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PERIODICITY IN A DELAYED RATIO-DEPENDENT PREDATOR-PREY SYSTEM WITH EXPLOITED TERM

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Abstract. With the help of the coincidence degree and the related continuation theorem, we explore the existence of at least two periodic solutions of a delayed ratio-dependent predator-prey system with exploited term. Some easily verifiable sufficient criteria are established for the existence of at least two positive periodic solutions.

1. Introduction

As is well-known, the traditional Lotka-Volterra type predator-prey model with prey-dependent functional response fails to model the interference among predators. To overcome the shortcoming, Arditi and Ginzburg [1] proposed the following ratio-dependent predator-prey model

$$\begin{aligned} x'(t) &= x(a-bx) - \frac{cxy}{my+x}, \\ y'(t) &= -dy + \frac{fxy}{my+x}, \end{aligned} \tag{1.1}$$

which incorporates mutual interference by the predatorss. For a detailed justification of (1.1) and its merits versus the prey-dependent functional response model, we refer to [1]. In addition, system (1.1) and its non-autonomous variation have been studied by many authors and seen great progress, see, for example, [5, 6, 10, 15] and the references therein. Beretta and Kuang [2] introduced a single discrete time delay into the predator equation in system (1.1), namely

$$\begin{aligned} x'(t) &= x(a-bx) - \frac{cxy}{my+x}, \\ y'(t) &= y\Big[-d + \frac{fx(t-\tau)}{my(t-\tau)+x(t-\tau)}\Big], \end{aligned} \tag{1.2}$$

and carried out systematic work on the global qualitative analysis of (1.2). In paper [4], Fan and Wang studied a more general delayed ratio-dependent predator-prey

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model

$$
\begin{aligned}
x'(t) &= x\Big[a(t) - b(t)\int_{-\infty}^{t} k(t-s)x(s)ds\Big] - \frac{c(t)xy}{my+x},\\
y'(t) &= y\Big[-d(t) + \frac{f(t)x(t-\tau(t))}{my(t-\tau(t)) + x(t-\tau(t))}\Big],
\end{aligned}
\tag{1.3}
$$

where x, y denote prey and predator density, respectively. m is a constant that denotes the half capturing saturation constant, $a \in C(\mathbb{R}, \mathbb{R})$, b, c, d, f, τ in $C(\mathbb{R}, \mathbb{R}_+)$, $\mathbb{R}^+ = [0, +\infty), k(s) : \mathbb{R}^+ \to \mathbb{R}^+$ is a measurable, normalized function such that $\int_0^{+\infty} k(s)ds = 1$. For a detailed discussion of the biological significance of the parameters in (1.3), refer to [3, 4, 9, 12, 13].

A very basic and important ecological problem associated with the study of multi-species population interactions is the existence of positive periodic solutions due to various seasonal effects present in real life situations. Although much progress has been seen in the study of such problems, there are relatively fewer results on the models with exploited term. Therefore, the major objective of this paper is to investigate the existence of periodic solutions of the following system

$$
\begin{aligned}
x'(t) &= x\Big[a(t) - b(t)\int_{-\infty}^{t} k(t-s)x(s)ds\Big] - \frac{c(t)xy}{m(t)y+x} - h(t),\\
y'(t) &= y\Big[-d(t) + \frac{f(t)x(t-\tau(t))}{m(t)y(t-\tau(t)) + x(t-\tau(t))}\Big],
\end{aligned}
\tag{1.4}
$$

where h is an exploitation term standing for harvesting or hunting.

An outline of this paper is given as follows. In section 2, we present some preliminaries including the famous coincidence degree theory and a basic lemma. In section 3, by using the coincidence degree theory, we will establish some sufficient conditions for the existence of positive periodic solutions of system (1.4). At last, an example is given to verify and support our theoretical result.

2. Preliminaries

Let us begin by introducing some terminology and results.

If g is a real continuously bounded function defined on $\mathbb{R}$, we set

$$
\bar{g} = \frac{1}{\omega}\int_0^{\omega} g(t)dt, \quad g^L = \min_{t\in[0,\omega]} g(t), \quad g^M = \max_{t\in[0,\omega]} g(t).
$$

In system (1.4), we always assume that a, $d : \mathbb{R} \to \mathbb{R}$ and b, c, m, f, $h, \tau : \mathbb{R} \to \mathbb{R}^+$ are ω-periodic and $\bar{a} > 0$, $\bar{d} > 0$, where ω, a fixed positive integer, denotes the prescribed common period of the parameters in system (1.4). Moreover, for biological reasons, we only consider solutions $(x(t), y(t))$ with $x(0) > 0, y(0) > 0$.

For the reader's convenience, we now recall Mawhin's coincidence degree which our study is based upon. Let X, Z be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Z$ a linear mapping, $N : X \to Z$ is a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim\ker L = \operatorname{codim}\operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $ImP = \ker L, \operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q)$. It follows that $L|\operatorname{Dom} L \cap \ker P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω be an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and

$K_P(I-Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \operatorname{Im} Q \to \ker L$.

Lemma 2.1 (Continuation Theorem [8]. *] Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega}$. Suppose*

(i) *For each $\lambda \in (0,1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
(ii) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \ker L$ and*

$$deg\{JQN, \Omega \cap \ker L, 0\} \neq 0.$$

Then the equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \bar{\Omega}$.

Lemma 2.2. *If $\bar{f} > \bar{d}$ and $\bar{a} - \overline{(c/m)} > 2\sqrt{\bar{b}\bar{h}}$, then the following algebraic equations*

$$\begin{gathered}
\bar{a} - \bar{b}\exp\{u\} - \frac{1}{\omega}\int_0^\omega \frac{c(t)\exp\{v\}}{m(t)\exp\{v\} + \exp\{u\}}dt - \frac{\bar{h}}{\exp\{u\}} = 0 \\
-\bar{d} + \frac{1}{\omega}\int_0^\omega \frac{f(t)\exp\{u\}}{m(t)\exp\{v\} + \exp\{u\}}dt = 0
\end{gathered} \tag{2.1}$$

have two solutions.

Proof. Consider the function

$$f(z) = -\bar{d} + \frac{1}{\omega}\int_0^\omega \frac{f(t)}{m(t)z+1}dt.$$

It is easily seen that $f(z)$ is decreasing with z and

$$f(0) = \bar{f} - \bar{d} > 0, \quad \lim_{z\to+\infty} f(z) = -\bar{d} < 0,$$

then it follows that there exists a unique z^* such that $f(z^*) = 0$. Substituting $z^* = \exp\{v-u\}$ into the first equation in (2.1), we have

$$\bar{a} - \bar{b}\exp\{u\} - \frac{1}{\omega}\int_0^\omega \frac{c(t)z^*}{m(t)z^*+1}dt - \frac{\bar{h}}{\exp\{u\}} = 0. \tag{2.2}$$

Obviously, it is a quadratic equation with respect to $\exp\{u\}$, then it has two solutions, denoted by u_1 and u_2 with $u_1 < u_2$. Moreover, one can easily see that

$$\bar{a} - \bar{b}\exp\{u\} - (\overline{\frac{c}{m}} - \frac{\bar{h}}{\exp\{u\}} < 0.$$

Solving the inequality, produces

$$\begin{gathered}
\exp\{u_1\} < \frac{\bar{a} - \overline{(c/m)} - \sqrt{[\bar{a} - \overline{(c/m)}]^2 - 4\bar{b}\bar{u}}}{2\bar{b}}, \\
\exp\{u_2\} > \frac{\bar{a} - \overline{(c/m)} + \sqrt{[\bar{a} - \overline{(c/m)}]^2 - 4\bar{b}\bar{u}}}{2\bar{b}},
\end{gathered}$$

which implies (2.1) has two solutions and this completes the proof. $\square$

3. Main results

In this section, we devote ourselves to establishing easily verifiable sufficiency criteria for the existence of at least two positive periodic solutions of system (2.1) by employing the coincidence degree and the related continuation theorem introduced in the previous section.

Theorem 3.1. *If $\bar{f} > \bar{d}$ and $(a - c/m)^L > 2\sqrt{b^M h^M}$, then system* (1.4) *has at least two positive ω-periodic solutions.*

Proof. Let $x(t) = \exp\{u(t)\},\ y(t) = \exp\{v(t)\}$. Then system (1.4) can be written as

$$\begin{aligned} u'(t) &= a(t) - b(t)\int_{-\infty}^{t} k(t-s)\exp\{u(s)\}ds \\ &\quad - \frac{c(t)\exp\{v(t)\}}{m(t)\exp\{v(t)\} + \exp\{u(t)\}} - \frac{h(t)}{\exp\{u(t)\}}, \\ v'(t) &= -d(t) + \frac{f(t)\exp\{u(t-\tau(t))\}}{m(t)\exp\{v(t-\tau(t))\} + \exp\{u(t-\tau(t))\}}. \end{aligned} \tag{3.1}$$

It is easy to see that if system (3.1) has an ω-periodic solution $(u^*, v^*)^T$, then $(x^*, y^*)^T = (\exp\{u^*\}, \exp\{v^*\})^T$ is a positive ω-periodic solution of system (1.4). To this end, it suffices to prove that system (3.1) has at least two ω-periodic solutions.

For $\lambda \in (0,1)$, we consider the following system

$$\begin{aligned} u'(t) &= \lambda\Big[a(t) - b(t)\int_{-\infty}^{t} k(t-s)\exp\{u(s)\}ds \\ &\quad - \frac{c(t)\exp\{v(t)\}}{m(t)\exp\{v(t)\} + \exp\{u(t)\}} - \frac{h(t)}{\exp\{u(t)\}}\Big], \\ v'(t) &= \lambda\Big[-d(t) + \frac{f(t)\exp\{u(t-\tau(t))\}}{m(t)\exp\{v(t-\tau(t))\} + \exp\{u(t-\tau(t))\}}\Big]. \end{aligned} \tag{3.2}$$

Suppose that $(u(t), v(t))^T$ is an arbitrary ω-periodic solution of system (3.2) for a certain $\lambda \in (0,1)$. Integrating on both sides of (3.2) over the interval $[0,\omega]$, leads to

$$\begin{aligned} \bar{a}\omega = \int_0^\omega \Big[b(t)\int_{-\infty}^{t} k(t-s)\exp\{u(s)\}ds \\ + \frac{c(t)\exp\{v(t)\}}{m(t)\exp\{v(t)\} + \exp\{u(t)\}} + \frac{h(t)}{\exp\{u(t)\}}\Big]dt, \end{aligned} \tag{3.3}$$

$$\bar{d}\omega = \int_0^\omega \Big[\frac{f(t)\exp\{u(t-\tau(t))\}}{m(t)\exp\{v(t-\tau(t))\} + \exp\{u(t-\tau(t))\}}\Big]dt. \tag{3.4}$$

From these two equations, it follows that

$$\begin{aligned} \int_0^\omega |u'(t)|dt &\le \int_0^\omega |a(t)|dt + \int_0^\omega \Big[b(t)\int_{-\infty}^{t} k(t-s)\exp\{u(s)\}ds \\ &\quad + \frac{c(t)\exp\{v(t)\}}{m(t)\exp\{v(t)\} + \exp\{u(t)\}} + \frac{h(t)}{\exp\{u(t)\}}\Big]dt \\ &=: (\bar{A} + \bar{a})\omega, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\int_0^\omega |v'(t)|dt \\
&\le \int_0^\omega |d(t)|dt + \int_0^\omega \Big[\frac{f(t)\exp\{u(t-\tau(t))\}}{m(t)\exp\{v(t-\tau(t))\}+\exp\{u(t-\tau(t))\}}\Big]dt := (\bar{D}+\bar{d})\omega.
\end{aligned} \tag{3.6}$$

Choose $\xi_i, \eta_i \in I_\omega, i = 1, 2$, such that

$$u(\xi_1) = \min_{t\in[0,\omega]}\{u(t)\}, \quad u(\eta_1) = \max_{t\in[0,\omega]}\{u(t)\}, \tag{3.7}$$

$$v(\xi_2) = \min_{t\in[0,\omega]}\{v(t)\}, \quad v(\eta_2) = \max_{t\in[0,\omega]}\{v(t)\}. \tag{3.8}$$

By (3.3) and (3.7), we obtain

$$\bar{a}\omega > \int_0^\omega b(t)\exp\{u(\xi_1)\}dt = \exp\{u(\xi_1)\}\bar{b}\omega,$$

which reduces to $u(\xi_1) < \ln\{\frac{\bar{a}}{\bar{b}}\}$. This inequality and (3.5) give

$$u(t) \le u(\xi_1) + \int_0^\omega |u'(t)|dt < \ln\{\frac{\bar{a}}{\bar{b}}\} + (\bar{A}+\bar{a})\omega := \rho_1. \tag{3.9}$$

Multiplying the first equality of (3.2) by $\exp\{u(t)\}$, and integrating over $[0,\omega]$, we have

$$\begin{aligned}
\int_0^\omega a(t)\exp\{u(t)\}dt = \int_0^\omega \Big[& b(t)\exp\{u(t)\}\int_{-\infty}^t k(t-s)\exp\{u(s)\}ds \\
&+ \frac{c(t)\exp\{v(t)\}}{m(t)\exp\{v(t)\}+\exp\{u(t)\}} + h(t)\Big]dt
\end{aligned}$$

Again from (3.3) and (3.7), it follows that

$$\exp\{u(\eta_1)\}\bar{a}\omega \ge \int_0^\omega a(t)\exp\{u(t)\}dt > \int_0^\omega h(t)dt = \bar{h}\omega,$$

which implies $u(\eta_1) > \ln\{\frac{\bar{h}}{\bar{a}}\}$. Therefore, by (3.5) and (3.7), we obtain

$$u(t) \ge u(\eta_1) - \int_0^\omega |u'(t)|dt > \ln\{\frac{\bar{h}}{\bar{a}}\} - (\bar{A}+\bar{a})\omega := \rho_2. \tag{3.10}$$

Similarly, from (3.4) and (3.8), we derive that

$$\bar{d}\omega < \int_0^\omega \frac{f(t)\exp\{u(\eta_1)\}}{m(t)\exp\{v(\xi_2)\}}dt = \frac{1}{\exp\{v(\xi_2)\}}\overline{(\frac{f}{m})}\exp\{u(\eta_1)\}\omega\,.$$

Then by (3.9), we have $v(\xi_2) < \ln\{\frac{\bar{a}}{\bar{b}\bar{d}}\overline{(\frac{f}{m})}\} + (\bar{A}+\bar{a})\omega$. This, together with (3.6), gives

$$v(t) \le v(\xi_2) + \int_0^\omega |v'(t)|dt < \ln\{\frac{\bar{a}}{\bar{b}\bar{d}}\overline{(\frac{f}{m})}\} + (\bar{A}+\bar{a})\omega. \tag{3.11}$$

Moreover, from (3.4), (3.7) and (3.8), we get

$$\bar{d}\omega > \int_0^\omega \frac{f(t)\exp\{u(\xi_1)\}}{m(t)\exp\{v(\eta_2)\}+\exp\{u(\xi_1)\}}dt > \frac{\exp\{u(\xi_1)\}\bar{f}\omega}{m^M\exp\{v(\eta_2)\}+\exp\{u(\xi_1)\}}\,.$$

Then, by (3.10), we obtain

$$v(\eta_2) > \ln\big\{\frac{(\bar{f}-\bar{d})\bar{u}}{m^M\bar{a}\bar{d}}\big\} - (\bar{A}+\bar{a})\omega,$$

which, together with (3.6), produces

$$v(t) \geq v(\eta_2) - \int_0^{\omega} |v'(t)|dt > \ln\big\{\frac{(\bar{f}-\bar{d})\bar{u}}{m^M\bar{a}\bar{d}}\big\} - (\bar{A}+\bar{a}+\bar{D}+\bar{d})\omega := \rho_4. \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$|v(t)| < |\rho_3| + |\rho_4| + 1 := B_1. \tag{3.13}$$

From (3.7) and the first equality of (3.1), we also have

$$\begin{aligned} &a(\eta_1) - b(\eta_1)\int_{-\infty}^{\eta_1} k(\eta_1 - s)\exp\{u(s)\}ds \\ &- \frac{c(\eta_1)\exp\{v(\eta_1)\}}{m(\eta_1)\exp\{v(\eta_1)\}+\exp\{u(\eta_1)\}} - \frac{h(\eta_1)}{\exp\{u(\eta_1)\}} = 0, \end{aligned}$$

which implies

$$b(\eta_1)\exp\{2u(\eta_1)\} - (a(\eta_1) - \frac{c(\eta_1)}{m(\eta_1)})\exp\{u(\eta_1)\} + h(\eta_1) > 0.$$

Solving the inequality, we have

$$\exp\{u(\eta_1)\} < \frac{(a-c/m)^L - \sqrt{[(a-c/m)^L]^2 - 4b^M u^M}}{2b^M} =: \delta_-,$$

or

$$\exp\{u(\eta_1)\} > \frac{(a-c/m)^L + \sqrt{[(a-c/m)^L]^2 - 4b^M u^M}}{2b^M} =: \delta_+.$$

That is, $u(\eta_1) < \ln\delta_-$ or $u(\eta_1) > \ln\delta_+$. Similarly, we can obtain $u(\xi_1) < \ln\delta_-$ or $u(\xi_1) > \ln\delta_+$. These, together with (3.9) and (3.10), we obtain

$$\rho_2 < u(t) < \ln\delta_-, \quad \text{or} \quad \ln\delta_+ < u(t) < \rho_1. \tag{3.14}$$

By Lemma 2.2, the following algebraic equations

$$\begin{aligned} &\bar{a} - \bar{b}\exp\{u\} - \frac{1}{\omega}\int_0^{\omega}\frac{c(t)\exp\{v\}}{m(t)\exp\{v\}+\exp\{u\}}dt - \frac{\bar{h}}{\exp\{u\}} = 0 \\ &-\bar{d} + \frac{1}{\omega}\int_0^{\omega}\frac{f(t)\exp\{u\}}{m(t)\exp\{v\}+\exp\{u\}}dt = 0 \end{aligned}$$

have two solutions, denoted by $(u_1, v_1)^T$ and $(u_2, v_2)^T$ $(v_1 < v_2)$ and satisfying

$$\rho_2 < u_1 < \ln\delta_-, \quad \text{or} \quad \ln\delta_+ < u_2 < \rho_1. \tag{3.15}$$

Clearly, $\rho_1, \rho_2, B_1, \delta_-, \delta_+$ are independent of λ.

Now let us take $X = Y = \{(u(t), v(t))^T \in C(\mathbb{R}, \mathbb{R}^2) | u(t+\omega) = u(t), v(t+\omega) = v(t)\}$ and $\|(u(t), v(t))^T\| = \max_{t\in[0,\omega]}|u(t)| + \max_{t\in[0,\omega]}|v(t)|$. Then X is a Banach space equipped with the norm $\|\cdot\|$.

Let $L(u(t), v(t))^T = (u'(t), v'(t))^T$ and $N : X \to X$, where

$$\begin{aligned} &N\begin{pmatrix} u(t) \\ v(t)\end{pmatrix} \\ &= \begin{bmatrix} a(t) - b(t)\int_{-\infty}^t k(t-s)\exp\{u(s)\}ds - \frac{c(t)\exp\{v(t)\}}{m(t)\exp\{v(t)\}+\exp\{u(t)\}} - \frac{h(t)}{\exp\{u(t)\}} \\ -d(t) + \frac{f(t)\exp\{u(t-\tau(t))\}}{m(t)\exp\{v(t-\tau(t))\}+\exp\{u(t-\tau(t))\}} \end{bmatrix}. \end{aligned}$$

Define projectors P and Q by

$$P\begin{pmatrix} u(t) \\ v(t)\end{pmatrix} = Q\begin{pmatrix} u(t) \\ v(t)\end{pmatrix} = \begin{pmatrix} \frac{1}{\omega}\int_0^{\omega} u(t)dt \\ \frac{1}{\omega}\int_0^{\omega} v(t)dt \end{pmatrix}, \quad \begin{pmatrix} u(t) \\ v(t)\end{pmatrix} \in X.$$

Obviously, $\operatorname{Im} P = \ker L$ and $\operatorname{Im} L = \ker Q = \{(u(t), v(t))^T \in X : \bar{u} = \bar{v} = 0\}$ is closed in X, and $\dim \ker L = 2 = \mathit{codim} \operatorname{Im} L$. Thus, L is a Fredholm operator of index zero. Furthermore, the generalized inverse (to L) is as follows

$$K_P : \operatorname{Im} L \to \operatorname{Dom} L \cap \ker P, \quad K_P \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \int_0^t u(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t u(s)dsdt \\ \int_0^t v(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t v(s)dsdt \end{pmatrix}.$$

Now, we reach the point where we search for appropriate open bounded subsets $\Omega_i,\ i = 1, 2$ for the application of the continuation theorem. To this end, we take $B_2 = |v_1| + |v_2|$, and define

$$\Omega_1 = \{(u(t), v(t))^T \in X : \rho_2 < u(t) < \ln \delta_-, \max_{t\in[0,\omega]} |v(t)| < B_1 + B_2\},$$
$$\Omega_2 = \{(u(t), v(t))^T \in X : \ln \delta_+ < u(t) < \rho_1, \max_{t\in[0,\omega]} |v(t)| < B_1 + B_2\}.$$

Clearly, both Ω_1 and Ω_2 are open subsets of X and $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \phi$ in view of $\delta_- < \delta_+$. From (3.15), we see that $(u_1, v_1)^T \in \Omega_1, (u_2, v_2)^T \in \Omega_2$.

By using the Arzela-Ascoli theorem, it is not difficult to show that $QN(\Omega_i)$ and $K_P(I - Q)N(\Omega_i)$, $i = 1, 2$, are compact. Therefore, N is L-compact on $\Omega_i, i = 1, 2$.

Since we are concerned with periodic solutions $(u(t), v(t))^T$ confined in $\operatorname{Dom} L$, system (3.2) can be regarded as the following operator equation $L(u(t), v(t))^T = \lambda N(u(t), v(t))^T$, which is system (3.1) when $\lambda = 1$. According to the previous estimation of periodic solution of (3.2), we have proved requirement (i) of Lemma 2.1.

When $(u, v)^T \in \partial\Omega_i \cap \ker L, i = 1, 2$, and $(u, v)^T$ is a constant vector in $\mathbb{R}^2$. From (3.13) and (3.15) and Lemma 2.2, it follows that

$$QN \begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} \bar{a} - \bar{b} \exp\{u\} - \frac{1}{\omega} \int_0^\omega \frac{c(t) \exp\{v\}}{m(t) \exp\{v\} + \exp\{u\}} dt - \frac{\bar{h}}{\exp\{u\}} \\ \\ -\bar{d} + \frac{1}{\omega} \int_0^\omega \frac{f(t) \exp\{u\}}{m(t) \exp\{v\} + \exp\{u\}} dt \end{bmatrix} \neq 0.$$

Moreover, direct calculation shows that

$$\deg(JQN, \Omega_i \cap \ker L, 0) \neq 0, \quad i = 1, 2,$$

where $\deg(\cdot)$ is the Brouwer degree and the J is the identity mapping since $\operatorname{Im} Q = \ker L$.

By now, we have proved that each $\Omega_i (i = 1, 2)$ satisfies all the requirements of Lemma 2.2. Hence, system (3.1) has at least one ω-periodic solution in each of Ω_1 and Ω_2. The proof is completed. □

Next, we consider the ratio-dependence predator-prey system with distributed delays

$$\begin{aligned} x'(t) &= x\Big[a(t) - b(t) \int_{-\tau}^0 x(t + s)d\mu(s)\Big] - \frac{c(t)xy}{m(t)y + x} - h(t), \\ y'(t) &= y\Big[- d(t) + \frac{f(t) \int_{-\sigma}^0 x(t + s)d\nu(s)}{m(t) \int_{-\sigma}^0 y(t + s)d\nu(s) + \int_{-\sigma}^0 x(t + s)d\nu(s)}\Big], \end{aligned} \tag{3.16}$$

where τ, σ are positive constants and μ, ν are nondecreasing functions such that

$$\mu(0^+) - \mu(-\tau^-) = 1, \quad \nu(0^+) - \nu(-\sigma^-) = 1.$$

Theorem 3.2. *If $\bar{f} > \bar{d}$ and $(a - c/m)^L > 2\sqrt{b^M h^M}$, then system* (3.16) *has at least two positive ω-periodic solutions.*

The proof of the above theorem is similar to that of Theorem 3.1 and hence is omitted here.

Remark 3.3. From the proofs of Theorem 3.1 and 3.2, it is seen that the conclusion of Theorem 3.2 remains valid if some or all of the $\tau's$ and $\sigma's$ are ∞.

In system (1.4), when the distributed delay in the prey equation is replaced by the periodic delay, that is, system (1.4) is rewritten as

$$\begin{aligned} x'(t) &= x\big[a(t) - b(t)x(t-\tau(t))\big] - \frac{c(t)xy}{m(t)y+x} - h(t), \\ y'(t) &= y\Big[-d(t) + \frac{f(t)x(t-\tau(t))}{m(t)y(t-\tau(t)) + x(t-\tau(t))}\Big], \end{aligned} \tag{3.17}$$

the result remains valid.

Theorem 3.4. *If $\bar{f} > \bar{d}$ and $(a - c/m)^L > 2\sqrt{b^M h^M}$, then system* (3.17) *has at least two positive ω-periodic solutions.*

Observing system (3.17), we can see that the delay is a function of t. In real life, delay is not only depends on time, but also on states. Therefore, we now propose the predator-prey model

$$\begin{aligned} x'(t) &= x\big[a(t) - b(t)x(t-\tau(t,x(t),y(t)))\big] - \frac{c(t)xy}{m(t)y+x} - h(t), \\ y'(t) &= y\Big[-d(t) + \frac{f(t)x(t-\tau(t))}{m(t)y(t-\tau(t,x(t),y(t))) + x(t-\tau(t,x(t),y(t)))}\Big]. \end{aligned} \tag{3.18}$$

By a similar discussion, we can obtain the following result.

Theorem 3.5. *If $\bar{f} > \bar{d}$ and $(a - c/m)^L > 2\sqrt{b^M h^M}$, then system* (3.18) *has at least two positive ω-periodic solutions.*

Especially, when there is no exploited term, that is $h(t) \equiv 0$, the system (1.4) reduces to

$$\begin{aligned} x'(t) &= x\Big[a(t) - b(t)\int_{-\infty}^{t} k(t-s)x(s)ds\Big] - \frac{c(t)xy}{m(t)y+x}, \\ y'(t) &= y\Big[-d(t) + \frac{f(t)x(t-\tau(t))}{m(t)y(t-\tau(t)) + x(t-\tau(t))}\Big]. \end{aligned} \tag{3.19}$$

Employing the powerful and effective coincidence degree method, we can obtain the following theorem.

Theorem 3.6. *If $\bar{f} > \bar{d}$ and $\bar{a} > \overline{(c/m)}$, then system* (3.19) *has at least one positive ω-periodic solutions.*

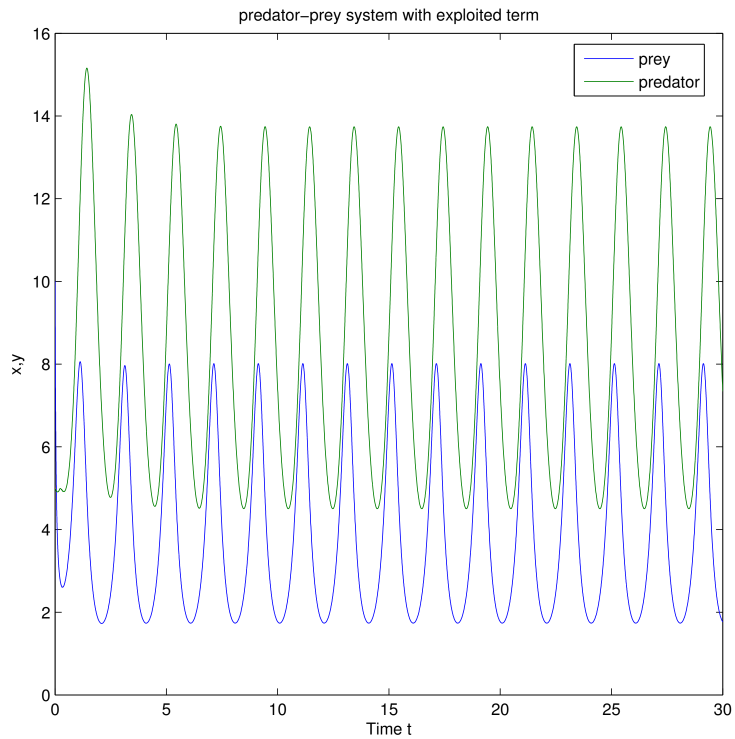
Remark 3.7. In system (3.19), when m(t)=m is constant, system (3.19) reduces to the system which is studied by Fan and Wang [4] and Theorem 3.5 reduces to the corresponding result in [4].

At last, When $h(t) \equiv 0$, systems (3.16)- (3.18) can be reduced to corresponding simpler systems and we have the following conclusion.

Theorem 3.8. *If $\bar{f} > \bar{d}$ and $\bar{a} > \overline{(c/m)}$, then systems* (3.16)-(3.18) *have at least one positive ω-periodic solutions, respectively.*

4. Numerical example

In this section, an example is given to illustrate our result.



Take $a(t) = 6+\sin(\pi t)$, $b(t) = \frac{3}{2}+\cos(\pi t)$, $c(t) = 2+\cos(\pi t)$, $m(t) = 1+\frac{1}{2}\cos(\pi t)$, $h(t) = \frac{1}{2} + \frac{1}{4}\sin(\pi t)$, $d(t) = 1 + \cos(\pi t)$, $f(t) = 3 + \sin(\pi t)$. Moreover, the delay kernel $k(t)$ is chosen as a delta function in the form of $k(t) = \delta(t)$ and $\tau(t) \equiv 0.2$, then system (1.4) becomes

$$\begin{aligned} x'(t) &= x(t)\Big[6 + \sin(\pi t) - (\frac{3}{2} + \cos(\pi t))x(t)\Big] \\ &\quad - \frac{(2+\cos(\pi t))x(t)y(t)}{(1+\frac{1}{2}\cos(\pi t))y(t) + x(t)} - (\frac{1}{2} + \frac{1}{4}\sin(\pi t)), \\ y'(t) &= y(t)\Big[-(1+\cos(\pi t)) + \frac{(3+\sin(\pi t))x(t-0.2)}{(1+\frac{1}{2}\cos(\pi t))y(t-0.2) + x(t-0.2)}\Big]. \end{aligned} \tag{4.1}$$

In this case, all the parameters are 2-periodic functions. By simple calculations, we have

$$\bar{f} = 3, \quad \bar{d} = 1, \quad (a - \frac{c}{m})^L = 3, \quad b^M = \frac{5}{2}, \quad h^M = \frac{3}{4}. \tag{4.2}$$

Therefore, (4.2) shows that conditions of Theorem 3.1 hold and so system (4.2) has at least two positive 2-periodic solutions. With initial values $x(0) = 10$, $y(0) = 5$ and $t \in [0, 30]$, the above figure shows that the existence of periodic solutions.

References

[1] R. Arditi, L. R. Ginzburg. *Coupling in predator-prey dynamics: Ratio-dependence*, J. Theoretical Biology, 139 (1989) 311-326.

[2] E. Beretta, Y. Kuang. *Global analysis on some delayed ratio-dependent predator-prey systems*, Nonlinear Anal, 32 (1998) 381-408.

[3] J. M. Cushing. *Integro-differential equations and delay models in population dynamics*, Lecture notes in Biomathematics, Vol. 20, Spring-Verlag, Berlin, 1977.

[4] M. Fan, K.Wang. *Periodicity in a delayed ratio-dependent predator-prey systems*, J. math. Anal. Appl, 262 (2001) 179-190.

[5] M. Fan, K. Wang. *Periodic solutions of a discrete time nonautonomous ratio-dependent predator-prey system*, Math. comput. model, 35 (2002) 951-961.

[6] M. Fan, Q. Wang, X. F. Zou. *Dynamics of a nonautonomous ratio-dependent predator-prey system*, Pro. Roy. Soc. Edinburgh Sect. A (2003) 97-118.

[7] H. I. Freedman, R. M. Mathsen. *Persistence in predator prey systems with ratio-dependent predator-influence*, Bull. Math. Biol, 55 (1993) 817-827.

[8] R. E. Gaines, J. L. Mawhin. *Coincidence degree and nonlinear differential equations*, Springer-Verlag, Berlin, (1977).

[9] S. B. Hsu, T. W. Huang. *Global stability for a class of predator-prey systems*, SIAM J. Appl. Math, 55, No. 3 (1995) 763-783.

[10] S. B. Hsu, T. W. Huang, Y. Kuang. *Global analysis of the Michaelis-Menten type ratio-dependent predator-prey system*, J. Math. Biol, 42 (2003) 489-506.

[11] S. B. Hsu, T. W. Huang, Y. Kuang. *A ratio-dependent food chain model and its applications to biological control*, J. Math. Biol, 181(2003)55-83.

[12] Y. K. Li. *Periodic solutions of a periodic delay predator-prey system*, Proc. Amer. Math. Soc, 127 (1999) 1331-1335.

[13] E. Sáez, E. González-Olivares. *Dynamics of a predator-pey model*, SIAM J. Appl. Math, 59, No. 5 (1999) 1867-1878.

[14] D. S. Tian, X. W. Zeng. *Existence of at least two periodic solutions of a ratio-dependent predator-prey model with exploited term*, Acta Math. Appl. Sin, English series, 21, No. 3 (2005) 489-494.

[15] Qian Wang, Meng Fan, Ke Wang. *Dynamics of a class of nonautonomous semi-ratio-dependent predator-prey systems with functional responses*, J. Math. Anal. Appl, 278, No. 2 (2003) 443-471.

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