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POSITIVE SOLUTIONS FOR SEMIPOSITONE FOURTH-ORDER TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract. In this paper we investigate the existence of positive solutions of the following nonlinear semipositone fourth-order two-point boundary-value problem with second derivative:

$$u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 \le t \le 1,$$
$$u'(1) = u''(1) = u'''(1) = 0, \quad ku(0) = u'''(0),$$

where $-6 < k < 0$, $f \ge -M$, and M is a positive constant. Our approach relies on the Krasnosel'skii fixed point theorem.

1. Introduction

Recently an increasing interest in studying the existence of positive solutions for fourth-order two-point boundary value problems is observed. Among others we refer to [1, 2, 3, 4, 5, 6, 7, 8, 9].

In this paper we consider the positive solutions of the following nonlinear semipositone fourth-order two-point boundary value problem with second derivative:

$$\begin{gathered} u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 \le t \le 1, \\ u'(1) = u''(1) = u'''(1) = 0, \quad ku(0) = u'''(0), \end{gathered} \tag{1.1}$$

where $-6 < k < 0$, f is continuous and there exists $M > 0$ such that $f \ge -M$. This implies that f is not necessarily nonnegative, monotone, superlinear and sublinear. And also this assumption implies that the problem (1.1) is semipositone .

The purpose of this paper is to establish the existence of positive solutions of problem (1.1) by using Krasnosel'skii fixed point theorem in cones.

The rest of this paper is organized as follows: in section 2, we present some preliminaries and lemmas. Section 3 is devoted to proving the existence of positive solutions of problem (1.1). An example is considered in section 4 to illustrate our main results.

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2. Preliminaries and lemmas

Let $C^2[0,1]$ be the Banach space with norm $\|u\|_0 = \max\{\|u\|, \|u''\|\}$, where

$$\|u\| = \max_{0\le t\le 1} |u(t)|, \quad u \in C[0,1].$$

By routine calculation, we easily obtain the following Lemma.

Lemma 2.1. *If $k \neq 0$, then*

$$\begin{gathered} u^{(4)}(t) = h(t), \quad 0 \le t \le 1, \\ u'(1) = u''(1) = u'''(1) = 0, \quad ku(0) = u'''(0), \end{gathered}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)h(s)ds,$$

where the Green function is

$$G(t,s) = -\frac{1}{6}\begin{cases} \frac{6}{k} + s^3, & 0 \le s \le t \le 1, \\ \frac{6}{k} - (s-t)^3 + s^3, & 0 \le t \le s \le 1. \end{cases}$$

Remark 2.2. If $-6 < k < 0$, then

$$0 < (1+\frac{k}{6})G(0,s) \le G(t,s) \le G(0,s) = \max_{0\le t\le 1} G(t,s) = -\frac{1}{k} \tag{2.1}$$

in closed bounded region $D = \{(t,s) : 0 \le t \le 1, 0 \le s \le 1\}$.

Let

$$p(t) := \int_0^1 G(t,s)ds = \frac{1}{24}t^4 - \frac{1}{6}t^3 + \frac{1}{4}t^2 - \frac{1}{6}t - \frac{1}{k}, \quad 0 \le t \le 1.$$

Since

$$\begin{gathered} p'(t) = \frac{1}{6}t^3 - \frac{1}{2}t^2 + \frac{1}{2}t - \frac{1}{6} = -\frac{1}{6}(1-t)^3 \le 0, \quad 0 \le t \le 1, \\ p''(t) = \frac{1}{2}t^2 - t + \frac{1}{2} = \frac{1}{2}(1-t)^2 \ge 0, \quad 0 \le t \le 1, \end{gathered}$$

we have

$$\|p\| = \max_{0\le t\le 1} p(t) = p(0) = -\frac{1}{k}, \quad \min_{0\le t\le 1} p(t) = p(1) = -\frac{1}{k} - \frac{1}{24}, \tag{2.2}$$

$$\|p''\| = \max_{0\le t\le 1} |p''(t)| = \frac{1}{2}. \tag{2.3}$$

Our approach is based on the following Krasnosel'skii fixed point theorem.

Lemma 2.3. *Let X be a Banach space, and $K \subset X$ be a cone in X. Assume Ω_1, Ω_2 are bounded open subsets of K with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $F : K \to K$ be a completely continuous operator such that either*

(1) $\|Fu\| \le \|u\|, u \in \partial\Omega_1$, *and* $\|Fu\| \ge \|u\|, u \in \partial\Omega_2$, *or*
(2) $\|Fu\| \ge \|u\|, u \in \partial\Omega_1$, *and* $\|Fu\| \le \|u\|, u \in \partial\Omega_2$.

Then F has a fixed point in $\overline{\Omega}_2 \setminus \Omega_1$.

To apply the Krasnosel'skii fixed point theorem, we need to construct a suitable cone. Let

$$\begin{aligned} C_0^2[0,1] = \{u \in C^2[0,1] : u(t) \ge 0,\ u''(t) \ge 0,\ 0 \le t \le 1, \\ u'(1) = u''(1) = u'''(1) = 0,\ ku(0) = u'''(0)\}. \end{aligned}$$

It is easy to check that the following set P is a cone in $C^2[0,1]$:

$$P = \big\{u \in C_0^2[0,1] : \min_{0\le t\le 1} u(t) \ge (1+\frac{k}{6})\|u\|\big\},$$

where $-6 < k < 0$. For convenience, let

$$\alpha(r) = \max\{f(t,u,v) : (t,u,v) \in D_1(r)\}, \tag{2.4}$$

$$\beta(r) = \min\{f(t,u,v) : (t,u,v) \in D_2(r)\}, \tag{2.5}$$

where

$$D_1(r) = \big\{(t,u,v) : 0 \le t \le 1,\ \frac{M}{k} \le u \le r + (\frac{1}{k} + \frac{1}{24})M,\ -\frac{M}{2} \le v \le r\big\},$$

$$\begin{aligned} D_2(r) = \big\{(t,u,v) : \frac{1}{4} \le t \le \frac{3}{4},\ (\frac{1}{k} + \frac{175}{6144})M \le u \le r + (\frac{1}{k} + \frac{85}{2048})M, \\ -\frac{9}{32}M \le v \le r - \frac{1}{32}M\big\}. \end{aligned}$$

$$C_1 = \min\Big\{\Big[\max_{0\le t\le 1}\int_0^1 G(t,s)ds\Big]^{-1}, \Big[\max_{0\le t\le 1}\int_0^1 |G''(t,s)|ds\Big]^{-1}\Big\} = \min\{-k, 2\},$$

$$\begin{aligned} C_2 &= \max\Big\{\Big[\max_{0\le t\le 1}\int_{\frac14}^{\frac34} G(t,s)ds\Big]^{-1}, \Big[\max_{0\le t\le 1}\int_{\frac14}^{\frac34} |G''(t,s)|ds\Big]^{-1}\Big\} \\ &= \max\Big\{\big(-\frac{1}{2k} + \frac{1}{6144}\big)^{-1}, \frac{32}{9}\Big\}. \end{aligned}$$

Obviously, $0 < C_1 < C_2$.

3. Main results

Theorem 3.1. *Let $-6 < k < 0$. Assume that*

$$f : [0,1] \times [\frac{M}{k}, +\infty) \times [-\frac{M}{2}, +\infty) \to [-M, +\infty) \tag{3.1}$$

is continuous, where $M > 0$ is a constant. Suppose there exist two positive numbers r_1 and r_2 with $\min\{r_1, r_2\} > \frac{-6}{6k+k^2}M$ such that

$$\alpha(r_1) \le r_1 C_1 - M, \quad \beta(r_2) \ge r_2 C_2 - M, \tag{3.2}$$

where α, β are as in (2.4) and (2.5), respectively. Then problem (1.1) *has at least one positive solution.*

Proof. Let $u_0(t) = Mp(t), 0 \le t \le 1$. Then by (2.1) and (2.3) we have

$$(-\frac{1}{k} - \frac{1}{24})M \le u_0(t) \le -\frac{M}{k}, \quad 0 \le u_0''(t) \le \frac{1}{2}M, \quad 0 \le t \le 1. \tag{3.3}$$

Consider the fourth-order two-point boundary-value problem

$$\begin{gathered} u^{(4)}(t) = f(t, u(t) - u_0(t), u''(t) - u_0''(t)) + M, \quad 0 \le t \le 1, \\ u'(1) = u''(1) = u'''(1) = 0, \\ ku(0) = u'''(0), \end{gathered} \tag{3.4}$$

This problem is equivalent to the integral equation

$$u(t)=\int_0^1 G(t,s)[f(s,u(s)-u_0(s),u''(s)-u_0''(s))+M]ds.$$

For $u\in C_0^2[0,1]$, we define the operator A as follows

$$(Au)(t)=\int_0^1 G(t,s)[f(s,u(s)-u_0(s),u''(s)-u_0''(s))+M]ds,\quad 0\le t\le 1.$$

Computing the second derivative of $(Au)(t)$, we obtain

$$(Au)''(t)=\int_t^1 (s-t)[f(s,u(s)-u_0(s),u''(s)-u_0''(s))+M]ds,\quad 0\le t\le 1.$$

Noticing (3.3) and that $u\in C_0^2[0,1]$, we have

$$\frac{M}{k}\le u(t)-u_0(t)<+\infty,$$
$$-\frac{1}{2}M\le u''(t)-u_0''(t)<+\infty,\quad 0\le t\le 1.$$

Thus, from (3.1) we get

$$(Au)(t)\ge 0,\quad (Au)''(t)\ge 0,\quad t\in[0,1].$$

By the definition of $G(t,s)$,

$$G'(1,s)=G''(1,s)=G'''(1,s)=0,\quad \text{and}\quad G'''(0,s)=kG(0,s)=-1,$$

which implies that

$$(Au)'(1)=(Au)''(1)=(Au)'''(1)=0,\quad \text{and}\quad k(Au)(0)=(Au)'''(0).$$

Hence, $A:C_0^2[0,1]\to C_0^2[0,1]$. Moreover, for each $t\in[0,1]$, (By (2.1) we have

$$\begin{aligned}(Au)(t)&=\int_0^1 G(t,s)[f(s,u(s)-u_0(s),u''(s)-u_0''(s))+M]ds\\ &\ge (1+\frac{k}{6})\int_0^1 G(0,s)[f(s,u(s)-u_0(s),u''(s)-u_0''(s))+M]ds\\ &\ge (1+\frac{k}{6})\max_{0\le t\le 1}\int_0^1 G(t,s)[f(s,u(s)-u_0(s),u''(s)-u_0''(s))+M]ds\\ &=(1+\frac{k}{6})\|Au\|.\end{aligned}$$

Thus, $A:P\to P$.

We can check that A is completely continuous by routine method. Since $C_1<C_2$, it is easy to check that $r_1\ne r_2$. Without loss of generality, we assume $r_1<r_2$. Let

$$\Omega_1=\{u\in P:\|u\|_0<r_1\},\quad \Omega_2=\{u\in P:\|u\|_0<r_2\}.$$

If $u\in\partial\Omega_1$, then $\|u\|_0=r_1$. So, $\|u\|\le r_1$ and $\|u''\|\le r_1$. This implies

$$0\le u(t)\le r_1\quad 0\le u''(t)\le r_1,\quad 0\le t\le 1.$$

By (2.2), for $0\le t\le 1$, we have

$$\frac{1}{k}M\le u(t)-u_0(t)\le r_1+\big(\frac{1}{k}+\frac{1}{24}\big)M,\quad -\frac{1}{2}M\le u''(t)-u_0''(t)\le r_1.$$

By (3.2),

$$f(t,u(t)-u_0(t),u''(t)-u_0''(t))\le \alpha(r_1)\le r_1C_1-M,\quad 0\le t\le 1.$$

It follows that

$$\begin{aligned}\|Au\| &= \max_{0\le t\le 1}\int_0^1 G(t,s)[f(s,u(s)-u_0(s),u''(s)-u_0''(s))+M]ds\\ &\le r_1C_1\max_{0\le t\le 1}\int_0^1 G(t,s)ds\le r_1,\end{aligned}$$

$$\begin{aligned}\|(Au)''\| &= \max_{0\le t\le 1}\int_0^1 |G''(t,s)|[f(s,u(s)-u_0(s),u''(s)-u_0''(s))+M]ds\\ &\le r_1C_1\max_{0\le t\le 1}\int_0^1 |G''(t,s)|ds\le r_1.\end{aligned}$$

Therefore, $\|Au\|_0\le r_1=\|u\|_0$.

If $u\in\partial\Omega_2$, then $\|u\|_0=r_2$. So, $\|u\|\le r_2$ and $\|u''\|\le r_2$. This implies that

$$0\le u(t)\le r_2,\quad 0\le u''(t)\le r_2,\quad 0\le t\le 1.$$

Since

$$\begin{gathered}-\frac{85}{2048}-\frac{1}{k}=p(\frac{3}{4})\le p(t)\le p(\frac{1}{4})=-\frac{175}{6144}-\frac{1}{k},\quad \frac14\le t\le\frac34,\\ \frac{1}{32}\le p''(t)=\frac12(1-t)^2\le\frac{9}{32},\quad \frac14\le t\le\frac34,\end{gathered}$$

we have

$$(\frac1k+\frac{175}{6144})M\le u(t)-u_0(t)\le r_2+(\frac1k+\frac{85}{2048})M,\quad \frac14\le t\le\frac34,$$

and

$$-\frac{9}{32}M\le u''(t)-u_0''(t)\le r_2-\frac{M}{32},\quad \frac14\le t\le\frac34.$$

Thus, by (3.2) we obtain

$$f(t,u(t)-u_0(t),u''(t)-u_0''(t))\ge\beta(r_2)\ge r_2C_2-M,\quad \frac14\le t\le\frac34.$$

From this,

$$\begin{aligned}\|Au\| &\ge \max_{0\le t\le 1}\int_{\frac14}^{\frac34} G(t,s)[f(s,u(s)-u_0(s),u''(s)-u_0''(s))+M]ds\\ &\ge r_2C_2\max_{0\le t\le 1}\int_{\frac14}^{\frac34} G(t,s)ds\ge r_2,\end{aligned}$$

and

$$\begin{aligned}\|(Au)''\| &\ge \max_{0\le t\le 1}\int_{\frac14}^{\frac34} G''(t,s)[f(s,u(s)-u_0(s),u''(s)-u_0''(s))+M]ds\\ &\ge r_2C_2\max_{0\le t\le 1}\int_{\frac14}^{\frac34} G''(t,s)ds\ge r_2.\end{aligned}$$

It follows that $\|Au\|_0\ge r_2=\|u\|_0$. By Lemma 2.3, we assert that the operator A has at least one fixed point $\overline{u}\in P$ with $r_1\le\|\overline{u}\|_0\le r_2$. This implies that (3.4) has at least one solution $\overline{u}\in P$ with $r_1\le\|\overline{u}\|_0\le r_2$.

Let $u_*(t) = \overline{u}(t) - u_0(t)$, $0 \le t \le 1$. We will check that u_* is a solution of the problem (1.1). In fact, since $A\overline{u} = \overline{u}$, we have

$$\begin{aligned}
u_*(t) + u_0(t) = \overline{u}(t) = (A\overline{u})(t)& \\
&= \int_0^1 G(t,s)[f(s,\overline{u}(s) - u_0(s), \overline{u}''(s) - u_0''(s)) + M]ds \\
&= \int_0^1 G(t,s) f(s, u_*(s), u_*''(s))ds + u_0(t).
\end{aligned}$$

It follows that

$$u_*(t) = \int_0^1 G(t,s) f(s, u_*(s), u_*''(s))ds, \quad 0 \le t \le 1.$$

In other words, u_* is a solution of (1.1). Therefore, the problem (1.1) has at least one solution u_* satisfying $u_* + u_0 \in P$ and $r_1 \le \|u^* + u_0\|_0 \le r_2$.

Since $r_1 = \min\{r_1, r_2\} > -\frac{6}{6k+k^2}M$, we have

$$\begin{aligned}
u_*(t) &= [u_*(t) + u_0(t)] - u_0(t) = [u_*(t) + u_0(t)] - Mp(t) \\
&\ge (1 + \frac{k}{6})\|u_*(t) + u_0(t)\| + \frac{M}{k} \\
&\ge (1 + \frac{k}{6})[r_1 + \frac{6}{6k+k^2}M] > 0, \quad 0 \le t \le 1,
\end{aligned}$$

which implies that u_* is a positive solution of (1.1). □

Using Theorem 3.1, we can prove following result.

Theorem 3.2. *Let $-6 < k < 0$. Assume that*

$$f : [0,1] \times [\frac{M}{k}, +\infty) \times [-\frac{M}{2}, +\infty) \to [-M, +\infty) \tag{3.5}$$

is continuous, where $M \ge 0$ is a constant. Suppose that there exist three positive numbers $r_1 < r_2 < r_3$ with $r_1 > -\frac{6}{6k+k^2}M$ such that one of the following conditions is satisfied:

(1) $\alpha(r_1) \le r_1C_1 - M$, $\beta(r_2) > r_2C_2 - M$, $\alpha(r_3) \le r_3C_1 - M$;
(2) $\beta(r_1) \ge r_1C_2 - M$, $\alpha(r_2) < r_2C_1 - M$, $\beta(r_3) \ge r_3C_2 - M$.

Then problem (1.1) has at least two positive solutions.

4. Examples

Example 4.1. Consider the boundary-value problem

$$\begin{gathered}
u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 \le t \le 1, \\
u'(1) = u''(1) = u'''(1) = 0, \quad -2u(0) = u'''(0),
\end{gathered} \tag{4.1}$$

where $f : [0,1] \times [-1, +\infty) \times [-1, +\infty) \to [-2, +\infty)$ is defined by

$$f(t,u,v) = \begin{cases}
t^2 + \sqrt{u+1} + 9\sqrt{v+1} - 2, & (t,u,v) \in [0,1] \times [-1, -\frac{1}{2}] \times [-1, -\frac{1}{2}], \\
t^2 + \frac{u}{4} + 9\sqrt{v+1} + \frac{\sqrt{2}}{2} - \frac{15}{8}, & (t,u,v) \in [0,1] \times [-\frac{1}{2}, \infty) \times [-1, -\frac{1}{2}], \\
t^2 + \sqrt{u+1} + \frac{v}{5} + \frac{9}{2}\sqrt{2} - \frac{19}{10}, & (t,u,v) \in [0,1] \times [-1, -\frac{1}{2}] \times [-\frac{1}{2}, \infty), \\
t^2 + \frac{u}{4} + \frac{v}{5} + 5\sqrt{2} - \frac{71}{40}, & (t,u,v) \in [0,1] \times [-\frac{1}{2}, \infty) \times [-\frac{1}{2}, \infty).
\end{cases}$$

Thus, $k=-2$, $M=2$, $C_1=2$ and $C_2=\frac{6144}{1537}$. For

$$D_1(r)=\big\{(t,u,v):0\le t\le 1,\ -1\le u\le r-\frac{11}{12},\ -1\le v\le r\big\},$$

$$D_2(r)=\big\{(t,u,v):\frac{1}{4}\le t\le \frac{3}{4},\ -\frac{2897}{3072}\le u\le r-\frac{939}{1024},\ -\frac{9}{16}\le v\le r-\frac{1}{16}\big\}.$$

By simple computations, we obtain

$$\begin{aligned}\alpha(6)&=\max\{f(t,u,v):(t,u,v)\in D_1(6)\}\\&=\max\big\{f(1,\frac{61}{12},6),\ f(1,\frac{61}{12},-\frac{1}{2}),\ f(1,-\frac{1}{2},6),\ f(1,-\frac{1}{2},-\frac{1}{2})\big\}\\&=f(1,\frac{61}{12},6)=8.76<10=6C_1-M,\end{aligned}$$

and

$$\begin{aligned}&\beta(\frac{13}{8})\\&=\min\big\{f(t,u,v):(t,u,v)\in D_2(\frac{13}{8})\big\}\\&=\min\big\{f(\frac{1}{4},-\frac{2897}{3072},-\frac{9}{16}),\ f(\frac{1}{4},-\frac{2897}{3072},-\frac{1}{2}),\ f(\frac{1}{4},-\frac{1}{2},-\frac{9}{16}),\ f(\frac{1}{4},-\frac{1}{2},-\frac{1}{2})\big\}\\&=f(\frac{1}{4},-\frac{2897}{3072},-\frac{9}{16})=4.76>4.49=\frac{13}{8}C_2-M.\end{aligned}$$

Take $r_1=6$ and $r_2=\frac{13}{8}$. Then (3.2) holds. Moreover, we have

$$\min\{r_1,r_2\}=\frac{13}{8}>\frac{3}{2}=-\frac{6}{6k+k^2}M.$$

So, by Theorem 3.1, problem (4.1) has at least one positive solution.

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