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RESONANT PROBLEM FOR SOME SECOND-ORDER DIFFERENTIAL EQUATION ON THE HALF-LINE

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Abstract. We prove the existence of at least one solution to a nonlinear second-order differential equation on the half-line, with the boundary conditions $x'(0) = 0$ and with the first derivative vanishing at infinity. Our main tool is the multi-valued version of the Miranda Theorem.

1. Introduction

Most nonlinear differential, integral or, more generally, functional equations have the form $Lx = N(x)$, where L is a linear and N nonlinear operator, in appropriate Banach spaces.

We have no problem if L is a linear Fredholm operator of index 0. Then the kernel of the linear part of the above equation is trivial. It means that there exists an integral operator and we can apply known topological methods to prove the existence theorems [1, 5, 12].

If kernel L is nontrivial then the equation is called resonant and one can manage the problem by using the coincidence degree in that case [13].

But, if the domain is unbounded (for example the half-line) the operator is usually non-Fredholm (the range of L is not a closed subspace in any reasonable Banach space) Such problems have been studied by different methods in many papers. We mention only [2, 7, 8, 16, 17, 18]. For instant, the perturbation method was developed in a series of papers [10, 10, 9, 19, 20].

We consider the following asymptotic boundary-value problem (BVP) on the half-line

$$x'' = f(t, x, x'), \quad x'(0) = 0, \quad \lim_{t\to\infty} x'(t) = 0.$$

The problem is resonant, since the corresponding homogeneous linear problem: $x'' = 0, \quad x'(0) = \lim_{t\to\infty} x'(t) = 0$ has nontrivial solutions - constant functions.

Similar problem was considered in [17]. There, the asymptotic boundary condition $\lim_{t\to\infty} x'(t) = 0$ is replaced by $x \in H^2(\mathbb{R}_+)$ that is close but not the same as our one and assumptions are of completely different kind.

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Our problem has been already studied in [21]. In this paper, we have obtained the existence result in a completely different way than by using standard methods for resonant problems. This enables us to get it under weak assumptions: a linear growth condition and a sign condition for the nonlinear term f. Similar assumptions appear also for other boundary-value problems.

But there we assumed also that function f is Lipschitz continuous. It was an artificial condition. We needed it only to show that defined there mapping g is a function and then to apply the Miranda Theorem [14, p. 124]. In this paper we omit the assumption of Lipschitz continuity. In this way the mapping g becomes a multifunction. Consequently, to get the Theorem about the existence of at least one solution to the resonant problem we have to prove the multi-valued version of Miranda Theorem (see Appendix, Theorem 3.1).

In this paper we apply Theorem 3.1 only in case $k = 1$. But as we know the Theorems of the type of Miranda's theorem has many applications for instance in control theory [6]. That's why it seams that Theorem 3.1 can be very useful as well.

2. The main result

Let us consider an asymptotic BVP

$$x'' = f(t, x, x'), \quad x'(0) = 0, \quad \lim_{t\to\infty}, x'(t) = 0 \tag{2.1}$$

where $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

The following assumption will be needed in this paper:

(i) $|f(t, x, y)| \le b(t)|y| + c(t)$, where $b, c \in L^\infty(0, \infty)$;

(ii) there exists $M > 0$ such that $xf(t, x, y) \ge 0$ for $t \ge 0$, $y \in \mathbb{R}$ and $|x| \ge M$.

First, we consider problem

$$y' = f(t, c + \int_0^t y, y), \quad y(0) = 0, \tag{2.2}$$

for fixed $c \in \mathbb{R}$. Observe that (2.2) is equivalent to an initial value problem

$$x'' = f(t, x, x'), \quad x(0) = c, \quad x'(0) = 0. \tag{2.3}$$

Since f is continuous, then by assumption (i) and the Local Existence Theorem we get that problem (2.3) has at least one local solution. We can write (2.2) as

$$y_c(t) = \int_0^t f\Big(s, c + \int_0^s y_c(u)du, y_c(s)\Big)ds \tag{2.4}$$

Set

$$B := \int_0^\infty b(s)ds, \quad C := \int_0^\infty c(s)ds. \tag{2.5}$$

By (i) and (2.5) we get

$$|y_c(t)| \le \int_0^t (b(s)|y_c(s)| + c(s))ds \le C + \int_0^t b(s)|y_c(s)|ds. \tag{2.6}$$

Now, due to Gronwall's Lemma [14, p. 17], we have

$$|y_c(t)| \le C\exp \int_0^t b(s)ds$$

Hence, by the Theorem on a Priori Bounds [14, p. 146], (2.3) has a global solution for $t \geq 0$. We obtain that (2.2) has a global solution for $t \geq 0$. Moreover, by assumption (i) and (2.5), we have

$$|y_c(t)| \leq C\exp \int_0^t b(s)ds \leq C\exp \int_0^\infty b(s)ds = C\exp B < \infty \tag{2.7}$$

Hence all global solutions are bounded for $t \geq 0$.

The function $t \mapsto f(t, c + \int_0^t y(u)du, y(t))$ is absolutely integrable; i.e.,

$$\forall_{\varepsilon>0} \quad \exists_{M>0} \quad \Big| \int_M^\infty f\Big(t, c + \int_0^t y(u)du, y(t)\Big)dt\Big| < \varepsilon.$$

In particular, there exists a limit $\lim_{t\to\infty} y_c(t)$, for every c. Thus all solutions of (2.2) have finite limits at $+\infty$.

Denote by $BC(\mathbb{R}_+, \mathbb{R})$ the space of continuous and bounded functions with supremum norm and by $BCL(\mathbb{R}_+, \mathbb{R})$ its closed subspace of continuous and bounded functions which have finite limits at $+\infty$.

Let us consider the nonlinear operator $F : BCL(\mathbb{R}_+, \mathbb{R}) \to BCL(\mathbb{R}_+, \mathbb{R})$ given by

$$F(c, x)(t) = \int_0^t f(s, c + \int_0^s x(u)du, x(s))ds, \tag{2.8}$$

where $c \in \mathbb{R}$ is fixed. It is easy to see that F is well defined. By using the Lebesgue Dominated Convergence Theorem one can prove the continuity of F.

The following theorem gives a sufficient condition for compactness in the space $BC(\mathbb{R}_+, \mathbb{R})$, ([15]).

Theorem 2.1. *If $A \subset BC(\mathbb{R}_+, \mathbb{R})$ satisfies following conditions:*

(1) *There exists $M > 0$, that for every $x \in A$ i $t \in [0, \infty)$ we have $|x(t)| \leq L$;*
(2) *for each $t_0 \geq 0$, the family A is equicontinuous at t_0;*
(3) *for each $\varepsilon > 0$ there exists $T > 0$ and $\delta > 0$ such that if $|x(T) - y(T)| \leq \delta$, then $|x(t) - y(t)| \leq \varepsilon$ for $t \geq T$ and all $x, y \in A$.*

Then A is relatively compact in $BC(\mathbb{R}_+, \mathbb{R})$.

Now, we can prove that operator F is completely continuous.

Proposition 2.2. *Under assumption (i) operator F is completely continuous.*

Proof. We shall show that the image of $A := \{x \in BCL(\mathbb{R}_+, \mathbb{R}) \mid \|x\|_{BC(\mathbb{R}_+,\mathbb{R})} \leq M\}$ under F is relatively compact. Condition (1) of Theorem 2.1 is satisfied, since $|F(c, x)(t)| \leq C + MB$.

Now, we prove condition (2). We show that for any $t_0 \geq 0$ and $\varepsilon > 0$ there exists $\delta > 0$ that for each $x \in B$ if $|t - t_0| < \delta$, then $|F(c, x)(t) - F(c, x)(t_0)| < \varepsilon$. Let us choose an arbitrary $\varepsilon > 0$. By (i) there exist δ_1, $\delta_2 > 0$ such that

$$\text{if } |t - t_0| < \delta_1, \text{ then } \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} b(s)ds < \frac{\varepsilon}{2M},$$
$$\text{if } |t - t_0| < \delta_2, \text{ then } \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} c(s)ds < \frac{\varepsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for $|t - t_0| < \delta$, we get

$$\begin{aligned}
|F(c,x)(t) - F(c,x)(t_0)| &\le \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} |f(s, c + \int_0^s x(u)du, x(s))|ds \\
&\le M \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} b(s)ds + \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} c(s)ds \\
&< M\frac{\varepsilon}{2M} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

It remains to prove condition (3). By assumption (i) for every $\varepsilon > 0$ there exist t_1, t_2 large enough that

$$\int_{t_1}^{\infty} b(s)ds < \frac{\varepsilon}{6M}, \quad \int_{t_2}^{\infty} c(s)ds < \frac{\varepsilon}{6}.$$

Let $T = \max\{t_1, t_2\}$ and $\delta := \varepsilon/3$. If $|F(c,x)(T) - F(c,y)(T)| \le \delta$, then for $t \ge T$ we get

$$\begin{aligned}
&|F(c,x)(t) - F(c,y)(t)| \\
&\le |F(c,x)(T) - F(c,y)(T)| + \int_T^{\infty} \big|f(s, c + \int_0^s x(u)du, x(s))\big|ds \\
&\quad + \int_T^{\infty} \big|f(s, c + \int_0^s y(u)du, y(s))\big|ds \\
&\le |F(c,x)(T) - F(c,y)(T)| + 2\int_T^{\infty} Mb(s)ds + 2\int_T^{\infty} c(s)ds \\
&\le \frac{\varepsilon}{3} + 2\frac{\varepsilon}{6M} + 2\frac{\varepsilon}{6} = \varepsilon.
\end{aligned}$$

The proof is complete. $\square$

Note that the solutions of (2.2) are fixed points of operator F defined by (2.8). Let $\operatorname{fix} F(c,\cdot)$ denote the set of fixed points of operator F, where c is given. Let us consider the multifunction $g : \mathbb{R} \to 2^{\mathbb{R}}$ given by

$$g(c) := \{\lim_{t\to\infty} y_c(t) : y_c \in \operatorname{fix} F(c,\cdot)\}.$$

We will show that g is upper semicontinuous map from $\mathbb{R}$ into its compact connected subsets. To prove this we will define two maps: φ and Φ.

Let us consider the function $\varphi : BCL(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}$ given by

$$\varphi(y_c) = \lim_{t\to\infty} y_c(t).$$

It is easily seen that function φ is continuous. Set

$$\Phi : \mathbb{R} \ni c \to \operatorname{fix} F(c,\cdot) \subset BCL(\mathbb{R}_+, \mathbb{R}).$$

We can now formulate the following result.

Proposition 2.3. *Let assumption* (i) *hold. Then the multi-valued map Φ is upper semicontinuous, that is for each $c_0 \in \mathbb{R}$ and for any neighborhood $U \subset BCL(\mathbb{R}_+, \mathbb{R})$ of $\Phi(c_0)$ there exists a neighborhood V of c_0 such that $\Phi(c) \subset U$, for all $c \in V$.*

Proof. Suppose, contrary to our claim, that Φ is not upper semicontinuous; i. e., for some $c_0 \in \mathbb{R}$ there exist neighborhood $U \subset BCL(\mathbb{R}_+, \mathbb{R})$ of $\operatorname{fix} F(c_0,\cdot)$ and sequence $c_n \to c_0$ and $x_n \in \operatorname{fix} F(c_n,\cdot) \setminus U$.

By (2.7) we get that the solutions of (2.2) are equibounded for any c. Hence the sequence (x_n) is bounded. Moreover, we have

$$x_n = F(c_n, x_n). \tag{2.9}$$

Proposition 2.2 yields that operator F is completely continuous. Then, by (2.9), (x_n) is relatively compact. Hence, from the sequence (x_n), we can extract a subsequence (x_{n_l}) which is convergent to some x_0 in $BCL(\mathbb{R}_+, \mathbb{R})$. Moreover, $c_{n_k} \to c_0$. Letting $k \to \infty$ in the equality

$$x_{n_k} = F(c_{n_k}, x_{n_k}),$$

we get

$$x_0 = F(c_0, x_0).$$

Hence, $x_0 \in \operatorname{fix} F(c_0, \cdot) \subset U$. This contradicts the fact that $x_{n_k} \in \operatorname{fix} F(c_{n_k}, \cdot) \setminus U$, for every n_k, and completes the proof. □

Proposition 2.4. *Let assumption (i) hold. Then the set-valued map Φ has compact and connected values.*

Proof. By (2.7) we know that the set of solutions of IVP (2.2) is equibounded for any c. Now, the proof that $\operatorname{fix} F(c, \cdot)$ is relatively compact in $BCL(\mathbb{R}_+, \mathbb{R})$ follows by the same method as in Proposition 2.2.

We next show that $\operatorname{fix} F(c, \cdot)$ is closed. Let (y_n) be an arbitrary convergent sequence such that $y_n \in \operatorname{fix} F(c, \cdot)$, and let $y_n \to y$. We have

$$F(c, y_n)(t) = \int_0^t f(s, c + \int_0^s y_n(u)du, y_n(s))ds. \tag{2.10}$$

The sequence (y_n) is bounded, since it is convergent. Moreover y_n is uniformly convergent to y on $[0, \infty)$. By (i) and the Lebesgue Dominated Convergence Theorem, letting $n \to \infty$, we get

$$\lim_{n\to\infty} F(c, y_n)(t) = \int_0^t f(s, c + \int_0^s y(u)du, y(s))ds.$$

On the other hand, $y_n = F(c, y_n)$ and consequently

$$y(t) = \lim_{n\to\infty} y_n(t) = \lim_{n\to\infty} F(c, y_n)(t) = F(c, y)(t)$$

Hence $y \in \operatorname{fix} F(c, \cdot)$. From the above it follows that $\operatorname{fix} F(c, \cdot)$ is compact.

It is left is to show that $\operatorname{fix} F(c, \cdot)$ is connected in $BCL(\mathbb{R}_+, \mathbb{R})$. On the contrary, suppose that the set is not connected. Since $\operatorname{fix} F(c, \cdot)$ is compact, there exist compact sets A and B such that $A, B \neq \emptyset$, $A \cap B = \emptyset$ and $A \cup B = \operatorname{fix} F(c, \cdot)$. Let $\varepsilon := \operatorname{dist}(A, B)$, $\varepsilon > 0$. Then

$$\forall_{y \in A, z \in B} \quad \|y - z\| \geq \varepsilon. \tag{2.11}$$

By (2.7) there exists $T > 0$ such that for any $y \in \operatorname{fix} F(c, \cdot)$ we get

$$\int_T^\infty \left| f\big(t, c + \int_0^t y(u)du, y(t)\big)\right| dt < \frac{1}{3}\varepsilon. \tag{2.12}$$

Let $y \in A$, i $z \in B$. Now, consider the functions y and z cut to the compact set $[0, T]$ and set $y|_{[0,T]}$ and $z|_{[0,T]}$. By Kneser's Theorem [11, p. 413], the set $\operatorname{fix} F(c, \cdot)|_{[0,T]}$

is connected in $C([0,T],\mathbb{R})$. From this, there exist $x_1,\dots,x_k$ in $\operatorname{fix} F(c,\cdot)|_{[0,T]}$ such that $x_1 = y\,|_{[0,T]}$, $x_k = z\,|_{[0,T]}$ and

$$\|x_i - x_{i+1}\|_{C([0,T],\mathbb{R})} < \frac{1}{3}\varepsilon. \tag{2.13}$$

Hence, at least two sequel x_i, x_{i+1} in $\operatorname{fix} F(c,\cdot)|_{[0,T]}$ are such that $x_i \in A$ i $x_{i+1} \in B$. Moreover, $x_i, x_{i+1} \in \operatorname{fix} F(c,\cdot)$. By (2.11), (2.12) and (2.13) we get a contradiction. Indeed

$$\begin{aligned}
\varepsilon &\le \|x_i - x_{i+1}\|_{BCL(\mathbb{R}_+,\mathbb{R})} \\
&\le \max\{\|x_i - x_{i+1}\|_{C([0,T],\mathbb{R})}, \sup_{t\ge T} |x_i(t) - x_{i+1}(t)|\} \\
&\le \|x_i - x_{i+1}\|_{C([0,T],\mathbb{R})} + \sum_{j=i}^{i+1} \int_T^\infty |f(t, c + \int_0^t x_j(u)du, x_j(t))|dt \\
&< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon.
\end{aligned}$$

Hence, $\operatorname{fix} F(c,\cdot)$ is connected in $BCL(\mathbb{R}_+,\mathbb{R})$, which proves the Proposition. □

Proposition 2.5. *Let assumption (i) hold. Then the multifunction g is upper semicontinuous map from $\mathbb{R}$ into its compact intervals.*

Proof. By Proposition 2.4 we know that Φ has compact and connected values. Now, by continuity of φ, the set

$$\{\lim_{t\to\infty} y_c(t) \mid y_c \in \operatorname{fix} F(c,\cdot)\}$$

is compact and connected subset of $\mathbb{R}$ for every c. From this, we conclude that multifunction $g(c) = (\varphi \circ \Phi)(c)$ maps $\mathbb{R}$ into its compact intervals. Moreover, g is upper semicontinuous as a superposition of set-valued map with compact values and continuous function [4, p. 47]. □

We can now formulate our main result.

Theorem 2.6. *Under assumption (i)-(ii) problem* (2.1) *has at least one solution.*

Proof. Let $y_c \in \operatorname{fix} F(c,\cdot)$ be the bounded global solution of (2.2) and

$$g(c) := \{\lim_{t\to\infty} y_c(t) \mid y_c \in \operatorname{fix} F(c,\cdot)\}.$$

Observe that $x(t) = c + \int_0^t y_c(s)ds$ is a solution of (2.1) if there exists an $c \in \mathbb{R}^k$ such that $0 \in g(c)$.

We shall show that g satisfies assumptions of the multi-valued version of Miranda Theorem for $k = 1$ (see Appendix).

By Proposition 2.5 we get that multifunction g is upper semicontinuous map from $\mathbb{R}$ into its compact (so convex) intervals.

Let $c = M + 1$, where $M > 0$ is the constant from assumption (ii). Set $\widehat{M} := M + 1$. We will show that $y_{\widehat{M}}(t) \ge 0$ for $t \ge 0$ and all $y_{\widehat{M}} \in \operatorname{fix} F(\widehat{M},\cdot)$.

By (2.2) we have $y_{\widehat{M}}(0) = 0$. Assume that for some t and $y_{\widehat{M}} \in \operatorname{fix} F(\widehat{M},\cdot)$ we have $y_{\widehat{M}}(t) < 0$. Then there exist $t_* := \inf\{t \mid y_{\widehat{M}}(t) < 0\}$ such that $y_{\widehat{M}}(t_*) = 0$

and $y_{\widehat{M}}(t) \geq 0$ for $t < t_*$ (if $t_* \neq 0$). By continuity of $y_{\widehat{M}}(t)$ there exists $t_1 > t_*$ such that $\int_{t_*}^{t_1} |y_{\widehat{M}}(t)|dt \leq 1$. Hence, we get

$$x(t) = c + \int_{t_*}^{t} y_{\widehat{M}}(s)ds = \widehat{M} + \int_{t_*}^{t} y_{\widehat{M}}(s)ds \geq M \quad \text{for} \quad t \in [t_*, t_1]\,.$$

Now, by (ii) we have

$$x(t)f(t, x(t), y(t)) = x(t)y'_{M+1}(t) \geq 0.$$

Hence $y'_{\widehat{M}}(t) \geq 0$ for $t \in [t_*, t_1]$. It means that $y_{\widehat{M}}(t)$ is nondecreasing on $[t_*, t_1]$. Since $y_{\widehat{M}}(t_*) = 0$ we get a contradiction. Hence $y_{\widehat{M}}(t) \geq 0$ for $t \geq 0$. In consequence, if $d \in g(\widehat{M}) = \lim_{t\to\infty} y_{M+1}(t)$, then $d \geq 0$.

To prove that if $d \in g(-\widehat{M})$, then $d \leq 0$ we proceed analogously. Hence, by multi-valued version of Miranda Theorem, there exists an $c \in [-\widehat{M}, \widehat{M}]$ such that, $0 \in g(c)$. This completes the proof. □

3. Appendix

The theorem below is a generalization of the well known Miranda Theorem [14, p. 214] which gives zeros of point-valued continuous maps.

Theorem 3.1. *Let g be an upper semicontinuous map from the hypercube $[-\widehat{M}, \widehat{M}]^k$ into convex and compact subsets of $\mathbb{R}^k$ and satisfying for $d = (d_1, \dots, d_k) \in \mathbb{R}^k$ the conditions*

$$\text{if } d \in g(x_1, \dots, x_{i-1}, \widehat{M}, x_{i+1}, \dots, x_k), \text{ then } d_i \geq 0 \tag{3.1}$$

and

$$\text{if } d \in g(x_1, \dots, x_{i-1}, -\widehat{M}, x_{i+1}, \dots, x_k), \text{ then } d_i \leq 0, \tag{3.2}$$

for every $i = 1, \dots, k$. Then there exists $\widetilde{x} \in [-\widehat{M}, \widehat{M}]^k$ such that $0 \in g(\widetilde{x})$.

Proof. To prove this theorem, first of all suppose that the inequalities in (3.1) and (3.2) hold in the strict sense. Let $g_i = \mathrm{P}_i g$ for $i = 1, \dots, k$, where P_i is the projection of multifunction g on i-th axis. By (3.1) and (3.2) for $i = 1, \dots, k$ we have

$$\begin{gathered} g_i(x_1, \dots, x_{i-1}, \widehat{M}, x_{i+1}, \dots, x_k) \subset (0, \infty), \\ g_i(x_1, \dots, x_{i-1}, -\widehat{M}, x_{i+1}, \dots, x_k) \subset (-\infty, 0). \end{gathered}$$

It is easy to see that g_i is upper semicontinuous map from $[-\widehat{M}, \widehat{M}]^k$ into compact convex intervals for every i.

By (3.1) and the fact that g_i is upper semicontinuous there exists $\gamma_i > 0$ such that for any $x \in [-\widehat{M}, \widehat{M}]^k$, where $x_i \in (\widehat{M} - \gamma_i, \widehat{M}]$, we get $g_i(x) \subset (0, \infty)$, for every $i = 1, \dots, k$. Similarly, by (3.2) and the fact that g_i is upper semicontinuous there exists $\beta_i > 0$ such that for any $x \in [-\widehat{M}, \widehat{M}]^k$, where $x_i \in [-\widehat{M}, -\widehat{M} + \beta_i)$, we have $g_i(x) \subset (-\infty, 0)$, for every $i = 1, \dots, k$.

An upper semicontinuous map with compact values from a compact space has compact graph ([3]). Hence

$$\widehat{g} := \sup\{|d| \mid d \in g_i(x), x \in [-\widehat{M}, \widehat{M}]^k, i = 1, \dots, k\} < \infty.$$

Let $\delta := \min\{\beta_1, \dots, \beta_k, \gamma_1, \dots, \gamma_k, \widehat{M}\}$. Set $\varepsilon := \frac{\delta}{\widehat{g}}$ and consider the set-valued mapping given by

$$F_i(x) = x_i - \varepsilon g_i(x).$$

Then, for any $x_i \in [-\widehat{M}+\delta, \widehat{M}-\delta]$ and $y \in g_i(x)$ we have

$$-\widehat{M} = -\widehat{M}+\delta-\delta = -\widehat{M}+\delta-\varepsilon\widehat{g} \le x_i - \varepsilon y \le \widehat{M}-\delta+\varepsilon\widehat{g} = \widehat{M}-\delta+\delta = \widehat{M}.$$

For $x_i \in [-\widehat{M}, -\widehat{M}+\delta)$, if $y \in g_i(x)$, then $y<0$ and from this $-\varepsilon y > 0$. We get

$$-\widehat{M} \le x_i \le x_i - \varepsilon y \le -\widehat{M}+\delta+\varepsilon\widehat{g} \le -\widehat{M}+2\delta \le \widehat{M}.$$

Next, for $x_i \in (\widehat{M}-\delta, \widehat{M}]$, if $y \in g_i(x)$, then $y>0$ and in this way $-\varepsilon y < 0$. We get

$$-\widehat{M} \le \widehat{M}-2\delta \le \widehat{M}-\delta-\varepsilon\widehat{g} \le x_i-\varepsilon y \le x_i \le \widehat{M}.$$

Now, let us consider the multi-valued mapping

$$F(x) = x - \varepsilon g(x),$$

where $\varepsilon := \frac{\delta}{\widehat{g}}$ and $x \in [-\widehat{M}, \widehat{M}]^k$. By the assumptions of multifunction g, F is upper semicontinuous with convex and compact values in $\mathbb{R}^k$. However, we know more. We get that F maps from the hypercube $[-\widehat{M}, \widehat{M}]^k$ into its compact and convex sets. Indeed, the projection $\mathrm{P}_i F = F_i$ of F on i-th axis is compact interval contained in $[-\widehat{M}, \widehat{M}]$ for every $i = 1, \dots, k$.

Hence, by Kakutani's Theorem ([3]), there exists $\overline{x} \in [-\widehat{M}, \widehat{M}]^k$ such that $\overline{x} \in F(\overline{x})$. On the other hand

$$F(\overline{x}) = \overline{x} - \varepsilon g(\overline{x}).$$

Thus $0 \in F(\overline{x}) - \overline{x} = -\varepsilon g(\overline{x})$, and from this $0 \in g(\overline{x})$.

Now, suppose that the inequalities in (3.1) and (3.2) hold in a weak sense. Then for the following set-valued mapping

$$H_n(x) = g(x) + \frac{1}{n}x, \quad n \in \mathbb{N}$$

we have sharp inequalities. By the first part of the proof it follows that there exists $\widetilde{x}_n \in [-\widehat{M}, \widehat{M}]^k$ such that $0 \in H_n(\widetilde{x}_n)$, for every n. Hence $0 \in g(\widetilde{x}_n) + \frac{1}{n}\widetilde{x}_n$. Since sequence $(\widetilde{x}_n)$ is bounded, we can extract a convergent subsequence of (x_n). Let $(\widetilde{x}_n)$ denote the subsequence and let $\widetilde{x}_n \to \widetilde{x}$.

It suffices to show that $0 \in g(\widetilde{x})$. Suppose, contrary to our claim, that $0 \notin g(\widetilde{x})$. Let $\eta := \operatorname{dist}(0, g(\widetilde{x}))$. Since $g(\widetilde{x})$ is compact, we have that $\eta > 0$. Choose neighborhood $U := \{y \mid |y-z| < \frac{1}{3}\eta, z \in g(\widetilde{x})\}$ of $g(\widetilde{x})$. For every n we have $0 \in g(\widetilde{x}_n) + \frac{1}{n}\widetilde{x}_n$. From this there exists $y_n \in g(\widetilde{x}_n)$ such that $y_n + \frac{1}{n}\widetilde{x}_n = 0$, $n \in \mathbb{N}$. In particular, g is upper semicontinuous at $\widetilde{x}$. Hence, there exists N_1 such that for every $n \ge N_1$ we have $g(\widetilde{x}_n) \subset U$.

Moreover, $\frac{1}{n}\widetilde{x}_n \to 0$, as $n \to \infty$. Hence, there exists N_2 such that for any $n \ge N_2$ we have $|\frac{1}{n}\widetilde{x}_n| < \frac{1}{3}\eta$. Set $N = \max\{N_1, N_2\}$. Then for $n \ge N$ we get $|y_n| = |-\frac{1}{n}\widetilde{x}_n| < \frac{1}{3}\eta$. On the other hand, $y_n \in g(\widetilde{x}_n) \subset U$, which is impossible. Hence $0 \in g(\widetilde{x})$ and the proof is complete. □

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