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SPACE DIMENSION CAN PREVENT THE BLOW-UP OF SOLUTIONS FOR PARABOLIC PROBLEMS

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Abstract. In the present paper, we investigate the preventive role of space dimension for semilinear parabolic problems. Conditions guaranteeing the absence of the blow-up of the solutions are formulated.

1. Introduction and Main Results

Consider the equation

$$u_t - \alpha \Delta u = f(u) \quad \text{in } Q_T = (0,T) \times \{|\mathbf{x}| < R\}, \quad \mathbf{x} \in \mathbb{R}^n \tag{1.1}$$

coupled with initial condition

$$u(0,\mathbf{x}) = \phi(|\mathbf{x}|), \tag{1.2}$$

where $\phi(R) = 0$, $|\phi'(|\mathbf{x}|)| \le K-$a constant, and one of the two boundary conditions:

$$u\big|_{S_T} = 0, \quad \text{or} \tag{1.3}$$

$$-\alpha \frac{\partial u}{\partial \nu}\big|_{S_T} = \kappa u\big|_{S_T}, \quad S_T = (0,T) \times \{|\mathbf{x}| = R\}. \tag{1.4}$$

Here the heat conductivity coefficient α and the heat transfer coefficient κ are strictly positive constants. Concerning the function f we assume that

$$|f(\xi)| \le f(\eta) \quad \text{for all } \xi \text{ and } \eta \text{ such that } |\xi| \le \eta. \tag{1.5}$$

For example, functions $f(u) = |u|^{p-1}u$ for arbitrary $p \ge 1$ (or u^p if defined) as well as $f(u) = e^u$, $f(u) = \ln(|u|+1)$ or $f(u) = |u|^p$ satisfy condition (1.5).

It is well known that for the above problems the phenomenon of blowing up of the solution may occur, i.e. there exists t^* such that $|u(t,\mathbf{x}^*)| \to +\infty$ when $t \to t^*$ at least for one $\mathbf{x}^* \in \{|\mathbf{x}| \le R\}$ (see, [2, 3] and the references there). The goal of the present paper is to show that the space dimension can prevent blow-up.

Introduce constants $C(n)$ and $\Sigma(n)$:

$$C(n) = \frac{n + e^{1-n} - 2}{(n-1)^2}, \quad \Sigma(n) = \frac{1 - e^{1-n}}{n-1}.$$

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Assume that

$$\alpha \geq \frac{f(KR)R}{K} C(n), \tag{1.6}$$

$$\kappa \geq \frac{\alpha f(KR)\Sigma(n)}{\alpha K - f(KR)RC(n)}. \tag{1.7}$$

Obviously condition (1.7) makes sense only if in (1.6) we have strict inequality. One can easily see that

$$\lim_{n\to+\infty} C(n) = 0, \quad \lim_{n\to 1} C(n) = \frac{1}{2},$$
$$\lim_{n\to+\infty} \Sigma(n) = 0, \quad \lim_{n\to 1} \Sigma(n) = 1,$$

hence when the dimension n grows the restrictions (1.6) and (1.7) on α and κ becomes weaker.

Our results are formulated as follows.

Theorem 1.1. *Suppose that $f(u)$ is Hölder continuous function. If conditions* (1.5), (1.6) *hold then for arbitrary $T > 0$ there exists a classical solution of problem* (1.1)–(1.3) *and*

$$\max_{Q_T} |u(t, \mathbf{x})| \leq KR.$$

Furthermore, if $f(u)$ is Lipschitz continuous, the solution is unique.

Theorem 1.2. *Suppose that $f(u)$ is Hölder continuous function. If conditions* (1.5)–(1.7) *hold and $\phi'(R) = 0$, then for arbitrary $T > 0$ there exists a classical solution of problem* (1.1), (1.2), (1.4) *and*

$$\max_{Q_T} |u(t, \mathbf{x})| \leq KR.$$

Furthermore, if $f(u)$ is Lipschitz continuous, the solution is unique.

Example 1.3. Consider the equation

$$u_t - \Delta u = u^2 \quad \text{in } (0, T) \times \{|\mathbf{x}| < 1\}. \tag{1.8}$$

Condition (1.6) takes the form

$$1 \geq K\, C(n).$$

Obviously, for arbitrary K we can select n_K such that

$$1 \geq KC(n_K).$$

Hence for any $n \geq n_K$ the solution of problem (1.8), (1.2), (1.3) can not blow-up.

Example 1.4. Consider the equation

$$u_t - \Delta u = e^u \quad \text{in } (0, T) \times \{|\mathbf{x}| < 1\}. \tag{1.9}$$

Condition (1.6) takes the form

$$1 \geq \frac{e^K}{K}\, C(n).$$

Here also we can easily find n_K such that for any $n \geq n_K$ the solution of problem (1.9), (1.2), (1.3) can not blow-up.

Example 1.5. Consider problem (1.8), (1.2), (1.4). For arbitrary K we can select n_K such that $1 > KC(n_K)$ and for arbitrary $\kappa > 0$ we find n_κ such that

$$\kappa \geq \frac{K\Sigma(n_\kappa)}{1 - KC(n_\kappa)}.$$

Thus we conclude that for $n \geq \max\{n_K, n_\kappa\}$ the solution of problem (1.8), (1.2), (1.4) can not blow-up.

Note that if $\kappa = 0$ there is no heat flow through the boundary and the solution blows up.

2. Proof of Theorems 1.1 and 1.2

It is well known (see, for example, [1]) that the solvability of the above problems follows from the a priori estimate of the $\max |u|$. Hence our goal is to obtain this estimate.

Proof of Theorem 1.1. In (t, r) variables, where $r = |\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}$, problem (1.1) - (1.3) takes the form

$$u_t - \alpha\big(u_{rr} + \frac{n-1}{r}u_r\big) = f(u) \quad \text{in } Q_T^* = \{(t, r) : t \in (0, T), 0 < r < R\}, \tag{2.1}$$

$$u(0, r) = \phi(r), \quad \text{where } \phi(R) = 0,\ |\phi'(r)| \leq K, \tag{2.2}$$

$$u_r(t, 0) = 0, \quad u(t, R) = 0. \tag{2.3}$$

Consider the auxiliary equation

$$u_t - \alpha\big(u_{rr} + \frac{n-1}{r}u_r\big) = \bar{f(u)} \quad \text{in } Q_T^*, \tag{2.4}$$

where

$$\bar{f(u)} = \begin{cases} f(u), & \text{for } |u| \leq KR \\ f(KR), & \text{for } u > KR \\ f(-KR), & \text{for } u < -KR. \end{cases} \tag{2.5}$$

The existence of a classical solution of problem (2.4), (2.2), (2.3) follows, for example, from [4].

Our goal is to prove the a priori estimate $|u(t, r)| \leq KR$ for the solution of the auxiliary problem and consequently to show that equations (2.1) and (2.4) coincide. Consider the equation

$$h'' + \frac{n-1}{R}h' = -\frac{f(KR)}{\alpha} \tag{2.6}$$

coupled with the boundary condition $h(0) = KR$. Obviously, the function

$$h(r) = KR - C_1 + C_1 e^{\frac{1-n}{R}r} - \frac{f(KR)R}{\alpha(n-1)}r$$

satisfies (2.6) and the boundary condition $h(0) = KR$. For our purpose we need the function $h(r)$ to be nonnegative, nonincreasing and concave. The restrictions $h'(r) \leq 0$ or

$$h'(r) = \frac{1-n}{R}C_1 e^{\frac{1-n}{R}r} - \frac{f(KR)R}{\alpha(n-1)} \leq 0$$

implies

$$C_1 \geq -\frac{f(KR)R^2}{\alpha(n-1)^2}\,.$$

Also restriction $h(r) \geq 0$ (actually $h(R) \geq 0$) implies
$$C_1 \leq -\frac{f(KR)R^2 - \alpha(n-1)KR}{\alpha(n-1)(1-e^{1-n})}.$$
Condition on α in Theorem 1.1 guarantees that
$$-\frac{f(KR)R^2}{\alpha(n-1)^2} \leq -\frac{f(KR)R^2 - \alpha(n-1)KR}{\alpha(n-1)(1-e^{1-n})}.$$
To satisfy condition $h''(r) \leq 0$, we select
$$C_1 = -\frac{f(KR)R^2}{\alpha(n-1)^2}.$$
Thus we take
$$h(r) = KR + \frac{f(KR)R}{\alpha(n-1)^2}\big[R(1-e^{\frac{1-n}{R}r}) - (n-1)r\big].$$
Define the operator
$$L \equiv \frac{\partial}{\partial t} - \alpha\Big(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\Big).$$
Denote by Γ_T the parabolic boundary of Q_T^* (i.e., $\Gamma_T = \partial Q_T^* \setminus \{t = T, 0 < r < R\}$). For $v(t,r) \equiv u(t,r) - h(r)$ we have
$$\begin{aligned}
Lv &\equiv v_t - \alpha(v_{rr} + \frac{n-1}{r}v_r)\\
&= f(\bar{u}) + \alpha(h'' + \frac{n-1}{r}h')\\
&< f(\bar{u}) + \alpha(h'' + \frac{n-1}{R}h')\\
&= f(\bar{u}) - f(KR) \leq 0 \quad \text{in } \bar{Q}_T^* \setminus \Gamma_T.
\end{aligned} \tag{2.7}$$
Here we use the fact that $h'(r)$ is strictly negative in $(0,R)$. Note that from (1.5) and (2.5) follows that
$$-f(KR) \leq f(\bar{u}) \leq f(KR).$$
Obviously $v(0,r) = \phi(r) - h(r) \leq 0$ since $h''(r) \leq 0$, $h(0) = KR$ and $h(R) \geq 0$, besides $u(t,R) - h(R) \leq 0$. Taking into account (2.7) and the fact that $v_r(t,0) = 0$ we conclude that v can not attain its maximum neither in $\bar{Q}_T^* \setminus \Gamma_T$ nor on $\{0 < t < T, r = 0\}$, hence
$$u(t,r) \leq h(r) \leq KR.$$
Let us obtain the lower estimate. For $w(t,r) \equiv u(t,r) + h(r)$ we have
$$\begin{aligned}
Lw &= w_t - \alpha(w_{rr} + \frac{n-1}{r}w_r)\\
&= f(\bar{u}) - \alpha(h'' + \frac{n-1}{r}h')\\
&> f(\bar{u}) - \alpha(h'' + \frac{n-1}{R}h')\\
&= f(\bar{u}) + f(KR) \geq 0 \quad \text{in } \bar{Q}_T^* \setminus \Gamma_T.
\end{aligned} \tag{2.8}$$
Obviously $w \geq 0$ for $t = 0$ and for $r = R$. Taking into account (2.8) and the fact that $w_r(t,0) = 0$ we conclude that w can not attain its minimum neither in $\bar{Q^*}_T \setminus \Gamma_T$ nor on $\{0 < t < T, r = 0\}$, hence
$$u(t,r) \geq -h(r) \geq -KR.$$

Thus

$$|u(t,r)| \le KR.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. In (t,r) variables condition (1.4) takes the form

$$u_r(t,0) = 0, \quad -\alpha u_r(t,R) = \kappa u(t,R). \tag{2.9}$$

Consider the auxiliary problem (2.4), (2.2), (2.9). The existence of a classical solution of this problem follows, for example, from [4]. Our goal is to prove the a priori estimate $|u(t,r)| \le KR$ for the solution of problem (2.4), (2.2), (2.9).

As it follows from the proof of Theorem 1.1 the function $v \equiv u(t,r) - h(r)$ can not attain its positive maximum in $\bar{Q}_T^* \setminus \Gamma_T$. Suppose that function $u(t,r) - h(r)$ attains its positive maximum on the right boundary of the domain, in this case we have $u(t,R) > h(R) > 0$, besides, from the boundary condition (2.9) and from condition (1.7) we conclude that

$$v_r(t,r)\big|_{r=R} = u_r(t,r) - h'(r)\big|_{r=R} = -\frac{\kappa}{\alpha}u(t,R) - h'(R) < -\frac{\kappa}{\alpha}h(R) - h'(R) \le 0,$$

which is impossible. Taking into account that $v(0,r) = \phi(r) - h(r) \le 0$ and the fact that due to the condition $v_r(t,0) = 0$ positive maximum cannot be obtained on $\{0 < t < T, r = 0\}$ we conclude that

$$u(t,r) \le h(r) \le KR.$$

Let us obtain the lower estimate. We have that function $w \equiv u(t,r) + h(r)$ can not attain its negative minimum in $\bar{Q}_T^* \setminus \Gamma_T$. Suppose that the function $u(t,r)+h(r)$ attains its negative minimum on the right boundary of the domain, in this case we have $u(t,R) < -h(R)$, besides, from boundary condition (2.9) and from condition (1.7) we conclude that

$$w_r(t,r)\big|_{r=R} = u_r(t,r) + h'(r)\big|_{r=R} = -\frac{\kappa}{\alpha}u(t,R) + h'(R) > \frac{\kappa}{\alpha}h(R) + h'(R) \ge 0,$$

which is impossible. Taking into account that $w(0,r) = \phi(r) + h(r) \ge 0$ and the fact that due to the condition $w_r(t,0) = 0$ negative minimum cannot be obtained on $\{0 < t < T, r = 0\}$ we conclude that

$$u(t,r) \ge h(r) \ge -KR.$$

Thus $|u(t,r)| \le KR$. This completes the proof of Theorem 1.2. □

The uniqueness in Theorems 1.1 and 1.2 can be proved by standard arguments based on maximum principle.

Remark 2.1. *Consider the linear case $f(u) = \lambda u$ with λ positive. For the solution of equation*

$$u_t = \alpha \Delta u + \lambda u \tag{2.10}$$

coupled with conditions (1.2), (1.3) *we have the standard estimate*

$$|u(t,\mathbf{x})| \le e^{\lambda t} \max |\phi(\mathbf{x})|.$$

Let us apply Theorem 1.1 to this case. Inequality (1.6) *takes the form*

$$\alpha \ge \lambda R^2 C(n). \tag{2.11}$$

Thus if (2.11) *is fulfilled then for the solution of problem* (2.10), (1.2), (1.3) *the estimate, independent of* t,

$$|u(t, \mathbf{x})| \le KR$$

holds.

Remark 2.2. Consider the sublinear case, $q \in (0,1)$. As mentioned above the function $f(u) = |u|^q$ (as well as $f(u) = u^q$ if defined) satisfies condition (1.5). Consider the equation

$$u_t - \alpha\Delta u = |u|^q \quad (\text{or } u^q) \quad \text{in } Q_T \tag{2.12}$$

coupled with conditions (1.2), (1.3). Inequality (1.6) takes the form

$$\alpha \ge \frac{R^{1+q}C(n)}{K^{1-q}}. \tag{2.13}$$

Obviously for any $\alpha > 0$ we can always select $K \ge \max|\phi'(|\mathbf{x}|)|$ big enough such that (2.13) is fulfilled. Thus from Theorem 1.1 it follows that the classical solution $u(t, \mathbf{x})$ of problem (2.12), (1.2), (1.3) exists and $|u(t, \mathbf{x})| \le KR$ where K is selected so that (2.13) is fulfilled.

Similarly, we can consider the equation

$$u_t - \alpha\Delta u = \ln(|u| + 1) \quad \text{in } Q_T$$

and obtain the existence of a classical solution of problem (1.2), (1.3) satisfying the inequality $|u(t, \mathbf{x})| \le KR$ where $K \ge \max|\phi'(|\mathbf{x}|)|$ is such that

$$\alpha \ge \frac{\ln(KR + 1)R}{K} C(n).$$

References

[1] O. A. Ladyzhenskaja, V. A. Solonnikov, N. N. Uralceva; *Linear and quasilinear equations of parabolic type.* (in Russian) Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23 American Math. Society, Providence, R.I. 1967 xi+648 pp.

[2] P. Quittner, Ph. Souplet; *Superlinear Parabolic Problems: Blow-up, Global Existence and Steady States*, Birkhauser, 2007.

[3] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, A. P. Mikhailov; *Blow-up in quasilinear parabolic equations*, de Gruyter Expositions in Mathematics, 19. Walter de Gruyter & Co., Berlin, 1995, 535 pp.

[4] Al. S. Tersenov, Ar. S. Tersenov; *Global solvability for a class of quasilinear parabolic problems*, Indiana Univ. Math. J. 50 (2001), no. 4, 1899–1913.

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