\infty)$, as t goes to infinity then

$$\lim_{t\to\infty} \frac{\nu(0)}{\nu(t)} u_j(x,t) = \frac{\int_0^\infty u_{0j}(y) e^{-\frac{w_0(y)}{\nu(0)}} \{e^{-\frac{(x-y)^2}{2\tau(\infty)}} - e^{-\frac{(x+y)^2}{2\tau(\infty)}}\} dy}{\int_0^\infty e^{-\frac{w_0(y)}{\nu(0)}} \{e^{-\frac{(x-y)^2}{2\tau(\infty)}} + e^{-\frac{(x+y)^2}{2\tau(\infty)}}\} dy}.$$

4. Explicit solution for (1.7) and its asymptotic behaviour

In this section we consider a system of partial differential equations of the form

$$\begin{gathered} u_t + (\frac{u^2}{2})_x - \frac{\nu'(t)}{\nu(t)} u = \frac{\nu(t)}{2} u_{xx}, \\ v_t + (uv)_x - \frac{\nu'(t)}{\nu(t)} v = \frac{\nu(t)}{2} v_{xx}, \\ w_t + (\frac{v^2}{2} + uw)_x - \frac{\nu'(t)}{\nu(t)} w = \frac{\nu(t)}{2} w_{xx}, \end{gathered} \tag{4.1}$$

in $-\infty < x < \infty$, $t > 0$, supplemented with an initial condition at $t = 0$

$$u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x). \tag{4.2}$$

Assuming that $\nu(t) > 0$ for $t \geq 0$ and two times continuously differentiable and $u_0(x), v_0(x), w_0(x)$ are in $L^p(R^1)$ for some $1 \leq p \leq \infty$, we find explicit solution for the problem (4.1) and (4.2). Under addition conditions on the data we study the asymptotic behaviour of its solution as t goes to infinity. The limit depends on whether $\int_0^\infty \nu(s)ds$ is finite or infinite. Given $u_0(x), v_0(x), w_0(x)$ we introduce the functions

$$U_0(x) = \int_0^x u_0(y)dy, V_0(x) = \int_0^x v_0(y)dy, W_0(x) = \int_0^x w_0(y)dy. \tag{4.3}$$

$$p_1(x) = e^{-\frac{U_0(y)}{\nu(0)}}, p_2(x) = -\frac{V_0(y)}{2\nu(0)} e^{-\frac{U_0(y)}{\nu(0)}}, p_3(x) = (\frac{V_0(y)^2}{2\nu(0)^2} - \frac{W_0(y)}{\nu(0)}) e^{-\frac{U_0(y)}{\nu(0)}}. \tag{4.4}$$

We shall show explicit formula for the solution of (4.1) and (4.2) can be represented in terms of

$$a_j(x,t) = \frac{1}{\sqrt{2\pi\tau(t)}} \int_{-\infty}^{+\infty} p_j(y) e^{-\frac{(x-y)^2}{2\tau(t)}} dy, j = 1, 2, 3. \tag{4.5}$$

With the independent variable

$$\tau(t) = \int_0^t \nu(s)ds, \xi = \frac{x}{\sqrt{\tau(t)}}, \tag{4.6}$$

we introduce the following function to describe the asymptotic form of the solution.

$$\alpha_j(\xi) = p_j(\infty) \int_{-\infty}^{\xi} e^{-y^2/2} dy + p_j(-\infty) \int_{\xi}^{\infty} e^{-y^2/2} dy, j = 1, 2, 3. \tag{4.7}$$

For $\tau(\infty) < \infty$, we introduce

$$\beta_j(x) = \int_{-\infty}^{+\infty} p_j(y) e^{-\frac{(x-y)^2}{2\tau(\infty)}} dy, \tag{4.8}$$

We shall prove the following result.

Theorem 4.1. *(a). Assume that the initial data $u_0(x), v_0(x), w_0(x)$ are in $L^p(R^1)$ for some $1 \leq p \leq \infty$, then the functions $(u(x,t), v(x,t), w(x,t))$ defined by*

$$u = -\nu(t)(\log(a_1))_x, \quad v = -\nu(t)(\frac{a_2}{a_1})_x, \quad w = \nu(t)(-\frac{a_3}{a_1} + \frac{{a_2}^2}{2{a_1}^2})_x. \tag{4.9}$$

are two times continuously differentiable and is a classical solution of the initial value problem (4.1) and (4.2).
(b) When $u_0, v_0, w_0 \in L^1(R^1)$, this solution has the following asymptotic behaviour as t tends to infinity.

When $\tau(t) \to \infty$ as $t \to \infty$

$$\lim_{t\to\infty} \frac{\sqrt{\tau(t)}}{\nu(t)}.u(x,t) = -\frac{{\alpha_1}'(\xi)}{\alpha(\xi)}, \tag{4.10}$$

$$\lim_{t\to\infty} \frac{\sqrt{\tau(t)}}{\nu(t)}.v(x,t) = -\frac{{\alpha_2}'(\xi)}{\alpha(\xi)}) + \frac{\alpha_2(\xi)}{\alpha_1(\xi)}.\frac{{\alpha_1}'(\xi)}{\alpha_1(\xi)}, \tag{4.11}$$

$$\lim_{t\to\infty} \frac{\sqrt{\tau(t)}}{\nu(t)}.w(x,t) = -\frac{{\alpha_3}'(\xi)}{\alpha_1(\xi)} + \frac{\alpha_3(\xi)}{\alpha_1(\xi)}.\frac{{\alpha_1}'(\xi)}{\alpha_1(\xi)} + \frac{\alpha_2(\xi)}{\alpha_1(\xi)}.\frac{{\alpha_2}'(\xi)}{\alpha_1(\xi)} - \frac{\alpha_2(\xi)}{\alpha_1(\xi)}.\frac{\alpha_2(\xi)}{\alpha_1(\xi)}.\frac{{\alpha_1}'(\xi)}{\alpha_1(\xi)}, \tag{4.12}$$

uniformly with respect to the variable $\xi = \frac{x}{\sqrt{\tau(t)}}$ *on bounded sets.*

When $\tau(\infty) < \infty$*, we have*

$$\lim_{t\to\infty} \frac{\sqrt{\tau(t)}}{\nu(t)}.u(x,t) = -\frac{\beta_1'(x)}{\beta_1(x)}, \tag{4.13}$$

$$\lim_{t\to\infty} \frac{\sqrt{\tau(t)}}{\nu(t)}.v(x,t) = -\frac{\beta_2'(x)}{\beta_1(x)}) + \frac{\beta_2(x)}{\beta_1(x)}.\frac{\beta_1'(x)}{\beta_1(x)}, \tag{4.14}$$

$$\begin{aligned} \lim_{t\to\infty} \frac{\sqrt{\tau(t)}}{\nu(t)}.w(x,t) = &-\frac{\beta_3'(x)}{\beta_1(x)} + \frac{\beta_3(x)}{\beta_1(x)}.\frac{\beta_1'(x)}{\beta_1(x)} + \frac{\beta_2(x)}{\beta_1(x)}.\frac{\beta_2'(x)}{\beta_1(x)} \\ &- \frac{\beta_2(x)}{\beta_1(x)}.\frac{\beta_2(x)}{\beta_1(x)}.\frac{\beta_1'(x)}{\beta_1(x)}, \end{aligned} \tag{4.15}$$

Proof. First we note that if (U,V,W) is a solution of

$$\begin{gathered} U_t + (\frac{U_x^2}{2}) - \frac{\nu'(t)}{\nu(t)}U = \frac{\nu(t)}{2}U_{xx}, \\ V_t + (U_xV_x) - \frac{\nu'(t)}{\nu(t)}V = \frac{\nu(t)}{2}V_{xx}, \\ W_t + (\frac{V_x^2}{2} + U_xV_x) - \frac{\nu'(t)}{\nu(t)}W = \frac{\nu(t)}{2}W_{xx}, \end{gathered} \tag{4.16}$$

with initial condition

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad W(x,0) = W_0(x), \tag{4.17}$$

where $U_0(x), V_0(x), W_0(x)$ are given by (4.3) then (u,v,w) defined by

$$u = U_x, v = V_x, w = W_x \tag{4.18}$$

is the solution to (4.1) and (4.2). Now we make the transformation

$$\begin{gathered} a(x,t) = e^{-\frac{U(x,t)}{\nu(t)}}, \\ b(x,t) = -\frac{V(x,t)}{\nu(t)}e^{-\frac{U(x,t)}{\nu(t)}}, \\ c(x,t) = (\frac{V(x,t)^2}{2\nu(t)^2} - \frac{W(x,t)}{\nu(t)})e^{-\frac{U(x,t)}{\nu(t)}}. \end{gathered} \tag{4.19}$$

An easy computation shows that

$$a_t - \frac{\nu(t)}{2}a_{xx} = \frac{-e^{-\frac{U}{\nu(t)}}}{\nu(t)}(U_t - \frac{\nu'(t)}{\nu(t)}U + \frac{{U_x}^2}{2} - \frac{\nu(t)}{2}U_{xx}), \tag{4.20}$$

$$\begin{aligned} b_t - \frac{\nu(t)}{2}b_{xx} = &\frac{-e^{-\frac{U}{\nu(t)}}}{\nu(t)}[(V_t - \frac{\nu'(t)}{\nu(t)}V + U_xV_x - \frac{\nu(t)}{2}V_{xx}) \\ &+ \frac{V}{\nu^2(t)}(U_t - \frac{\nu'(t)}{\nu(t)}U + \frac{U_x^2}{2} - \frac{\nu(t)}{2}U_{xx})], \end{aligned} \tag{4.21}$$

$$\begin{aligned}c_t-\frac{\nu(t)}{2}c_{xx}=\frac{-e^{-\frac{U}{\nu(t)}}}{\nu(t)}&[(\frac{W}{\nu(t)}-\frac{V^2}{2\nu^2(t)})(U_t+(\frac{U_x^2}{2})-\frac{\nu'(t)}{\nu(t)}U-\frac{\nu(t)}{2}U_{xx})\\&+\frac{V}{\nu^2(t)}(V_t+(U_xV_x)-\frac{\nu'(t)}{\nu(t)}V-\frac{\nu(t)}{2}V_{xx}),\\&-\frac{1}{\nu(t)}(W_t+(\frac{V_x^2}{2}+U_xV_x)-\frac{\nu'(t)}{\nu(t)}W-\frac{\nu(t)}{2}W_{xx})]\end{aligned}\tag{4.22}$$

From (4.16)-(4.22), we get

$$u=-\nu(t)(\log(a))_x,\quad v=-\nu(t)(\frac{b}{a})_x,\quad w=\nu(t)(-\frac{c}{a}+\frac{b^2}{2a^2})_x.\tag{4.23}$$

is the solution of (4.1) with initial conditions (4.2) if a,b and c are solutions of the equation

$$a_t=\frac{\nu(t)}{2}a_{xx},\quad b_t=\frac{\nu(t)}{2}b_{xx},\quad c_t=\frac{\nu(t)}{2}c_{xx}\tag{4.24}$$

with initial conditions

$$\begin{gathered}a(x,0)=e^{-\frac{U_0(x)}{\nu(0)}},\quad b(x,0)=-\frac{V_0(x)}{\nu(0)}e^{-\frac{U_0(x)}{\nu(0)}},\\c(x,0)=(\frac{V_0(x)^2}{2\nu(0)^2}-\frac{W_0(x)}{\nu(0)})e^{-\frac{U_0(x)}{\nu(0)}}\end{gathered}\tag{4.25}$$

respectively. Solution to (4.24) and (4.25) is

$$a(x,t)=a_1(x,t),\quad b(x,t)=a_2(x,t),\quad c(x,t)=a_3(x,t)$$

where a_1,a_2,a_3 are given by (4.5) and hence the formula (4.9) follows from (4.23).

To see that the solution is infinite times continuously differentiable, it is enough to show that a_j defined by (4.5) is infinitely differentiable when u_0,v_0,w_0 are in L^p for some $1\leq p\leq\infty$. But this follows, by the applications of Holder's inequality, as in the proof of Theorem 3.1. This proves part (a) of the theorem.

To study the large time behaviour, we write (4.9) in a convenient way and follow the method of section 3. We have

$$u(x,t)=-\nu(t)\frac{a_{1x}}{a_1},\tag{4.26}$$

$$v(x,t)=-\nu(t)\frac{a_{2x}}{a_1}+\nu(t)\frac{a_2}{a_1}.\frac{a_{1x}}{a_1},\tag{4.27}$$

$$w(x,t)=-\nu(t)\frac{a_{3x}}{a_1}+\nu(t)\frac{a_3}{a_1}.\frac{a_{1x}}{a_1}+\nu(t)\frac{a_2}{a_1}.\frac{a_{2x}}{a_1}-\nu(t)\frac{a_2}{a_1}.\frac{a_2}{a_1}.\frac{a_{1x}}{a_1}\tag{4.28}$$

Setting $\xi=x/\sqrt{\tau(t)}$, and then making a change of variable $z=\frac{\sqrt{\tau(t)}\xi-y}{\sqrt{\tau(t)}}$ and renaming z as y, we get

$$a_j(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}p_j(\sqrt{\tau(t)}(\xi-y)e^{-y^2/2}dy.$$

Following the analysis of section 3, it follows that

$$\begin{aligned}\lim_{t\to\infty}\sqrt{2\pi}.a_j(x,t)&=p_j(\infty)\int_{-\infty}^{\xi}e^{-y^2/2}dy+p_j(-\infty)\int_{\xi}^{\infty}e^{-y^2/2}dy\\&=\alpha_j(\xi),\quad j=1,2,3.\end{aligned}\tag{4.29}$$

$$\begin{aligned}\lim_{t\to\infty}\sqrt{2\pi\tau(t)}.a_{j\,x}(x,t) &= (p_j(\infty)-p_j(-\infty))e^{-\xi^2/2}\\ &= {\alpha_j}'(\xi), j=1,2,3.\end{aligned}\tag{4.30}$$

These limits are valid uniformly for ξ belonging bounded subsets of R^1. Now we note that (4.26)-(4.28) can be written as

$$\frac{\sqrt{\tau(t)}}{\nu(t)}u(x,t) = -\sqrt{\tau(t)}\frac{a_{1x}}{a_1},\tag{4.31}$$

$$\frac{\sqrt{\tau(t)}}{\nu(t)}v(x,t) = -\sqrt{\tau(t)}\frac{a_{2x}}{a_1}+\frac{a_2}{a_1}.\sqrt{\tau(t)}\frac{a_{1x}}{a_1},\tag{4.32}$$

$$\frac{\sqrt{\tau(t)}}{\nu(t)}w(x,t) = -\sqrt{\tau(t)}\frac{a_{3x}}{a_1}+\frac{a_3}{a_1}.\sqrt{\tau(t)}\frac{a_{1x}}{a_1}+\frac{a_2}{a_1}.\sqrt{\tau(t)}\frac{a_{2x}}{a_1}-\frac{a_2}{a_1}.\frac{a_2}{a_1}.\sqrt{\tau(t)}\frac{a_{1x}}{a_1}\tag{4.33}$$

We observe that $\alpha(\xi)>0$ and hence letting t tends to infinity in (4.31) -(4.33) and using (4.29) and (4.30) we get the asymptotic form (4.10)-(4.12).

When $\tau(\infty)$ finite the asymptotic form (4.13) -(4.15) follows from (4.5), (4.8) and (4.31)-(4.33). The proof of the theorem is complete. □

Note that the proof of the theorem also shows that the boundary value problem for (4.1), with initial conditions $u(x,0)=u_0(x), v(x,0)=v_0(x), w(x,0)=w_0(x)$ for $x>0$ and boundary conditions $u(0,t)=v(0,t)=w(0,t)=0$ is given by

$$u=-\nu(t)(\log(b_1))_x,\quad v=-\nu(t)(\frac{b_2}{b_1})_x,\quad w=\nu(t)(-\frac{b_3}{b_1}+\frac{{b_2}^2}{2{b_1}^2})_x.\tag{4.34}$$

where the functions $b_j, j=1,2,3$ are given by

$$b_j(x,t)=\frac{1}{\sqrt{2\pi\tau(t)}}\int_{-\infty}^{+\infty}p_j(y)\{e^{-\frac{(x-y)^2}{2\tau(t)}}+e^{-\frac{(x+y)^2}{2\tau(t)}}\}dy,\quad j=1,2,3.\tag{4.35}$$

With $\tau(t)$ and $p_j(x)$ as given before by (4.6) (4.3)and (4.4). Further When $\tau(\infty)=\infty$, we have

$$\lim_{t\to\infty}\frac{\sqrt{\tau(t)}}{\nu(t)}.u(x,t)=0,\quad \lim_{t\to\infty}\frac{\sqrt{\tau(t)}}{\nu(t)}.v(x,t)=0,\quad \lim_{t\to\infty}\frac{\sqrt{\tau(t)}}{\nu(t)}.w(x,t)=0.$$

When $\tau(\infty)<\infty$, we have

$$\lim_{t\to\infty}\frac{\sqrt{\tau(t)}}{\nu(t)}.u(x,t),\quad \lim_{t\to\infty}\frac{\sqrt{\tau(t)}}{\nu(t)}.v(x,t),\quad \lim_{t\to\infty}\frac{\sqrt{\tau(t)}}{\nu(t)}.w(x,t)$$

are given by (4.13)-(4.15) with β_j replaced by B_j where

$$B_j(x)=\int_0^{+\infty}p_j(y)\{e^{-\frac{(x-y)^2}{2\tau(\infty)}}+e^{-\frac{(x-y)^2}{2\tau(\infty)}}\}dy.$$

5. Explicit solution of (1.10) and its asymptotic behaviour

Here we consider vector equivalent of Burgers equation with an additional geometrical term and variable viscosity namely

$$U_t+U.\nabla U-\frac{\nu'(t)}{\nu(t)}U=\frac{\nu(t)}{2}\Delta U.\tag{5.1}$$

Here we assume that $\nu(t) > 0$ for $t \geq 0$ and twice continuously differentiable. The special case $\nu(t)$ a positive constant was treated first by by Nerney et al [22] and by Joseph and Sachdev [14]. We generalize their results for variable $\nu(t)$, and write down explicit solution of the equation (5.1) with initial conditions of the form

$$U(x,0) = \nabla_x \phi_0(x). \tag{5.2}$$

We also study the asymptotic behaviour solutions for large time and vanishing viscosity limits. More precisely, we prove the following theorem.

Theorem 5.1. *(a). Under the assumptions, $\nabla_x \phi_0(x)$ bounded or integrable, the function*

$$U(x,t) = \frac{\nu(t)}{\nu(0)} \frac{\int_{R^n} \nabla_x \phi_0(y) \exp(-\frac{|x-y|^2}{2\tau(t)} - \frac{\phi_0(y)}{\nu(0)}) dy}{\int_{R^n} \exp(-\frac{|x-y|^2}{2\tau(t)} - \frac{\phi_0(y)}{\nu(0)}) dy}. \tag{5.3}$$

is twice continuously differentiable in the variables (x,t) and is exact solution of the initial value problem for (5.1) and (5.2), where $\tau(t) = \int_0^t \nu(s) ds$.
(b). Assume that

$$\phi_0(x) = \sum_1^n \phi_0^i(x_i) + o(1), |x| \to \infty, \tag{5.4}$$

where $\phi_0^i(x), i = 1, 2, \dots, n$ are differentiable functions from R^1 to R^1 and the limits

$$\lim_{x_i \to -\infty} \phi_0^i(x_i) = \phi_i^-, \lim_{x_i \to \infty} \phi_0^i(x_i) = \phi_i^+, \tag{5.5}$$

exist. Let

$$\begin{gathered} \xi = \frac{x}{\tau(t)^{1/2}}, \tau(t) = \int_0^t \nu(s) ds, k_i = \frac{(\phi_i^+ - \phi_i^-)}{\nu(0)} \\ g_i(\xi_i) = \exp(-\frac{k_i}{2}) \int_{-\infty}^{\xi_i} \exp(-\frac{z_i^2}{2}) dz_i + \exp(\frac{k_i}{2}) \int_{\xi_i}^{\infty} \exp(-\frac{z_i^2}{2}) dz_i, \end{gathered} \tag{5.6}$$

for $i = 1, 2, \dots n$. Then if $\tau(\infty) = \infty$

$$\lim_{t \to \infty} (\frac{\tau(t)^{1/2}}{\nu(t)}) U((\tau(t))^{1/2}\xi, t) = -(\frac{g_1'(\xi_1)}{g_1(\xi_1)}, \quad \frac{g_2'(\xi_2)}{g_2(\xi_2)}, \dots, \frac{g_n'(\xi_n)}{g_n(\xi_n)}). \tag{5.7}$$

If $\tau(\infty) < \infty$, then

$$\lim_{t \to \infty} \frac{\nu(0)}{\nu(t)} U(x,t) = \frac{\int_{R^n} \nabla_x \phi_0(y) \exp(-\frac{|x-y|^2}{2\tau(\infty)} - \frac{\phi_0(y)}{\nu(0)}) dy}{\int_{R^n} \exp(-\frac{|x-y|^2}{2\tau(\infty)} - \frac{\phi_0(y)}{\nu(0)}) dy}. \tag{5.8}$$

(c). Let $U(x,t)$ is the solution of (5.1) with the initial

$$U(x,0) = x \chi_{[|x| \leq l_0]}(x) \tag{5.9}$$

where $\chi_{[|x| \leq l_0]}(x)$ is the characteristic function of the of the ball $B(0, l_0) = [x : |x| \leq l_0]$. If $\tau(\infty) = \infty$ then we have

$$\lim_{t \to \infty} U(x,t) = U^{\infty}(x,t) \tag{5.10}$$

uniformly in the variable $\xi = \frac{x}{(\tau(t))^{1/2}}$ *belonging to a bounded subset of* R^n*, where* U^∞ *is given by*

$$U^\infty(x,t) = \frac{x/(\frac{\tau(t)}{\nu(t)})^{1/2}}{(\frac{\tau(t)}{\nu(t)})^{1/2}[1 + \frac{\tau(t)^{n/2}}{c_0}\exp(\frac{|x|^2}{2\tau(t)})]} \tag{5.11}$$

with

$$c_0 = \frac{\exp(\frac{l_0^2}{2\nu(0)})}{(2\pi)^{\frac{n}{2}}}[\int_{[|y|\le l_0]} \exp(-\frac{|z|^2}{2\nu(0)})dz - \exp(-\frac{l_0^2}{2\nu(0)})|B(0,l_0)|]. \tag{5.12}$$

Here $|B(0,l_0)|$ *denotes the volume, if space dimension* $n \ge 3$*, area if* $n = 2$ *and length if* $n = 1$*, of* $B(0,l_0)$*.*
(d). Further assume that $\nabla_x\phi_0(x) \in L^1(R)$ *then we have the following estimates*

$$\|U(x,t)\|_p = O(1)\tau^{\frac{-1}{2}(n-1/p)}. \tag{5.13}$$

(e). Assume $\nu(t) = \epsilon\nu_0(t)$*,* $\tau_0(t) = \int_0^t \nu_0(s)ds$ *and* U^ϵ *the corresponding solution of* (5.1) *and* (5.2)*, then*

$$\lim_{\epsilon\to} U^\epsilon(x,t) = \frac{\nu_0(t)}{\tau_0(t)}(x - y(x,\tau_0(t)) \tag{5.14}$$

where $y(x,\tau)$ *minimizes*

$$\min_{y\in R^n}(\frac{|x-y|^2}{2\tau_0(t)} - \frac{\phi_0(y)}{\nu(0)})\,. \tag{5.15}$$

Proof. If a solution U is sought as a gradient of some scalar function ϕ ,

$$U = \nabla_x\phi, \tag{5.16}$$

then equation (5.1) becomes

$$\nabla_x[\phi_t + \frac{|\nabla\phi|^2}{2} - \frac{\nu(t)'}{\nu(t)}\nabla\phi - \frac{\nu(t)}{2}\Delta\phi] = 0.$$

This leads to

$$\phi_t + \frac{|\nabla\phi|^2}{2} - \frac{\nu'(t)}{\nu(t)}\phi - \frac{\nu(t)}{2}\Delta\phi = f(t),$$

where $f(t)$ is an arbitrary function of t. Since we are interested in the space derivative $\nabla_x\phi$ and this is independent of $f(t)$, we let $f(t) = 0$. If we are given an initial data for U which is gradient of some scalar function ϕ_0 of the form, $U(x,0) = \nabla_x\phi_0(x)$, by (5.16), it is enough to find a solution ϕ of

$$\phi_t + \frac{|\nabla\phi|^2}{2} - \frac{\nu(t)'}{\nu(t)}\phi - \frac{\nu}{2}\Delta\phi = 0 \tag{5.17}$$

with initial condition

$$\phi(x,0) = \phi_0(x). \tag{5.18}$$

We may then use (5.16) to get the solution U of (5.1) and (5.2). To solve (5.17) and (5.18) we use the Hopf-Cole transformation

$$\theta(x,t) = \exp(-\frac{\phi}{\nu(t)}). \tag{5.19}$$

From (5.17) - (5.19) it follows that if θ is the solution of the linear problem with variable viscosity

$$\theta_t = \frac{\nu(t)}{2}\Delta\theta, \quad \theta(x,0) = \exp(-\frac{\phi_0(x)}{\nu(0)}). \tag{5.20}$$

then ϕ given by (5.19) gives the solution to (5.17) and (5.18). Solving (5.20), we have

$$\theta(x,t) = \frac{1}{(2\pi\tau(t))^{\frac{n}{2}}}\int_{R^n}\exp(-\frac{|x-y|^2}{2\tau(t)} - \frac{\phi_0(y)}{\nu(0)})dy. \tag{5.21}$$

From (5.16) and (5.19) we see that the solution to (5.1) and (5.2) is given by

$$U = -\nu(t)\frac{\nabla\theta}{\theta}. \tag{5.22}$$

Now

$$\begin{aligned}
\partial_{x_j}\theta(x,t) &= \frac{1}{(2\pi\tau(t))^{\frac{n}{2}}}\int_{R^n}\partial_{x_j}(\exp(-\frac{|x-y|^2}{2\tau(t)}))\exp(-\frac{\phi_0(y)}{\nu(0)})dy \\
&= -\frac{1}{(2\pi\tau(t))^{\frac{n}{2}}}\int_{R^n}\partial_{y_j}(\exp(-\frac{|x-y|^2}{2\tau(t)}))\exp(-\frac{\phi_0(y)}{\nu(0)})dy \\
&= \frac{1}{(2\pi\tau(t))^{\frac{n}{2}}}\int_{R^n}\exp(-\frac{|x-y|^2}{2\tau(t)})\partial_{y_j}(\exp(-\frac{\phi_0(y)}{\nu(0)}))dy \\
&= -\frac{1}{(2\pi\tau(t))^{\frac{n}{2}}}\int_{R^n}\frac{\partial_{y_j}\phi(y)}{\nu(0)}\exp(-\frac{|x-y|^2}{2\tau(t)})\exp-(\frac{\phi_0(y)}{\nu(0)})dy
\end{aligned} \tag{5.23}$$

From (5.22) and (5.23) explicit formula (5.3) for (5.1) and (5.2) follows. This function is smooth follows as in the proof of Theorem 3.1.

Next we study the asymptotic behaviour of this solution as $t \to \infty$. The asymptotic form depends on $\tau(\infty)$ is finite or not. First we study case when $\tau(\infty) = \infty$. To prove the asymptotic form, consider θ given by (5.21), where ϕ_0 satisfies the conditions (5.4) and (5.5). After a change of variable the expression for θ becomes

$$\begin{aligned}
\theta(x,t) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{R^n}\exp(-\frac{|z|^2}{2} - \frac{1}{\nu(0)}\phi_0(x-(\tau(t))^{1/2}z)dz \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{R^n}\exp(-\frac{|z|^2}{2} - \frac{1}{\nu(0)}\phi_0((\tau(t))^{1/2}(\xi - z))dz
\end{aligned}$$

Now using (5.4) we get

$$\begin{aligned}
\theta(x,t) &\approx \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{R^n}\exp(-\frac{|z|^2}{2} - \frac{1}{\nu(0)}\sum_1^n\phi_0^i((\tau(t))^{1/2}(\xi_i - z_i)))dz_i \\
&= \prod_i^n\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\exp(-\frac{z_i^2}{2} - \frac{1}{\nu(0)}\phi_0^i((\tau)^{1/2}(\xi_i - z_i)))dz_i
\end{aligned} \tag{5.24}$$

as $t \to \infty$ uniformly on bounded sets of ξ in R^n. Now take the i-th term in this product. As in section 2, following the argument of Hopf [7], we get

$$\begin{aligned}
&\int_{-\infty}^{\infty}\exp(-\frac{z_i^2}{2} - \frac{1}{\nu(0)}\phi_0^i((\tau(t))^{1/2}(\xi_i - z_i)))dz_i \\
&\approx \exp(-\frac{\phi_i^+}{\nu(0)})\int_{-\infty}^{\xi_i}\exp(-\frac{z_i^2}{2})dz_i + \exp(-\frac{\phi_i^-}{\nu(0)})\int_{\xi_i}^{\infty}\exp(-\frac{z_i^2}{2})dz_i.
\end{aligned} \tag{5.25}$$

Thus from (5.24) and (5.25), we have

$$\begin{aligned}(2\pi)^{\frac{n}{2}}\lim_{t\to\infty}\theta(\xi(\tau(t))^{1/2},t) = \prod_i^n &(\exp(-\frac{1}{\nu(0)}\phi_i^+)\int_{-\infty}^{\xi_i}\exp(-\frac{z_i^2}{2})dz_i \\ &+\exp(-\frac{1}{\nu(0)}\phi_i^-)\int_{\xi_i}^{\infty}\exp(-\frac{z_i^2}{2})dz_i).\end{aligned} \tag{5.26}$$

Similarly, we get

$$\begin{aligned}(2\pi)^{\frac{n}{2}}\lim_{t\to\infty}\tau(t)^{1/2}\theta_{x_l}(\xi(\tau(t))^{1/2},t) = \prod_{i\neq l}&(\exp(-\frac{1}{\nu(0)}\phi_i^+)\int_{-\infty}^{\xi_i}\exp(-\frac{z_i^2}{2})dz_i \\ &+\exp(-\frac{1}{\nu(0)}\phi_i^-)\int_{\xi_i}^{\infty}\exp(-\frac{z_i^2}{2})dz_i) \\ &\times(\exp(-\frac{\phi_l^+}{\nu(0)})-\exp(-\frac{\phi_l^-}{\nu(0)}))\exp(-\frac{\xi_l^2}{2})\end{aligned} \tag{5.27}$$

Since $\frac{(\tau(t))^{1/2}}{\nu(t)}U = -(\tau(t))^{1/2}\frac{\nabla_x\theta}{\theta}$ and θ is bounded away from 0, we have, from (5.26) and (5.27),

$$\begin{aligned}&\lim_{t\to\infty}\frac{(\tau(t))^{\frac{1}{2}}}{\nu(t)}U((\tau(t)^{1/2}\xi,t) \\ &= -\lim_{t\to\infty}\tau(t)^{1/2}(\frac{\theta^\nu_{x_1}}{\theta},\frac{\theta^\nu_{x_2}}{\theta},\dots,\frac{\theta^\nu_{x_n}}{\theta}) = -(\frac{g_1'(\xi_1)}{g_1(\xi_1)},\dots,\frac{g_n'(\xi_n)}{g_n(\xi_n)})\end{aligned}$$

which is (5.7). The case $\tau(\infty)$ is simple and the form (5.8) follows from (5.3).

Now we consider the equation (5.1) with antisymmetric initial data (5.9). Note that this initial data can be written as the gradient of ϕ_0 where

$$\phi_0(x) = (\frac{|x|^2}{2}\chi_{[|x|\leq l_0]}(x)+\frac{{l_0}^2}{2}(1-\chi_{[|x|\leq l_0]}(x))).$$

By (5.21) and (5.22), the solution is given by

$$U(x,t) = -\nu(t)\frac{\nabla Q}{Q}, \tag{5.28}$$

where $Q(x,t)$ can be written as

$$Q(x,t) = I_1+\exp(-\frac{l_0^2}{2\nu(0)})I_2 \tag{5.29}$$

where

$$\begin{gathered}I_1 = \frac{1}{(2\pi\tau(t))^{\frac{n}{2}}}\int_{[|y|\leq l_0]}\exp(-\frac{1}{2}[\frac{|y|^2}{\nu(0)}+\frac{|x-y|^2}{\tau(t)}])dy, \\ I_2 = \frac{1}{(2\pi\tau(t))^{\frac{n}{2}}}\int_{[|y|>l_0]}\exp(-\frac{|x-y|^2}{2\tau(t)}]dy.\end{gathered}$$

Here we take the case $\tau(\infty)=\infty$. The case $\tau(\infty)<\infty$ is already covered by (5.8). It can be easily checked by making use of error function and its asymptotics that, as $t\to\infty$,

$$I_1 \approx \frac{\exp(-\frac{|x|^2}{2\tau(t)})}{(2\pi\tau(t)^{\frac{n}{2}}}\int_{[|z|\leq l_0]}\exp(-\frac{z^2}{2\nu(0)})dz, I_2\approx[1-\frac{\exp(-\frac{|x|^2}{2\tau(t)})}{(2\pi\tau(t))^{\frac{n}{2}}}|B(0,l_0)|]. \tag{5.30}$$

Substituting these asymptotics in (5.29) , we get

$$
\begin{aligned}
&Q(x,t)\\
&\approx \exp(-\frac{l_0^2}{2\nu(0)}) + \frac{\exp(-\frac{|x|^2}{2\tau(t)})}{(2\pi\tau(t)^{\frac{n}{2}}}\Big[\int_{[|z|\le l_0]}\exp(-\frac{|z|^2}{2\nu(0)})dz - \exp(-\frac{l_0^2}{2\nu(0)})|B(0,l_0)|\Big].
\end{aligned} \tag{5.31}
$$

Similarly,

$$
\begin{aligned}
&\nabla_x Q(x,t)\\
&\approx -\frac{1}{\tau(t)^{1/2}}.\frac{x}{\tau(t)^{\frac{1}{2}}}.\frac{\exp(-\frac{|x|^2}{2\tau(t)})}{(2\pi\tau(t))^{\frac{n}{2}}}[\int_{[|z|\le l_0]}\exp(-\frac{|z|^2}{2\nu(0)})dz - \exp(-\frac{l_0^2}{2\nu(0)})|B(0,l_0)|].
\end{aligned} \tag{5.32}
$$

From (5.29), (5.30) and (5.32) we get

$$
\begin{aligned}
&U(x,t)\approx\\
&\frac{x}{\tau(t)^{1/2}}.\frac{\nu(t)}{\tau(t)^{1/2}}\frac{\frac{\exp(-\frac{|x|^2}{2\tau(t)})}{(2\pi\tau t)^{\frac{n}{2}}}[\int_{[|z|\le l_0]}\exp(-\frac{|z|^2}{2\nu(0)})dz - \exp(-\frac{l_0^2}{2\nu(0)})|B(0,l_0)|]}{\exp(-\frac{l_0^2}{2\nu}) + \frac{\exp(-\frac{|x|^2}{2\nu t})}{(2\pi\nu t)^{\frac{n}{2}}}[\int_{[|z|\le l_0]}\exp(-\frac{|z|^2}{2\nu})dz - \exp(-\frac{l_0^2}{2\nu})|B(0,l_0)|]}.
\end{aligned}
$$

On rearranging the terms we get

$$
U(x,t)\approx \frac{\frac{x}{(\frac{\tau(t)}{\nu(t)})^{1/2}}}{(\frac{\tau(t)}{\nu(t)})^{1/2}[1+\frac{\tau(t)^{\frac{n}{2}}}{c_0}\exp(\frac{|x|^2}{2\tau(t)})]} \tag{5.33}
$$

where c_0 is given by (5.12).

Now to prove the decay estimates, for general initial data $\nabla_x\phi_0(x)\in L^1(R^n)$. First note that in this case $\phi_0(x)$ is bounded function and hence there exists constants c_1, c_2 both positive such that the estimate

$$
c_1 = \inf_{x\in R^n}(e^{\frac{-\phi_0(x)}{\nu(0)}}) \le \theta \le \sup_{x\in R^n} e^{\frac{-\phi_0(x)}{\nu(0)}} = c_2.
$$

Hence from (5.22) from the expression (5.21) for θ we get

$$
|U(x,t)| \le \frac{c_2}{c_1}\frac{1}{(2\pi\tau(t))^{\frac{n}{2}}}\int_{R^n}|\nabla\phi(y)|\exp(-\frac{|x-y|^2}{2\tau(t)})dy. \tag{5.34}
$$

The right hand side integral is the convolution of heat kernel with $|\nabla\phi|$ and hence by the Young's inequality we get the estimate (5.13) This completes the proof of (5.14).

Now to study the inviscid the limit $\lim_{\epsilon\to 0} U^\epsilon(x,t)$ where U^ϵ is the solution of (5.1) and (5.2) given by the formula (5.3) when $\nu(t)=\epsilon\nu_0(t)$. First we note that

$$
U^\epsilon(x,t) = \nu_0(t)\int_{R^n}(\frac{x-y}{\tau_0(t)})d\mu^\epsilon_{(x,t)}(y) \tag{5.35}
$$

where, for each (x,t) and $\epsilon>0$, $d\mu^\epsilon_{(x,t)}(y)$ is a probability measure given explicitly by

$$
d\mu^\epsilon_{(x,t)}(y) = \frac{\exp(-\frac{1}{\epsilon}(\frac{|x-y|^2}{2\tau(t)}+\frac{\phi_0(y)}{\nu(0)})dy}{\int_{R^n}\exp(-\frac{1}{\epsilon}(\frac{|x-y|^2}{2\tau(t)}+\frac{\phi_0(y)}{\nu(0)})dy}. \tag{5.36}
$$

Following the argument of Hopf [7] and Lax [17] it can easily be seen that this measure tends to the δ-measure concentrated at $y(x,\tau(t))$, the minimizer of (5.15), which is unique for almost every (x,t). So for almost all (x,t) we get from (5.35) and (5.36)

$$\lim_{\epsilon\to 0} U^{\epsilon}(x,t) = \frac{(x - y_0(x,\tau(t))}{t}$$

where $y(x,\tau(t))$ is as before a minimizer of (5.15). $\square$

Remark 5.2. Here we observe that for Burgers equation, that is, when $n=1$ and $\nu(t)=\nu$, a constant, the parameter k_1 can be computed in terms of the mass of the initial data,

$$\nu k_1 = \phi_1^+ - \phi_1^- = \int_{x_0}^{\infty} u_0(y)dy - \int_{x_0}^{-\infty} u_0(y)dy = \int_{-\infty}^{\infty} u_0(y)dy$$

and then formula (5.17) with $n=1$ is exactly Hopf's result. Also we note that the limit function obtained in the result written in the (x,t) variable, namely

$$U(x,t) = (-\nu[\log(v_1(x_1,t))]_{x_1}, -\nu[\log(v_2(x_2,t))]_{x_2}, \dots, -\nu[\log(v_n(x_n,t))]_{x_n})$$

where for $i = 1,2,\dots n$,

$$v_i(x_i,t) = \exp(-\frac{k_i}{2}) \int_{-\infty}^{x_i/(\nu t)^{1/2}} \exp(-\frac{z_i^2}{2})dz_i + \exp(\frac{k_i}{2}) \int_{x_i/(\nu t)^{1/2}}^{\infty} \exp(-\frac{z_i^2}{2})dz_i$$

is an exact solution of (5.1).

Remark 5.3. It is easy to see that, for c_0 a constant

$$\theta(x,t) = \frac{1}{c_0} + \tau(t)^{-\frac{n}{2}} \exp(-\frac{|x|^2}{2\tau(t)})$$

is a solution of the of the heat equation with variable viscosity coefficient namely $\theta_t = \frac{\nu(t)}{2}\Delta\theta$. Its space gradient is

$$\nabla_x\theta(x,t) = -\frac{x}{\tau(t)}\tau(t)^{-\frac{n}{2}} \exp(-\frac{|x|^2}{2\tau(t)}).$$

By earlier discussion $U^{\infty} = -\nu(t)\frac{\nabla\theta}{\theta}$ is an exact solution of the vector Burgers equation (5.1). On simplification we get

$$U^{\infty}(x,t) = \frac{x/\frac{\tau(t)}{\nu(t)}^{1/2}}{\frac{\tau(t)}{\nu(t)}^{1/2}[1 + \frac{t^{\frac{n}{2}}}{c_0}\exp(\frac{|x|^2}{2t\nu})]}$$

For $n=1$ and $\nu(t)=\nu$, a constant this exact solution of the Burgers equation was discovered by Lighthill [20]. Sachdev, Joseph and Nair [28] showed that it can be obtained as time asymptotic of a pure initial vale problem. This solution is called the N-wave solution.

Remark 5.4. Note that the proof of part (e) of the theorem gives an explicit formula for the solution of the initial value problem for the Hamilton-Jacobi equation

$$\begin{gathered} \phi_t + \frac{1}{2}|\nabla\phi|^2 - \frac{\nu_0(t)'}{\nu_0(t)}\phi = 0, \\ \phi(x,0) = \phi_0(x) \end{gathered} \tag{5.37}$$

namely

$$\phi(x,t) = \min_y [\frac{\phi_0(y)}{\nu(0)} + \frac{\|x-y\|^2}{2\tau_0(t)}]. \tag{5.38}$$

Note that when $\nu_0(t)$ is a constant this explicit formula was derived for the viscosity solution of (5.37) earlier by other methods, see Lions [21]. Further, $\phi(x,t)$ is Lipschitz continuous when its initial data $\phi_0(x)$ is and is a solution to (5.37). Our analysis shows that the explicit formula (5.38) follows from the vanishing viscosity limit of

$$\begin{gathered} \phi_t + \frac{1}{2}|\nabla\phi|^2 - \frac{\nu_0'(t)}{\nu_0(t)}\phi = \epsilon\frac{\nu_0(t)}{2}\Delta\phi \\ \phi(x,0) = \phi_0(x). \end{gathered} \tag{5.39}$$

as the solution of (5.39) is given by the formula

$$\phi^\epsilon(x,t) = -\epsilon\nu_0(t)\log[\frac{1}{(2\pi\epsilon\nu_0(t)^{\frac{n}{2}}}\int_{R^n}\exp(-\frac{1}{\epsilon}(\frac{|x-y|^2}{2\tau_0(t)} + \frac{\phi_0(y)}{\nu_0(0)})dy.$$

By passing to the limit as ϵ goes to 0 we get exactly the formula (5.38).

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