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NON-OSCILLATORY BEHAVIOUR OF HIGHER ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

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Abstract. In this paper, we obtain sufficient conditions so that the neutral functional differential equation

$$\big[r(t)[y(t)-p(t)y(\tau(t))]'\big]^{(n-1)}+q(t)G(y(h(t)))=f(t)$$

has a bounded and positive solution. Here $n\geq 2$; q,τ,h are continuous functions with $q(t)\geq 0$; $h(t)$ and $\tau(t)$ are increasing functions which are less than t, and approach infinity as $t\to\infty$. In our work, $r(t)\equiv 1$ is admissible, and neither we assume that G is non-decreasing, that $xG(x)>0$ for $x\neq 0$, nor that G is Lipschitzian. Hence the results of this paper generalize many results in [1] and [4]–[8].

1. Introduction

In this paper we find sufficient conditions for the neutral delay differential equation (NDDE in short), of order $n\geq 2$,

$$\big[r(t)[y(t)-p(t)y(\tau(t))]'\big]^{(n-1)}+q(t)G(y(h(t)))=f(t) \tag{1.1}$$

to have a bounded positive solution which does not tend to zero as $t\to\infty$. Here $q,h,\tau\in C([0,\infty),R)$ such that $q(t)\geq 0, h(t)$ and $\tau(t)$ are increasing functions which are less thatn or equal to t, and approach ∞ as $t\to\infty$, $r\in C^{(n-1)}([0,\infty),(0,\infty))$, $p\in C^{(n)}([0,\infty),\mathbb{R})$, $G\in C(\mathbb{R},\mathbb{R})$.

We need some of the following assumptions in the sequel.

- (H1) There exists a bounded function $F(t)$ such that $F^{(n-1)}(t)=f(t)$.
- (H2) $\int_{t_0}^{\infty}t^{n-2}q(t)dt<\infty$.
- (H3) $\int_{t_0}^{\infty}\frac{dt}{r(t)}=\infty$.
- (H4) $\int_{t_0}^{\infty}\frac{dt}{r(t)}<\infty$.
- (H5) $\int_{t_0}^{\infty}(\frac{1}{r(t)}\int_t^{\infty}(s-t)^{n-2}q(s)ds)dt<\infty$.

Remark 1.1. Since $r(t)>0$, it follows that

- (i) either (H3) or (H4) holds exclusively.
- (ii) If (H3) holds then (H5) implies (H2) but not conversely.

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(iii) If (H4) holds then (H2) implies (H5) but not conversely.

The study of oscillation and non-oscillation properties of neutral delay differential equations has attracted the attention of many authors all over the world during the last two decades.In [1, 4, 5, 6, 7, 8] the authors have proved the existence of a bounded positive solution of neutral delay differential equations

$$(y(t)-p(t)y(t-\beta))' + q(t)G(y(t-\delta)) = f(t), \tag{1.2}$$

$$(y(t)-p(t)y(t-\beta))'' + q(t)G(y(t-\delta)) = f(t),, \tag{1.3}$$

$$(y(t)-p(t)y(t-\beta))^{(n)} + q(t)G(y(t-\delta)) = f(t), \tag{1.4}$$

where β and δ are constants. For that purpose the authors assume the following hypothesis.

- (H6) There exists a function $F(t)$ such that $F(t) \to 0$ as $t \to \infty$ and $F^n(t) = f(t)$.
- (H7) $\left| \int_{t_0}^{\infty} t^{n-1} f(t)dt \right| < \infty$.
- (H8) G is Lipschitzian in every interval of the form $[a, b]$, with $0 < a < b$.
- (H9) $xG(x) > 0$ for $x \neq 0$, and G is non-decreasing.

It is obvious that (H6) $\Leftrightarrow$ (H7) and (H1) is weaker than both (H6) and (H7).In (1.1) if we put $r(t) = 1$, $\tau(t) = t-\beta$, $h(t) = t-\delta$ then it reduces to (1.4). We find almost no result with the NDDE (1.1) in the literature. For example if $\tau(t) = t/2$ and $h(t) = t/3$ then the existing results fail to answer any thing. Since we formulate our results with (H1) and do not assume either (H8) or (H9), therefore our work extends, improves and generalizes some of the results of [1, 4, 5, 6, 7, 8]. While studying the existence of a positive solution of neutral delay differential equation (1.4) for $n \geq 2$, the authors take $p(t)$ in different ranges. But some how we find no result when $p(t) \equiv -1$, in these papers. However, in this work we consider $p(t)$ in different ranges including $p(t) = \pm 1$. Our results hold good when n is both odd or even but ≥ 2.

Let $T_y > 0$ and $T_0 = \min\{h(T_y), \tau(T_y)\}$. Suppose $\phi \in C([T_0 T_y], R)$. By a solution of (1.1), we mean a real valued continuous function $y \in C^{(n)}([T_0, \infty), R)$ such that $y(t) = \phi(t)$ for $T_0 \leq t \leq T_y$ and $y(t) - p(t)y(t-\tau)$ is differentiable, $r(t)(y(t) - p(t)y(t-\tau))'$ is $(n-1)$ times further differentiable and then for $t \geq T_y$ the neutral equation (1.1) is satisfied.Such a solution is said to be oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory.

So far as existence and uniqueness of solutions of (1.1) are concerned one may refer [3], but in this work we assume the existence of solutions of (1.1) and study its non-oscillatory behaviour.

2. Main Results

In this section we assume that there exists positive real numbers p, c, and d such that $p(t)$ satisfies one of the following conditions.

- (A1) $0 \leq p(t) \leq p < 1$.
- (A2) $-1 < -p \leq p(t) \leq 0$.
- (A3) $-d < p(t) \leq -c < -1$.
- (A4) $1 < c \leq p(t) < d$.

For our work we need the following Lemma from [3].

Lemma 2.1 (Krasnoselskiis Fixed point Theorem [2]). *Let X be a Banach space. Let Ω be a bounded closed convex subset of X and let S_1, S_2 be maps of Ω into X*

such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contraction and S_2 is completely continuous, then the equation

$$S_1x + S_2x = x$$

has a solution in Ω.

Theorem 2.2. *Let* (A1), (H1), (H4), (H5) *hold. Then there exists a bounded solution of* (1.1) *which is bounded below by a positive constant i.e there exists a solution of* (1.1) *which neither oscillates nor tends to zero as $t \to \infty$.*

Proof. Since $G \in C(\mathbb{R}, \mathbb{R})$, then let

$$\mu = \max\{G(x) : \frac{3}{5}(1-p) \le x \le 1\}. \tag{2.1}$$

From (H1) , we find $\alpha > 0$ and $t_1 > t_0 > 0$ such that

$$|F(t)| < \alpha \quad \text{for } t \ge t_1. \tag{2.2}$$

Then using (H4) we find $t_2 > t_1$ such that $t \ge t_2$ implies

$$\int_t^\infty \frac{1}{r(s)} ds < \frac{1-p}{10\,\alpha}. \tag{2.3}$$

From (2.2) and (2.3) it follows that for $t > t_3 > t_2$

$$\int_t^\infty \frac{|F(s)|}{r(s)} ds < \frac{1-p}{10}. \tag{2.4}$$

From (H5) we find $t_4 > t_3$ such that $t > t_4$ implies

$$\frac{\mu}{(n-2)!} \int_t^\infty \frac{1}{r(s)} \int_s^\infty (u-s)^{n-2} q(u) du\, ds < \frac{1-p}{10}. \tag{2.5}$$

Let $T > t_4$ and $T_0 = \min\{\tau(T), h(T)\}$. Then for $t \ge T$, (2.4) and (2.5) hold. Let $X = C([T_0, \infty), R)$ be the set of all continuous functions with norm $\|x\| = \sup_{t \ge T_0} |x(t)| < \infty$. Clearly X is a Banach space. Let

$$S = \big\{u \in BC([T_0, \infty), R) : \frac{3}{5}(1-p) \le u(t) \le 1\big\}, \tag{2.6}$$

with the supremum norm $\|u\| = \sup\{|u(t)| : t \ge T_0\}$. Clearly S is a closed, bounded and convex subset of $C([T_0, \infty), R)$. Define two maps A and $B : S \to X$ as follows. For $x \in S$,

$$Ax(t) = \begin{cases} Ax(T), & t \in [T_0, T]; \\ p(t)x(\tau(t)) + \frac{4(1-p)}{5}, & t \ge T, \end{cases} \tag{2.7}$$

and

$$Bx(t) = \begin{cases} Bx(T), & t \in [T_0, T]; \\ \frac{(-1)^{n-1}}{(n-2)!} \int_t^\infty \frac{1}{r(s)} \int_s^\infty (u-s)^{n-2} q(u) G(x(h(u))) du\, ds & \\ - \int_t^\infty \frac{F(s)}{r(s)} ds, & t \ge T. \end{cases} \tag{2.8}$$

First we show that if $x, y \in S$ then $Ax + By \in S$. In fact, for every $x, y \in S$ and $t \geq T$, we get

$$\begin{aligned}(Ax)(t) + (By)(t) &\leq p(t)x(\tau(t)) + \frac{4(1-p)}{5} - \int_t^\infty \frac{F(s)}{r(s)} ds \\ &\quad + \frac{(-1)^{n-1}}{(n-2)!} \int_t^\infty \frac{1}{r(s)} \int_s^\infty (u-s)^{n-2} q(u) G(y(h(u))) du\, ds \\ &\leq p + \frac{4(1-p)}{5} + \frac{1-p}{10} + \frac{1-p}{10} \leq 1.\end{aligned}$$

On the other hand for $t \geq T$,

$$\begin{aligned}(Ax)(t) + (By)(t) &\geq \frac{4(1-p)}{5} - \int_t^\infty \frac{F(s)}{r(s)} ds \\ &\quad + \frac{(-1)^{n-1}}{(n-2)!} \int_t^\infty \frac{1}{r(s)} \int_s^\infty (u-s)^{n-2} q(u) G(y(h(u))) du\, ds \\ &\geq \frac{4(1-p)}{5} - \alpha \int_t^\infty \frac{1}{r(s)} ds \\ &\quad - \frac{\mu}{(n-2)!} \int_t^\infty \frac{1}{r(s)} \int_s^\infty (u-s)^{n-2} q(u) du\, ds \\ &\geq \frac{4(1-p)}{5} - \frac{1-p}{10} - \frac{1-p}{10} = \frac{3}{5}(1-p).\end{aligned}$$

Hence

$$\frac{3}{5}(1-p) \leq (Ax)(t) + (By)(t) \leq 1$$

for $t \geq T$. So that $Ax + By \in S$ for all $x, y \in S$.

Next we show that A is a contraction in S. In fact, for $x, y \in S$ and $t \geq T$, we have

$$\begin{aligned}|(Ax)(t) - (Ay)(t)| &\leq |p(t)\{x(\tau(t)) - y(\tau(t))\}| \\ &\leq |p(t)||x(\tau(t)) - y(\tau(t))| \\ &\leq p \parallel x - y \parallel .\end{aligned}$$

Since $0 < p < 1$, we conclude that A is a contraction mapping on S.

We now show that B is completely continuous. First, we shall show that B is continuous. Let $x_k = x_k(t) \in S$ for $k = 1, 2, \dots.$ be such that $\sup_{t \geq T} |x_k(t) - x(t)| \to 0$ as $k \to \infty$.Because S is closed, $x = x(t) \in S$. For $t \geq T$, we have

$$\begin{aligned}&|(Bx_k)(t) - (Bx)(t)| \\ &\leq \frac{1}{(n-2)!} \int_t^\infty \frac{1}{r(s)} \int_s^\infty (u-s)^{n-2} q(u) |G(x(h(u))) - G(x_k(h(u)))| du\, ds.\end{aligned}$$

Since for all $t \geq T, x_k(t), k = 1, 2...,$ tend uniformly to $x(t)$ as $t \to \infty$ and G is continuous, therefore $|G(x(h(u))) - G(x_k(h(u)))| \to 0$ as $k \to \infty$. We conclude that $\lim_{k \to \infty} |(Bx_k)(t) - (Bx)(t)| = 0$. This means that B is continuous.

Next, we show that BS is relatively compact. It suffices to show that the family of functions $\{Bx : x \in S\}$ is uniformly bounded and equicontinuous on $[T_0, \infty)$. The uniform boundedness is obvious. For the equicontinuity, according to Levitan's result we only need to show that, for any given $\epsilon > 0, [T_0, \infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have change of amplitude less than ϵ. From (H5) and (H4), it follows that for any $\epsilon > 0$,we can find $T^* \geq T$ large enough so that

$$\frac{\mu}{(n-2)!}\int_{T^*}^{\infty}\frac{1}{r(s)}\int_s^{\infty}(u-s)^{n-2}q(u)du\,ds < \frac{\epsilon}{4},$$

and

$$\alpha\int_{T^*}^{\infty}\frac{ds}{r(s)} < \frac{\epsilon}{4}.$$

Then for $x \in S$ and $t_2 > t_1 \geq T^*$,

$$\begin{aligned}
&|(Bx)(t_2)-(Bx)(t_1)|\\
&= \Big|\frac{(-1)^n}{(n-2)!}\int_{t_2}^{\infty}\frac{1}{r(s)}\int_s^{\infty}(u-s)^{n-2}q(u)G(x(h(u)))du\,ds - \int_{t_2}^{\infty}\frac{F(s)}{r(s)}ds\\
&\quad - \frac{(-1)^n}{(n-2)!}\int_{t_1}^{\infty}\frac{1}{r(s)}\int_s^{\infty}(u-s)^{n-2}q(u)G(x(h(u)))du\,ds + \int_{t_1}^{\infty}\frac{F(s)}{r(s)}ds\Big|\\
&\leq \frac{\mu}{(n-2)!}\int_{t_1}^{\infty}\frac{1}{r(s)}\int_s^{\infty}(u-s)^{n-2}q(u)du\,ds + \alpha\int_{t_1}^{\infty}\frac{ds}{r(s)}\\
&\quad + \frac{\mu}{(n-2)!}\int_{t_2}^{\infty}\frac{1}{r(s)}\int_s^{\infty}(u-s)^{n-2}q(u)du\,ds + \alpha\int_{t_2}^{\infty}\frac{ds}{r(s)}\\
&< 4\frac{\epsilon}{4} = \epsilon.
\end{aligned}$$

For $x \in S$ and $T \leq t_1 < t_2 \leq T^*$,

$$\begin{aligned}
&|(Bx)(t_2)-(Bx)(t_1)|\\
&\leq \frac{\mu}{(n-2)!}\int_{t_1}^{t_2}\frac{1}{r(s)}\int_s^{\infty}(u-s)^{n-2}q(u)\,du\,ds + \alpha\int_{t_1}^{t_2}\frac{1}{r(s)}ds\\
&\leq \max_{T\leq s\leq T^*}\Big[\frac{\mu}{(n-2)!r(s)}\int_s^{\infty}(u-s)^{n-2}q(u)\,du + \frac{\alpha}{r(s)}\Big](t_2-t_1).
\end{aligned}$$

Thus there exists a $\delta > 0$ such that

$$|(Bx)(t_2)-(Bx)(t_1)| < \epsilon \quad \text{if } 0 < |t_2-t_1| < \delta.$$

For any $x \in S, T_0 \leq t_1 < t_2 \leq T$, it is easy to see that

$$|(Bx)(t_2)-(Bx)(t_1)| = 0 < \epsilon.$$

Therefore, $\{Bx : x \in S\}$ is uniformly bounded and equicontinuous on $[T_0,\infty)$ and hence BS is relatively compact. By Lemma 2.1, there is an $x_0 \in S$ such that $Ax_0 + Bx_0 = x_0$. It is easy to see that $x_0(t)$ is the required non oscillatory solution of the equation (1.1), which is bounded below by the positive constant $\frac{3(1-p)}{4}$. □

Corollary 2.3. *Let* (A1), (H1), (H2), (H4) *hold. Then there exists a bounded solution of* (1.1) *which is bounded below by a positive constant.*

Proof. By remark 1.1(iii) (H2) and (H4) imply (H5). Hence the proof follows from the proof of the above theorem,. □

Theorem 2.4. *Let* (A1), (H3), (H5) *hold. Suppose there exists $\alpha > 0$ such that for large t*

$$r(t) > \frac{1}{\alpha}, \tag{2.9}$$

and

$$\Big|\int_0^\infty F(t)dt\Big| < \infty \quad \textit{with} \quad F^{n-1}(t) = f(t). \tag{2.10}$$

Then there exists a bounded solution of (1.1) *which is bounded below by a positive constant.*

Proof. Using (2.9) and (2.10) we can get (2.4). Rest of the proof is similar to that of the Theorem 2.2. □

Corollary 2.5. *Let* (A1)*,* (H5)*,* (2.9)*,* (2.10) *hold. Then there exists a bounded solution of* (1.1) *which is bounded below by a positive constant.*

Proof. By Remark 1.1(i) we have either (H3) holds or (H4) holds. If (H3) holds then we proceed as in the proof of Theorem 2.4. On the other hand if (H4) holds then from (2.9) and (2.10) we get (2.4) and then proceed as in the proof of Theorem 2.2 to get the desired result. □

Remark 2.6. If in (H5) we take $r(t) \equiv 1$ then it reduces to

$$\int_{t_0}^\infty \int_t^\infty (u-t)^{n-2} q(u)du < \infty. \tag{2.11}$$

The above condition is required for our next result which follows from Corollary 2.5 when $r(t) \equiv 1$.

Corollary 2.7. *Inequality* (2.11) *is a sufficient condition for the nth order NDDE*

$$\big(y(t) - p(t)y(\tau(t))\big)^n + q(t)G(y(h(t))) = f(t) \tag{2.12}$$

to have a solution bounded below by a positive constant under the assumptions (A1), (2.9) *and* (2.10)*.*

Remark 2.8. We claim that the condition

$$\int_{t_0}^\infty u^{n-1} q(u)ds < \infty \tag{2.13}$$

implies (2.11). It is clear that (2.13) is equivalent to $\int_s^\infty (u-s)^{n-1} q(u)du < \infty$. Let

$$K(s) = \int_s^\infty (u-s)^{n-1} q(u)du.$$

This implies $\lim_{s\to\infty} K(s) = 0$. Differentiating, we get

$$K'(s) = -(n-1)\int_s^\infty (u-s)^{n-2} q(u)du.$$

From this integrating between s=t and s=T ,we obtain

$$\int_t^T K'(s)ds = -(n-1)\int_t^T \int_s^\infty (u-s)^{n-2} q(u)\,du\,ds.$$

Hence

$$-(n-1)\int_t^T \int_s^\infty (u-s)^{n-2} q(u)\,du\,ds = K(T) - K(t).$$

In the limit as $T \to \infty$, we obtain

$$\int_t^\infty \int_s^\infty (u-s)^{n-2} q(u)\,du\,ds = \frac{K(t)}{(n-1)} = \frac{1}{(n-1)}\int_t^\infty (u-t)^{n-1} q(u)du < \infty.$$

Hence the claim holds.

Remark 2.9. Corollary 2.7 improves [1, Theorem 1] and [4, Theorem 4.3] because the authors assumed G to satisfy (H9) and to be Lipschizian.

It may be noted in view of the Remark2.8 that the condition (2.11) used in Coprollary2.7 is weaker than the condition (2.13) used in [1, 4].

Theorem 2.10. *Let* (A2), (H1), (H4), (H5) *hold. Then there exists a bounded solution of* (1.1) *which is bounded below by a positive constant.*

Proof. We proceed as in the proof of the Theorem 2.2 with the following changes

$$\mu = \max\{|G(x)| : \frac{1-p}{10} \le x \le 1\}.$$

By (H1), (H4), (H5), we find T such that for $t \ge T$

$$\frac{\mu}{(n-2)!} \int_t^\infty \frac{1}{r(s)} \int_s^\infty (u-s)^{n-2} q(u) du\, ds < \frac{1-p}{10},$$

and

$$\int_t^\infty \frac{|F(t)|}{r(t)} dt < \alpha \int_t^\infty \frac{dt}{r(t)} < \frac{1-p}{10}.$$

Let $S = \{y \in X : \frac{1-p}{10} \le y(t) \le 1, t \ge T_0\}$.

$$(Ay)(t) = \begin{cases} \frac{7p+3}{10} + p(t)y(t-\tau) - \int_t^\infty \frac{F(s)}{r(s)} ds, & \text{for } t \ge T; \\ Ay(T), & \text{for } T_0 \le t \le T. \end{cases}$$

$$(By)(t) = \begin{cases} By(T), & \text{for } T_0 \le t \le T; \\ \frac{(-1)^{n-1}}{(n-2)!} \int_t^\infty \frac{1}{r(s)} \int_s^\infty (u-s)^{n-2} q(u) G(y(h(u))) du\, ds & \text{for } t \ge T. \end{cases}$$

Then as in Theorem2.2 we prove (i) $Ax + By \in S$ (ii) A is a contraction, and finally (iii) B is completely continuous. Then by Lemma 2.1 there is a fixed point x_0 in S such that $Ax_0 + Bx_0 = x_0$ which is the required solution bounded below by a positive constant. □

Remark 2.11. The above theorem substantially improves [8, Theorem 3.1] where the authors obtained a positive bounded solution of (1.1) with assumptions (A2), (H2), (H4), (H6), (H8), (H9). It may be noted that (H6) implies (H1) and (H2) with (H4)implies (H5). Further we did not require (H8) and (H9).

Theorem 2.12. *Let* (A2), (H3), (H5), (2.9) *and* (2.10) *hold. Then there exists a bounded solution of* (1.1) *which is bounded below by a positive constant.*

The proof of the above Theorem is similar to that of Theorem 2.10.

Definition 2.13. *For any* $t > t_0$*, define*

$$\tau_{-1}(t) = \{s \textit{ is a real number } : s \ge t \textit{ and } \tau(s) = t\}.$$

Remark 2.14. The function τ_{-1} defined above is the inverse function of $\tau(t)$. Since $\tau(t)$ is increasing it is one-one.Clearly $\tau_{-1}(\tau(t)) = t$ for $t > \tau_{-1}(t_0)$.

Theorem 2.15. *Let* (A3), (H1), (H4), (H5) *hold. Then there exists a bounded solution of* (1.1) *which is bounded below by a positive constant.*

Proof. If necessary increment d such that $d > 1 + \frac{2}{c}$. Choose positive numbers $\epsilon < \frac{c-1}{2}$, $h = (c-1) - \epsilon$ and $H = d - 1 + \frac{2\epsilon}{c}$. Then $H > h > 0$. Set

$$\mu = \max\{|G(x)| : h \le x \le H\}.$$

From (H4) and (H5) one can find $T > 0$ such that $t \ge T$ implies

$$\int_{\tau_{-1}(t)}^{\infty} \frac{F(s)}{r(s)} < \frac{\epsilon}{2},$$

and

$$\frac{\mu}{(n-2)!} \int_{\tau_{-1}(t)}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} (u-s)^{n-2} q(u) du\, ds < \frac{\epsilon}{2}.$$

Define

$$S = \{y(t) \in X : h \le y(t) \le H, t \ge T_0\}.$$

Define

$$Ax(t) = \begin{cases} Ax(T), & \text{if } t \in [T_0, T]; \\ \frac{x(\tau_{-1}(t))}{p(\tau_{-1}(t))} - \frac{cd-1}{p(\tau_{-1}(t))} + \frac{1}{p(\tau_{-1}(t))} \int_{\tau_{-1}(t)}^{\infty} \frac{F(s)}{r(s)} ds, & \text{if } t \ge T. \end{cases}$$

$$Bx(t) = \begin{cases} Bx(T), & \text{if } t \in [T_0, T]; \\ \frac{(-1)^n}{(n-2)!\, p(\tau_{-1}(t))} \int_{\tau_{-1}(t)}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} (u-s)^{n-2} q(u) G(y(h(u))) du\, ds, & \text{if } t \ge T. \end{cases}$$

We show that if $x, y \in S$, then $Ax + By \in S$. For $t \ge T$ we obtain

$$\begin{aligned} Ax + By =& \frac{-1}{p(\tau_{-1}(t))} \Big[- x(\tau_{-1}(t)) - \int_{\tau_{-1}(t)}^{\infty} \frac{F(s)}{r(s)} ds + (cd-1) \\ &+ \frac{(-1)^{n-1}}{(n-2)!} \int_{\tau_{-1}(t)}^{\infty} \frac{1}{r(s)} \Big(\int_{s}^{\infty} (u-s)^{n-2} q(u) G(y(h(u))) du \Big) ds \Big] \\ &\le \frac{1}{c} \big[-h + \frac{\epsilon}{2} + \frac{\epsilon}{2} + (cd-1) \big] \\ &= \frac{1}{c} \big[2\epsilon + c(d-1) \big] = (d-1) + \frac{2\epsilon}{c} \\ &\le H. \end{aligned}$$

Further,

$$\begin{aligned} Ax + By =& \frac{-1}{p(\tau_{-1}(t))} \Big[- x(\tau_{-1}(t)) - \int_{\tau_{-1}(t)}^{\infty} \frac{F(s)}{r(s)} ds + (cd-1) \\ &+ \frac{(-1)^{n-1}}{(n-2)!} \int_{\tau_{-1}(t)}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} (u-s)^{n-2} q(u) G(y(h(u))) du\, ds \Big] \\ &\ge \frac{1}{d} \big[-H - \frac{\epsilon}{2} + (cd-1) - \frac{\epsilon}{2} \big] \\ &= \frac{1}{d} \big[d(c-1) - \epsilon \frac{(c+2)}{c} \big] \\ &> c - 1 - \epsilon = h. \end{aligned}$$

Thus $Ax + By \in S$. Next we show that A is a contraction in S.In fact for $x, y \in S$ and $t \ge T$ we have

$$\|Ax - Ay\| \le |\frac{1}{p(\tau_{-1}(t))}| |x(\tau_{-1}(t)) - y(\tau_{-1}(t))| \le \frac{1}{c} \|x - y\|.$$

Hence A is a contraction because, $0 < \frac{1}{c} < 1$.Next we prove B is completely continuous as in the proof of Theoren2.2. Then by Lemma 2.1 there is a fixed point x_0 in S such that

$$Ax_0 + Bx_0 = x_0.$$

Writing $x_0 = y(t)$ and multiplying both sides of the above equation by $p(\tau_{-1}(t))$ we obtain,

$$\begin{aligned} p(\tau_{-1}(t))y(t) = y(\tau_{-1}(t)) &+ \int_{\tau_{-1}(t)}^{\infty} \frac{F(s)}{r(s)} ds - (cd-1) \\ &+ \frac{(-1)^n}{(n-2)!} \int_{\tau_{-1}(t)}^{\infty} \frac{1}{r(s)} \int_s^{\infty} (u-s)^{n-2} q(u) G(y(h(u))) du\, ds. \end{aligned}$$

Then we replace t by $\tau(t)$,use the fact that $\tau_{-1}(\tau(t)) = t$ and finally with some rearrangement of terms, obtain

$$\begin{aligned} y(t) - p(t)y(\tau(t)) = &- \int_t^{\infty} \frac{F(s)}{r(s)} + (cd-1) \\ &+ \frac{(-1)^{n-1}}{(n-2)!} \int_t^{\infty} \frac{1}{r(s)} \int_s^{\infty} (u-s)^{n-2} q(u) G(y(h(u))) du\, ds. \end{aligned}$$

First differentiating the above equation once and then multiplying bothsides by $r(t)$ and finally differentiating $n-1$ times ,we see that, x_0 is the required solution of (1.1), which is bounded below by a positive constant. □

Theorem 2.16. *Let* (A3), (H3), (H5), (2.9), (2.10) *hold. Then there exists a bounded solution of* (1.1) *which is bounded below by a positive constant.*

The proof of the above theorem is similar to that of the above theorem.

Corollary 2.17. *Let* (A3), (H5), (2.9), (2.10) *hold. Then there exists a bounded solution of* (1.1) *which is bounded below by a positive constant.*

Proof. In view of Remark 1.1 (i) the proof follows lines similar to those in Theorem 2.15 and 2.16.

The results for the range (A4) are similar to those under condition (A3). Hence we skip all proofs except the following one. □

Theorem 2.18. *Let* (A4), (H1), (H4), (H5) *hold. Then there exists a bounded solution of* (1.1) *which is bounded below by a positive constant.*

Proof. We proceed as in the proof of the Theorem2.15 with the following changes. Choose

$$\mu = \max\{|G(x)| : \frac{c-1}{d} \le x \le 2\}.$$

$$S = \{y \in X : \frac{c-1}{d} \le y \le 2\}.$$

From (H1), (H4) and (H5) we can find $T > 0$ such that $t \ge T$ implies

$$\int_{\tau_{-1}(t)}^{\infty} \frac{|F(s)|}{r(s)} < \frac{c-1}{2},$$

and

$$\frac{\mu}{(n-2)!} \int_{\tau_{-1}(t)}^{\infty} \frac{1}{r(s)} \int_s^{\infty} (u-s)^{n-2} q(u) du\, ds < \frac{c-1}{2}.$$

Define

$$Ax(t) = \begin{cases} Ax(T), & \text{if } t \in [T_0, T]; \\ \frac{x(\tau_{-1}(t))}{p(\tau_{-1}(t))} - \frac{2c-2}{p(\tau_{-1}(t))} + \frac{1}{p(\tau_{-1}(t))} \int_{\tau_{-1}(t)}^{\infty} \frac{F(s)}{r(s)} ds, & \text{if } t \geq T. \end{cases}$$

$$Bx(t) = \begin{cases} Bx(T), & \text{if } t \in [T_0, T]; \\ \frac{(-1)^n}{(n-2)!\, p(\tau_{-1}(t))} \int_{\tau_{-1}(t)}^{\infty} \frac{1}{r(s)} & \\ \times \int_s^{\infty} (u-s)^{n-2} q(u) G(y(h(u))) du\, ds, & \text{if } t \geq T. \end{cases}$$

For the rest of the proof we may refer the proofs of the Theorems 2.2 and 2.15. □

3. Positive solution for $p(t) = \pm 1$

In this section we find sufficient condition for the NDDE

$$(r(t)(y(t) + y(\tau(t)))')^{n-1} + q(t)G(y(h(t))) = f(t), \tag{3.1}$$

or

$$(r(t)(y(t) - y(\tau(t)))')^{n-1} + q(t)G(y(h(t))) = f(t), \tag{3.2}$$

to have a bounded positive solution.

The results with NDDE (3.1) are rare in the literature. We dont find such a result in [1] or [4, 5, 6, 7, 8]. To achieve our result we need the following Lemma.

Lemma 3.1 (Schauder's Fixed Point Theorem [2]). *Let Ω be a closed convex and nonempty subset of a Banach space X. Let $B : \Omega \to \Omega$ be a continuous mapping such that $B\Omega$ is a relatively compact subset of X. Then B has at least one fixed point in Ω. That is there exists an $x \in \Omega$ such that $Bx = x$.*

For $t \geq t_0$, define $\tau_{-1}^0(t) = t$, $\tau_{-1}^1(t) = \tau_{-1}(t)$, $\tau_{-1}^2(t) = \tau_{-1}(\tau_{-1}(t))$. For any positive integer $i > 2$, we define

$$\tau_{-1}^i(t) = \tau_{-1}(\tau_{-1}^{i-1}(t)).$$

Theorem 3.2. *Suppose* (H1), (H4), (H5) *hold. Then there exists a solution of* (3.1) *which is bounded below by a positive constant, that is, it neither oscillates nor tends to zero as t tends to ∞.*

Proof. We proceed as in the proof of Theorem 2.2 with the following changes. Let

$$\mu = \max\{|G(x)| : 1 \leq x \leq 5\}.$$

From (H1), (H4) and (H5) there exists $T > 0$ such that for $t \geq T$ implies

$$\frac{\mu}{(n-2)!} \Big| \int_t^{\infty} \frac{1}{r(s)} \int_s^{\infty} (u-s)^{n-2} q(u) du\, ds \Big| < 1, \tag{3.3}$$

and

$$\Big| \int_t^{\infty} \frac{F(s)}{r(s)} ds \Big| < 1. \tag{3.4}$$

For any continuous function $g(t)$, it is clear that

$$\sum_{l=1}^{\infty} \int_{\tau_{-1}^{2l-1}(t)}^{\tau_{-1}^{2l}(t)} g(s) ds < \int_t^{\infty} g(s) ds. \tag{3.5}$$

Hence using the above inequality in (3.3) and (3.4), we conclude that for $t \geq T$

$$\frac{\mu}{(n-2)!} \sum_{l=1}^{\infty} \int_{\tau_{-1}^{2l-1}(t)}^{\tau_{-1}^{2l}(t)} \frac{1}{r(s)} \int_s^{\infty} (u-s)^{n-2} q(u) du\, ds < 1, \tag{3.6}$$

and
$$\sum_{l=1}^{\infty}\int_{\tau_{-1}^{2l-1}(t)}^{\tau_{-1}^{2l}(t)}\frac{|F(s)|}{r(s)}ds<1. \tag{3.7}$$
Set $S=\{y\in X:1\le y(t)\le 5,\ t\ge T_0\}$. Next we define the mapping $B:S\to X$ as
$$By(t)=\begin{cases}By(T), & T_0\le t\le T;\\ 3-\sum_{l=1}^{\infty}\int_{\tau_{-1}^{2l-1}(t)}^{\tau_{-1}^{2l}(t)}\frac{F(s)}{r(s)}ds+\frac{(-1)^{n-1}}{(n-2)!}\sum_{l=1}^{\infty}\int_{\tau_{-1}^{2l-1}(t)}^{\tau_{-1}^{2l}(t)}\big(\frac{1}{r(s)} & \\ \times\int_s^{\infty}(u-s)^{n-2}q(u)G(y(h(u)))du\big)ds, & t\ge T.\end{cases}$$
Then using (3.6) and (3.7) we find that $By<5$ and $By>1$. Hence $By\in S$ for $y\in S$.Next we show BS is relatively compact as in the proof of Theorem 2.2. Then by Lemma 3.1 there is a fixed point y_0 in S such that $By_0=y_0$.Hence for $t\ge T$, we obtain
$$\begin{aligned}y_0(t)=&3+\frac{(-1)^{n-1}}{(n-2)!}\sum_{l=1}^{\infty}\int_{\tau_{-1}^{2l-1}(t)}^{\tau_{-1}^{2l}(t)}\frac{1}{r(s)}\int_s^{\infty}(u-s)^{n-2}q(u)G(y_0(h(u)))du\,ds\\&-\sum_{l=1}^{\infty}\int_{\tau_{-1}^{2l-1}(t)}^{\tau_{-1}^{2l}(t)}\frac{F(s)}{r(s)}ds.\end{aligned}$$
In the above we replace t by $\tau(t)$ and note that $\tau_{-1}^{m}(\tau(t))=\tau_{-1}^{m-1}(t)$ and $\tau_{-1}^{0}(t)=t$. Then It follows for $t\ge T$ that
$$\begin{aligned}y_0(\tau(t))=&3+\frac{(-1)^{n-1}}{(n-2)!}\sum_{l=1}^{\infty}\int_{\tau_{-1}^{2l-2}(t)}^{\tau_{-1}^{2l-1}(t)}\frac{1}{r(s)}\int_s^{\infty}(u-s)^{n-2}q(u)G(y_0(h(u)))du\,ds\\&-\sum_{l=1}^{\infty}\int_{\tau_{-1}^{2l-2}(t)}^{\tau_{-1}^{2l-1}(t)}\frac{F(s)}{r(s)}\,ds.\end{aligned}$$
Then for $t\ge T$ we have
$$\begin{aligned}y_0(t)+y_0(\tau(t))=&6+\frac{(-1)^{n-1}}{(n-2)!}\int_t^{\infty}\frac{1}{r(s)}\int_s^{\infty}(u-s)^{n-2}q(u)G(y_0(h(u)))du\,ds\\&-\int_t^{\infty}\frac{F(s)}{r(s)}ds.\end{aligned}$$
Differentiating the above equation first and then multiplying by $r(t)$ to both sides and after that differentiating again for $n-1$ times, we see that y_0 is the required solution of (3.1) which is bounded below by a positive constant.Hence this solution neither oscillates nor tends to zero as $t\to\infty$. Hence the theorem is proved. □

Corollary 3.3. *If* (H1), (H2), (H4) *hold, then there exists a positive solution of* (3.1) *which is bounded below by a positive constant.*

Proof. The proof follows from Remark 1.1 and the above Theorem. □

Theorem 3.4. *Let* (H3), (H5), (2.9) *and* (2.10) *hold. Then there exists a positive solution of* (3.1) *which is bounded below by a positive constant that is, it neither oscillates nor tends to zero as t tends to ∞.*

The proof of the above theorem is similar to that of Theorem 3.2.

Theorem 3.5. *Suppose* (H1) *hold. Assume for* $t \geq t_0$

$$\sum_{i=1}^{\infty} \int_{\tau_{-1}^{i}(t)}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} (u-s)^{n-2} q(u) du\, ds < \infty, \tag{3.8}$$

and

$$\sum_{i=1}^{\infty} \int_{\tau_{-1}^{i}(t)}^{\infty} \frac{1}{r(s)} ds < \infty. \tag{3.9}$$

Then (3.2) *has a solution bounded below by a positive constant.*

Proof. We proceed as in the proof of Theorem 3.2 with the following changes. Let

$$\mu = \max\{|G(x)| : 1 \leq x \leq 5\}.$$

Then from (H1), (3.8) and (3.9), there exists $T > 0$ such that for $t \geq T$

$$\frac{\mu}{(n-2)!} \sum_{i=1}^{\infty} \int_{\tau_{-1}^{i}(t)}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} (u-s)^{n-2} q(u) du\, ds < 1,$$

and

$$\sum_{i=1}^{\infty} \int_{\tau_{-1}^{i}(t)}^{\infty} \frac{F(s)}{r(s)} ds < 1.$$

Let $S = \{y \in X : 1 \leq y \leq 5,\, t \geq T_0\}$. Then define

$$By(t) = \begin{cases} By(T), & \text{for } t \in [T_0, T]; \\ 3 + \frac{(-1)^n}{(n-2)!} \sum_{i=1}^{\infty} \int_{\tau_{-1}^{i}(t)}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} (u-s)^{n-2} & \\ \times q(u) G(y(h(u))) du\, ds + \sum_{i=1}^{\infty} \int_{\tau_{-1}^{i}(t)}^{\infty} \frac{F(s)}{r(s)} ds, & \text{for } t \geq T. \end{cases}$$

Then as in Theorem 2.2 we prove (i) $By \in S$ for $y \in S$, and (ii) BS is relatively compact. Then by lemma 3.1 there exists a fixed point $y_0 \in S$ such that $By_0 = y_0$, Putting $y_0 = y(t)$, we get

$$\begin{aligned} y(t) = 3 - \frac{(-1)^{n-1}}{(n-2)!} \sum_{i=1}^{\infty} \int_{\tau_{-1}^{i}(t)}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} (u-s)^{n-2} q(u) G(y(h(u))) du\, ds \\ + \sum_{i=1}^{\infty} \int_{\tau_{-1}^{i}(t)}^{\infty} \frac{F(s)}{r(s)} ds. \end{aligned}$$

Then replacing t by $\tau(t)$ in the above and using $\tau_{-1}^{i}(\tau(t)) = \tau_{-1}^{i-1}(t)$, we may obtain $y(\tau(t))$. Consequently for $t \geq T$, we find

$$\begin{aligned} y(t) - y(\tau(t)) = &\frac{(-1)^{n-1}}{(n-2)!} \int_{t}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} (u-s)^{n-2} q(u) G(y(h(u))) du\, ds \\ &- \int_{t}^{\infty} \frac{F(s)}{r(s)} ds. \end{aligned}$$

We may differentiate the above and then multiply by $r(t)$ and then again differentiate $n-1$ times to arrive at (3.2). This solution is bounded below by a positive constant. $\square$

Remark 3.6. It is not difficult to verify that the above theorem still holds, if we replace (3.9) and (H1) by the following assumption

$$\sum_{i=1}^{\infty}\int_{\tau_{-1}^{i}(t)}^{\infty}\frac{1}{r(s)}\int_{s}^{\infty}(u-s)^{n-2}f(u)du\,ds<\infty. \tag{3.10}$$

Of course, in that case we have to modify the definition of the mapping B as follows.

$$By(t)=\begin{cases}By(T), & \text{for } t\in[T_0,T];\\ 3-\frac{(-1)^{n-1}}{(n-2)!}\sum_{i=1}^{\infty}\int_{\tau_{-1}^{i}(t)}^{\infty}\frac{1}{r(s)}\int_{s}^{\infty}(u-s)^{n-2} & \\ \times q(u)G(y(h(u)))du\,ds & \\ +\frac{(-1)^{n-1}}{(n-2)!}\sum_{i=1}^{\infty}\int_{\tau_{-1}^{i}(t)}^{\infty}\frac{1}{r(s)}\int_{s}^{\infty}(u-s)^{n-2}f(u)du\,ds, & \text{for } t\geq T.\end{cases}$$

If we put $r(t)=1$ in (3.8) and (3.10) then we obtain

$$\sum_{i=1}^{\infty}\int_{\tau_{-1}^{i}(t)}^{\infty}\int_{s}^{\infty}(u-s)^{n-2}q(u)du\,ds<\infty, \tag{3.11}$$

and

$$\sum_{i=1}^{\infty}\int_{\tau_{-1}^{i}(t)}^{\infty}\int_{s}^{\infty}(u-s)^{n-2}f(u)du\,ds<\infty. \tag{3.12}$$

Then from the above theorem the following result follows directly.

Corollary 3.7. *If* (3.11) *and* (3.12) *hold for $t>t_0$, then the NDDE*

$$(y(t)-y(t-\tau))^{(n)}+q(t)G(y(t-\sigma))=f(t) \tag{3.13}$$

has a solution, bounded below by a positive constant.

The above corollary improves and generalizes [5, Theorem 3.1] and [7, Theorem 2.5], because in these papers, the authors assume the following additional conditions that we don't require.

(i) n is odd.
(ii) G is non-decreasing and $xG(x)>0$ for $x\neq 0$.

Before we close this article we present an interesting example which illustrates most of the results of this paper.

Example 3.8. Consider NDDE

$$(r(t)(y(t)-py(t/2))')^{n-1}+\frac{1}{t^{n+2}}G(y(t/3))=0 \quad \text{for } t\geq t_0. \tag{3.14}$$

In this example suppose that p is any constant and $r(t)\equiv 1$ or $r(t)\equiv\frac{1}{t^2}$. If we compare this equation (3.14) with NDDE (1.1) then $\tau(t)=\frac{t}{2}$, $h(t)=\frac{t}{3}$ and $q(t)=\frac{1}{t^{n+2}}$. It is not difficult to verify that $q(t)$ satisfies (H2), (H5) and (3.8). Suppose that $G(u)=1-u^3$ and it is decreasing. Clearly the NDDE (3.14) has a positive solution $y(t)\equiv 1$. Hence this example illustrates all the results of this paper. However since G is decreasing and $\tau(t)$ is not of the form $t-k$, the existing results of [1, 4, 5, 6, 7, 8] are not applicable to (3.14).

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