# A FRICTIONAL CONTACT PROBLEM FOR AN ELECTRO-VISCOELASTIC BODY

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ABSTRACT. A mathematical model which describes the quasistatic frictional contact between a piezoelectric body and a deformable conductive foundation is studied. A nonlinear electro-viscoelastic constitutive law is used to model the piezoelectric material. Contact is described with the normal compliance condition, a version of Coulomb's law of dry friction, and a regularized electrical conductivity condition. A variational formulation of the model, in the form of a coupled system for the displacements and the electric potential, is derived. The existence of a unique weak solution of the model is established under a smallness assumption on the surface conductance. The proof is based on arguments of evolutionary variational inequalities and fixed points of operators.

## 1. INTRODUCTION

Considerable progress has been achieved recently in modeling, mathematical analysis and numerical simulations of various contact processes and, as a result, a general Mathematical Theory of Contact Mechanics (MTCM) is currently maturing. It is concerned with the mathematical structures which underlie general contact problems with different constitutive laws (i.e., different materials), varied geometries and settings, and different contact conditions, see for instance [5, 15, 18] and the references therein. The theory's aim is to provide a sound, clear and rigorous background for the constructions of models for contact between deformable bodies; proving existence, uniqueness and regularity results; assigning precise meaning to solutions; and the necessary setting for finite element approximations of the solutions.

There is a considerable interest in frictional or frictionless contact problems involving piezoelectric materials, see for instance [2, 9, 17] and the references therein. Indeed, many actuators and sensors in engineering controls are made of piezoelectric ceramics. However, there exists virtually no mathematical results about contact problems for such materials and there is a need to expand the MTCM to include the coupling between the mechanical and electrical material properties.

The piezoelectric effect is characterized by such a coupling between the mechanical and electrical properties of the materials. This coupling, leads to the appearance

of electric field in the presence of a mechanical stress, and conversely, mechanical stress is generated when electric potential is applied. The first effect is used in sensors, and the reverse effect is used in actuators.

On a nano-scale, the piezoelectric phenomenon arises from a nonuniform charge distribution within a crystal's unit cell. When such a crystal is deformed mechanically, the positive and negative charges are displaced by a different amount causing the appearance of electric polarization. So, while the overall crystal remains electrically neutral, an electric polarization is formed within the crystal. This electric polarization due to mechanical stress is called *piezoelectricity.* A deformable material which exhibits such a behavior is called a *piezoelectric material.* Piezoelectric materials for which the mechanical properties are elastic are also called *electro-elastic materials* and piezoelectric materials for which the mechanical properties are viscoelastic are also called *electro-viscoelastic materials.*

Only some materials exhibit sufficient piezoelectricity to be useful in applications. These include quartz, Rochelle salt, lead titanate zirconate ceramics, barium titanate, and polyvinylidene flouride (a polymer film), and are used extensively as switches and actuators in many engineering systems, in radioelectronics, electroacoustics and in measuring equipment. General models for electro-elastic materials can be found in [11, 12] and, more recently, in [1, 6, 13]. A static and a slip-dependent frictional contact problems for electro-elastic materials were studied in [2, 9] and in [16], respectively. A contact problem with normal compliance for electro-viscoelastic materials was investigated in [17]. In the last two references the foundation was assumed to be insulated. The variational formulations of the corresponding problems were derived and existence and uniqueness of weak solutions were obtained.

Here we continue this line of research and study a quasistatic frictionless contact problem for an electro-viscoelastic material, in the framework of the MTCM, when the foundation is conductive; our interest is to describe a physical process in which both contact, friction and piezoelectric effect are involved, and to show that the resulting model leads to a well-posed mathematical problem. Taking into account the conductivity of the foundation leads to new and nonstandard boundary conditions on the contact surface, which involve a coupling between the mechanical and the electrical unknowns, and represents the main novelty in this work.

The rest of the paper is structured as follows. In Section 2 we describe the model of the frictional contact process between an electro-viscoelastic body and a conductive deformable foundation. In Section 3 we introduce some notation, list the assumptions on the problem's data, and derive the variational formulation of the model. It consists of a variational inequality for the displacement field coupled with a nonlinear time-dependent variational equation for the electric potential. We state our main result, the existence of a unique weak solution to the model in Theorem 3.1. The proof of the theorem is provided in Section 4, where it is carried out in several steps and is based on arguments of evolutionary inequalities with monotone operators, and a fixed point theorem. The paper concludes in Section 5.

## 2. The model

We consider a body made of a piezoelectric material which occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$ and a unit outward normal $\boldsymbol{\nu}$. The body is acted upon by body forces of density $\mathbf{f}_0$ and has volume free electric

charges of density $q_0$. It is also constrained mechanically and electrically on the boundary. To describe these conditions, we assume a partition of $\Gamma$ into three open disjoint parts $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$, on the one hand, and a partition of $\Gamma_D \cup \Gamma_N$ into two open parts $\Gamma_a$ and $\Gamma_b$, on the other hand. We assume that $meas \, \Gamma_D > 0$ and $meas \, \Gamma_a > 0$; these conditions allow the use of coercivity arguments which guarantee the uniqueness of the solution for the model. The body is clamped on $\Gamma_D$ and, therefore, the displacement field $\mathbf{u} = (u_1, \ldots, u_d)$ vanishes there. Surface tractions of density $\mathbf{f}_N$ act on $\Gamma_N$. We also assume that the electrical potential vanishes on $\Gamma_a$ and a surface free electrical charge of density $q_b$ is prescribed on $\Gamma_b$. In the reference configuration the body may come in contact over $\Gamma_C$ with a conductive obstacle, which is also called the foundation. The contact is frictional and is modelled with the normal compliance condition and a version of Coulomb's law of dry friction. Also, there may be electrical charges on the part of the body which is in contact with the foundation and which vanish when contact is lost.

We are interested in the evolution of the deformation of the body and of the electric potential on the time interval $[0, T]$. The process is assumed to be isothermal, electrically static, i.e., all radiation effects are neglected, and mechanically quasistatic; i.e., the inertial terms in the momentum balance equations are neglected. We denote by $\mathbf{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$ the spatial and the time variable, respectively, and, to simplify the notation, we do not indicate in what follows the dependence of various functions on $\mathbf{x}$ and $t$. In this paper $i, j, k, l = 1, \ldots, d$, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding component of $\mathbf{x}$. A dot over a variable represents the time derivative.

We use the notation $\mathbb{S}^d$ for the space of second order symmetric tensors on $\mathbb{R}^d$ and " $\cdot$ " and $\| \cdot \|$ represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively, that is $\mathbf{u} \cdot \mathbf{v} = u_i v_i$, $\|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{1/2}$ for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, and $\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}$, $\|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2}$ for $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d$. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, by $u_\nu = \mathbf{u} \cdot \boldsymbol{\nu}$, $\mathbf{u}_\tau = \mathbf{u} - u_n \boldsymbol{\nu}$, $\sigma_\nu = \sigma_{ij} \nu_i \nu_j$, and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$.

The classical model for the process is as follows.

**Problem $\mathcal{P}$.** Find a displacement field $\mathbf{u} : \Omega \times [0, T] \rightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \rightarrow \mathbb{S}^d$, an electric potential $\varphi : \Omega \times [0, T] \rightarrow \mathbb{R}$ and an electric displacement field $\mathbf{D} : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ such that

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{E}^* \mathbf{E}(\varphi) \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$\mathbf{D} = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) + \beta \mathbf{E}(\varphi) \quad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$\text{Div } \boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \quad \text{in } \Omega \times (0, T), \tag{2.3}$$

$$\text{div } \mathbf{D} - q_0 = 0 \quad \text{in } \Omega \times (0, T), \tag{2.4}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \times (0, T), \tag{2.5}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{f}_N \quad \text{on } \Gamma_N \times (0, T), \tag{2.6}$$

$$-\sigma_\nu = p_\nu(u_\nu - g) \quad \text{on } \Gamma_C, \times(0, T), \tag{2.7}$$

$$\|\boldsymbol{\sigma}_\tau\| \leq p_\tau(u_\nu - g),$$

$$\dot{\mathbf{u}}_\tau \neq \mathbf{0} \Rightarrow \boldsymbol{\sigma}_\tau = -p_\tau(u_\nu - g)\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{on } \Gamma_C \times (0, T), \tag{2.8}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0, T), \tag{2.9}$$

$$\mathbf{D} \cdot \boldsymbol{\nu} = q_b \quad \text{on } \Gamma_b \times (0, T), \tag{2.10}$$

$$\mathbf{D} \cdot \boldsymbol{\nu} = \psi(u_\nu - g)\phi_L(\varphi - \varphi_0) \quad \text{on } \Gamma_C \times (0, T), \tag{2.11}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega. \tag{2.12}$$

We now describe problem (2.1)–(2.12) and provide explanation of the equations and the boundary conditions.

First, equations (2.1) and (2.2) represent the electro-viscoelastic constitutive law in which $\boldsymbol{\sigma} = (\sigma_{ij})$ is the stress tensor, $\boldsymbol{\varepsilon}(\mathbf{u})$ denotes the linearized strain tensor, $\mathcal{A}$ and $\mathcal{B}$ are the viscosity and elasticity operators, respectively, $\mathcal{E} = (e_{ijk})$ represents the third-order piezoelectric tensor, $\mathcal{E}^*$ is its transpose, $\boldsymbol{\beta} = (\beta_{ij})$ denotes the electric permittivity tensor, and $\mathbf{D} = (D_1, \ldots, D_d)$ is the electric displacement vector. Since we use the electrostatic approximation, the electric field satisfies $\mathbf{E}(\varphi) = -\nabla \varphi$, where $\varphi$ is the electric potential.

We recall that $\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$ and $\varepsilon_{ij}(\mathbf{u}) = (u_{i,j} + u_{j,i})/2$. The tensors $\mathcal{E}$ and $\mathcal{E}^*$ satisfy the equality

$$\mathcal{E}\boldsymbol{\sigma} \cdot \mathbf{v} = \boldsymbol{\sigma} \cdot \mathcal{E}^*\mathbf{v} \quad \forall \boldsymbol{\sigma} = (\sigma_{ij}) \in \mathbb{S}^d, \ \mathbf{v} \in \mathbb{R}^d,$$

and the components of the tensor $\mathcal{E}^*$ are given by $e^*_{ijk} = e_{kij}$.

A viscoelastic Kelvin-Voigt constitutive relation (see [5] for details) is given in (2.1), in which the dependence of the stress on the electric field is takes into account. Relation (2.2) describes a linear dependence of the electric displacement field $\mathbf{D}$ on the strain and electric fields; such a relation has been frequently employed in the literature (see, e.g., [1, 2, 13] and the references therein). In the linear case, the constitutive laws (2.1) and (2.2) read

$$\sigma_{ij} = a_{ijkl}\varepsilon_{k,l}(\dot{\mathbf{u}}) + b_{ijkl}\varepsilon_{kl}(\mathbf{u}) - e_{kij}\varphi_{,k},$$

$$D_i = e_{ijk}\varepsilon_{jk}(\mathbf{u}) + \beta_{ij}\varphi_{,j},$$

where $a_{ijkl}$, $b_{ijkl}$, $\beta_{ij}$ are the components of the tensors $\mathcal{A}$, $\mathcal{B}$ and $\boldsymbol{\beta}$, respectively, and $\varphi_{,j} = \partial\varphi/\partial x_j$.

Next, equations (2.3) and (2.4) are the steady equations for the stress and electric-displacement fields, respectively, in which "Div" and "div" denote the divergence operator for tensor and vector valued functions, i.e.,

$$\text{Div } \boldsymbol{\sigma} = (\sigma_{ij,j}), \quad \text{div } \mathbf{D} = (D_{i,i}).$$

We use these equations since the process is assumed to be mechanically quasistatic and electrically static.

Conditions (2.5) and (2.6) are the displacement and traction boundary conditions, whereas (2.9) and (2.10) represent the electric boundary conditions; the displacement field and the electrical potential vanish on $\Gamma_D$ and $\Gamma_a$, respectively, while the forces and free electric charges are prescribed on $\Gamma_N$ and $\Gamma_b$, respectively. Finally, the initial displacement $\mathbf{u}_0$ in (2.12) is given.

We turn to the boundary conditions (2.7), (2.8), (2.11) which describe the mechanical and electrical conditions on the potential contact surface $\Gamma_C$. The normal compliance function $p_\nu$, in (2.7), is described below, and $g$ represents the gap in the reference configuration between $\Gamma_C$ and the foundation, measured along the direction of $\boldsymbol{\nu}$. When positive, $u_\nu - g$ represents the interpenetration of the surface asperities into those of the foundation. This condition was first introduced in [10] and used in a large number of papers, see for instance [4, 7, 8, 14] and the references therein.

Conditions (2.8) is the associated friction law where $p_\tau$ is a given function. According to (2.8) the tangential shear cannot exceed the maximum frictional resistance $p_\tau(u_\nu - g)$, the so-called friction bound. Moreover, when sliding commences, the tangential shear reaches the friction bound and opposes the motion. Frictional contact conditions of the form (2.7), (2.8) have been used in various papers, see, e.g., [5, 14, 15] and the references therein.

Next, (2.11) is the electrical contact condition on $\Gamma_C$ which is the main novelty of this work. It represents a regularized condition which may be obtained as follows.

First, unlike previous papers on piezoelectric contact, we assume that the foundation is electrically conductive and its potential is maintained at $\varphi_0$. When there is no contact at a point on the surface (i.e., $u_\nu < g$), the gap is assumed to be an insulator (say, it is filled with air), there are no free electrical charges on the surface and the normal component of the electric displacement field vanishes. Thus,

$$u_\nu < g \;\Rightarrow\; \mathbf{D} \cdot \boldsymbol{\nu} = 0. \tag{2.13}$$

During the process of contact (i.e., when $u_\nu \geq g$) the normal component of the electric displacement field or the free charge is assumed to be proportional to the difference between the potential of the foundation and the body's surface potential, with $k$ as the proportionality factor. Thus,

$$u_\nu \geq g \;\Rightarrow\; \mathbf{D} \cdot \boldsymbol{\nu} = k\,(\varphi - \varphi_0). \tag{2.14}$$

We combine (2.13), (2.14) to obtain

$$\mathbf{D} \cdot \boldsymbol{\nu} = k\,\chi_{[0,\infty)}(u_\nu - g)\,(\varphi - \varphi_0), \tag{2.15}$$

where $\chi_{[0,\infty)}$ is the characteristic function of the interval $[0,\infty)$, that is

$$\chi_{[0,\infty)}(r) = \begin{cases} 0 & \text{if } r < 0, \\ 1 & \text{if } r \geq 0. \end{cases}$$

Condition (2.15) describes perfect electrical contact and is somewhat similar to the well-known Signorini contact condition. Both conditions may be over-idealizations in many applications.

To make it more realistic, we regularize condition (2.15) and write it as (2.11) in which $k\,\chi_{[0,\infty)}(u_\nu - g)$ is replaced with $\psi$ which is a regular function which will be described below, and $\phi_L$ is the truncation function

$$\phi_L(s) = \begin{cases} -L & \text{if } s < -L, \\ s & \text{if } -L \leq s \leq L, \\ L & \text{if } s > L, \end{cases}$$

where $L$ is a large positive constant. We note that this truncation does not pose any practical limitations on the applicability of the model, since $L$ may be arbitrarily large, higher than any possible peak voltage in the system, and therefore in applications $\phi_L(\varphi - \varphi_0) = \varphi - \varphi_0$.

The reasons for the regularization (2.11) of (2.15) are mathematical. First, we need to avoid the discontinuity in the free electric charge when contact is established and, therefore, we regularize the function $k\,\chi_{[0,\infty)}$ in (2.15) with a Lipschitz

continuous function $\psi$. A possible choice is

$$\psi(r) = \begin{cases} 0 & \text{if } r < 0, \\ k\delta r & \text{if } 0 \le r \le 1/\delta, \\ k & \text{if } r > \delta, \end{cases} \tag{2.16}$$

where $\delta > 0$ is a small parameter. This choice means that during the process of contact the electrical conductivity increases as the contact among the surface asperities improves, and stabilizes when the penetration $u_\nu - g$ reaches the value $\delta$. Secondly, we need the term $\phi_L(\varphi - \varphi_0)$ to control the boundednes of $\varphi - \varphi_0$.

Note that when $\psi \equiv 0$ in (2.11) then

$$\mathbf{D} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma_C \times (0, T), \tag{2.17}$$

which decouples the electrical and mechanical problems on the contact surface. Condition (2.17) models the case when the obstacle is a perfect insulator and was used in [2, 9, 16, 17]. Condition (2.11), instead of (2.17), introduces strong coupling between the mechanical and the electric boundary conditions and leads to a new and nonstandard mathematical model.

Because of the friction condition (2.8), which is non-smooth, we do not expect the problem to have, in general, any classical solutions. For this reason, we derive in the next section a variational formulation of the problem and investigate its solvability. Moreover, variational formulations are also starting points for the construction of finite element algorithms for this type of problems.

## 3. Variational formulation and the main result

We use standard notation for the $L^p$ and the Sobolev spaces associated with $\Omega$ and $\Gamma$ and, for a function $\zeta \in H^1(\Omega)$ we still write $\zeta$ to denote its trace on $\Gamma$. We recall that the summation convention applies to a repeated index.

For the electric displacement field we use two Hilbert spaces

$$\mathcal{W} = L^2(\Omega)^d, \quad \mathcal{W}_1 = \{\, \mathbf{D} \in \mathcal{W} : \operatorname{div} \mathbf{D} \in L^2(\Omega)\},$$

endowed with the inner products

$$(\mathbf{D}, \mathbf{E})_{\mathcal{W}} = \int_\Omega D_i E_i \, dx, \quad (\mathbf{D}, \mathbf{E})_{\mathcal{W}_1} = (\mathbf{D}, \mathbf{E})_{\mathcal{W}} + (\operatorname{div} \mathbf{D}, \operatorname{div} \mathbf{E})_{L^2(\Omega)},$$

and the associated norms $\|\cdot\|_{\mathcal{W}}$ and $\|\cdot\|_{\mathcal{W}_1}$, respectively. The electric potential field is to be found in

$$W = \{\zeta \in H^1(\Omega) : \zeta = 0 \text{ on } \Gamma_a\}.$$

Since $\operatorname{meas}\Gamma_a > 0$, the Friedrichs-Poincaré inequality holds, thus,

$$\|\nabla\zeta\|_{\mathcal{W}} \ge c_F \|\zeta\|_{H^1(\Omega)} \quad \forall\, \zeta \in W, \tag{3.1}$$

where $c_F > 0$ is a constant which depends only on $\Omega$ and $\Gamma_a$. On $W$, we use the inner product

$$(\varphi, \zeta)_W = (\nabla\varphi, \nabla\zeta)_{\mathcal{W}},$$

and let $\|\cdot\|_W$ be the associated norm. It follows from (3.1) that $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_W$ are equivalent norms on $W$ and therefore $(W, \|\cdot\|_W)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $c_0$, depending only on $\Omega$, $\Gamma_a$ and $\Gamma_C$, such that

$$\|\zeta\|_{L^2(\Gamma_C)} \le c_0 \|\zeta\|_W \quad \forall\, \zeta \in W. \tag{3.2}$$

We recall that when $\mathbf{D} \in \mathcal{W}_1$ is a sufficiently regular function, the Green type formula holds:

$$(\mathbf{D}, \nabla\zeta)_{L^2(\Omega)^d} + (\operatorname{div}\mathbf{D}, \zeta)_{\mathcal{W}} = \int_{\Gamma} \mathbf{D} \cdot \boldsymbol{\nu}\, \zeta\, da \quad \forall \zeta \in H^1(\Omega). \tag{3.3}$$

For the stress and strain variables, we use the real Hilbert spaces

$$Q = \{\boldsymbol{\tau} = (\tau_{ij}) : \; \tau_{ij} = \tau_{ji} \in L^2(\Omega)\} = L^2(\Omega)^{d \times d}_{sym},$$
$$Q_1 = \{\boldsymbol{\sigma} = (\sigma_{ij}) \in Q : \; \operatorname{div}\boldsymbol{\sigma} = (\sigma_{ij,j}) \in \mathcal{W}\},$$

endowed with the respective inner products

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_{\Omega} \sigma_{ij}\tau_{ij}dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{Q_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q + (\operatorname{div}\boldsymbol{\sigma}, \operatorname{div}\boldsymbol{\tau})_{\mathcal{W}},$$

and the associated norms $\|\cdot\|_Q$ and $\|\cdot\|_{Q_1}$. For the displacement variable we use the real Hilbert space

$$H_1 = \{\mathbf{u} = (u_i) \in \mathcal{W} : \boldsymbol{\varepsilon}(\mathbf{u}) \in Q\},$$

endowed with the inner product

$$(\mathbf{u}, \mathbf{v})_{H_1} = (\mathbf{u}, \mathbf{v})_{\mathcal{W}} + (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q,$$

and the norm $\|\cdot\|_{H_1}$.

When $\boldsymbol{\sigma}$ is a regular function, the following Green's type formula holds,

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + (\operatorname{Div}\boldsymbol{\sigma}, \mathbf{v})_{L^2(\Omega)^d} = \int_{\Gamma} \boldsymbol{\sigma}\nu \cdot \mathbf{v}\, da \quad \forall \mathbf{v} \in H_1. \tag{3.4}$$

Next, we define the space

$$V = \{\, \mathbf{v} \in H_1 : \mathbf{v} = \mathbf{0} \; \text{ on } \; \Gamma_D\}.$$

Since meas $\Gamma_D > 0$, Korn's inequality (e.g., [3, pp. 16–17]) holds and

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|_Q \geq c_K \|\mathbf{v}\|_{H_1} \quad \forall \mathbf{v} \in V, \tag{3.5}$$

where $c_K > 0$ is a constant which depends only on $\Omega$ and $\Gamma_D$. On the space $V$ we use the inner product

$$(\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q,$$

and let $\|\cdot\|_V$ be the associated norm. It follows from (3.5) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on $V$ and, therefore, the space $(V, (\cdot, \cdot)_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $\widetilde{c}_0$, depending only on $\Omega$, $\Gamma_D$ and $\Gamma_C$, such that

$$\|\mathbf{v}\|_{L^2(\Gamma_C)^d} \leq \widetilde{c}_0 \|\mathbf{v}\|_V \quad \forall \mathbf{v} \in V. \tag{3.6}$$

Finally, for a real Banach space $(X, \|\cdot\|_X)$ we use the usual notation for the spaces $L^p(0, T; X)$ and $W^{k,p}(0, T; X)$ where $1 \leq p \leq \infty, \; k = 1, 2, \ldots$; we also denote by $C([0, T]; X)$ and $C^1([0, T]; X)$ the spaces of continuous and continuously differentiable functions on $[0, T]$ with values in $X$, with the respective norms

$$\|x\|_{C([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X,$$
$$\|x\|_{C^1([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X + \max_{t \in [0,T]} \|\dot{x}(t)\|_X.$$

Recall that the dot represents the time derivative.

We now list the assumptions on the problem's data. The *viscosity operator* $\mathcal{A}$ and the *elasticity operator* $\mathcal{B}$ are assumed to satisfy the conditions:

$$
\begin{cases}
\text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\
\qquad \|\mathcal{A}(\mathbf{x}, \boldsymbol{\xi}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\xi}_2)\| \le L_{\mathcal{A}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\| \\
\qquad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\
\qquad (\mathcal{A}(\mathbf{x}, \boldsymbol{\xi}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge m_{\mathcal{A}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2 \\
\qquad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(d) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega, \\
\qquad \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\
\text{(e) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \mathbf{0}) \text{ belongs to } Q.
\end{cases}
\tag{3.7}
$$

$$
\begin{cases}
\text{(a) } \mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{B}} > 0 \text{ such that} \\
\qquad \|\mathcal{B}(\mathbf{x}, \boldsymbol{\xi}_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{\xi}_2)\| \le L_{\mathcal{B}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\| \\
\qquad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(c) The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \boldsymbol{\xi}) \text{ is measurable on } \Omega, \\
\qquad \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\
\text{(d) The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \mathbf{0}) \text{ belongs to } Q.
\end{cases}
\tag{3.8}
$$

Examples of nonlinear operators $\mathcal{A}$ and $\mathcal{B}$ which satisfy conditions (3.7) and (3.8) can be fond in [15, 18] and the many references therein.

The *piezoelectric tensor* $\mathcal{E}$ and the *electric permittivity tensor* $\boldsymbol{\beta}$ satisfy

$$
\begin{cases}
\text{(a) } \mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d. \\
\text{(b) } \mathcal{E}(\mathbf{x}, \boldsymbol{\tau}) = (e_{ijk}(\mathbf{x})\tau_{jk}) \quad \forall \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(c) } e_{ijk} = e_{ikj} \in L^\infty(\Omega).
\end{cases}
\tag{3.9}
$$

$$
\begin{cases}
\text{(a) } \boldsymbol{\beta} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d. \\
\text{(b) } \boldsymbol{\beta}(\mathbf{x}, \mathbf{E}) = (\beta_{ij}(\mathbf{x})E_j) \quad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(c) } \beta_{ij} = \beta_{ji} \in L^\infty(\Omega). \\
\text{(d) There exists } m_\beta > 0 \text{ such that } \beta_{ij}(\mathbf{x})E_i E_j \ge m_\beta \|\mathbf{E}\|^2 \\
\qquad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Omega.
\end{cases}
\tag{3.10}
$$

The *normal compliance functions* $p_r$ $(r = \nu, \tau)$ satisfy

$$
\begin{cases}
\text{(a) } p_r : \Gamma_C \times \mathbb{R} \to \mathbb{R}_+. \\
\text{(b) } \exists L_r > 0 \text{ such that } |p_r(\mathbf{x}, u_1) - p_r(\mathbf{x}, u_2)| \le L_r|u_1 - u_2| \\
\qquad \forall u_1, u_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_C. \\
\text{(c) } \mathbf{x} \mapsto p_r(\mathbf{x}, u) \text{ is measurable on } \Gamma_C, \text{ for all } u \in \mathbb{R}. \\
\text{(d) } \mathbf{x} \mapsto p_r(\mathbf{x}, u) = 0, \text{ for all } u \le 0.
\end{cases}
\tag{3.11}
$$

An example of a normal compliance function $p_\nu$ which satisfies conditions (3.11) is $p_\nu(u) = c_\nu u_+$ where $c_\nu \in L^\infty(\Gamma_C)$ is a positive surface stiffness coefficient, and $u_+ = \max\{0, u\}$. The choices $p_\tau = \mu p_\nu$ and $p_\tau = \mu p_\nu(1 - \delta p_\nu)_+$ in (2.8), where $\mu \in L^\infty(\Gamma_C)$ and $\delta \in L^\infty(\Gamma_C)$ are positive functions, lead to the usual or to a modified Coulomb's law of dry friction, respectively, see [5, 14, 19] for details. Here, $\mu$ represents the coefficient of friction and $\delta$ is a small positive material constant related to the wear and hardness of the surface. We note that if $p_\nu$ satisfies condition (3.11) then $p_\tau$ satisfies it too, in both examples. Therefore, we conclude that the results below are valid for the corresponding piezoelectric frictional contact models.

The *surface electrical conductivity* function $\psi$ satisfies:

$$
\begin{cases}
\text{(a) } \psi : \Gamma_C \times \mathbb{R} \to \mathbb{R}_+. \\
\text{(b) } \exists\, L_\psi > 0 \text{ such that } |\psi(\mathbf{x}, u_1) - \psi(\mathbf{x}, u_2)| \le L_\psi |u_1 - u_2| \\
\qquad \forall\, u_1, u_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_C. \\
\text{(c) } \exists\, M_\psi > 0 \text{ such that } |\psi(\mathbf{x}, u)| \le M_\psi \ \forall\, u \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_C. \\
\text{(e) } \mathbf{x} \mapsto \psi(\mathbf{x}, u) \text{ is measurable on } \Gamma_C, \text{ for all } u \in \mathbb{R}. \\
\text{(e) } \mathbf{x} \mapsto \psi(\mathbf{x}, u) = 0, \text{ for all } u \le 0.
\end{cases}
\tag{3.12}
$$

An example of a conductivity function which satisfies condition (3.12) is given by (2.16) in which case $M_\psi = k$. Another example is provided by $\psi \equiv 0$, which models the contact with an insulated foundation, as noted in Section 2. We conclude that our results below are valid for the corresponding piezoelectric contact models.

The forces, tractions, volume and surface free charge densities satisfy

$$
\mathbf{f}_0 \in W^{1,p}(0, T; L^2(\Omega)^d), \tag{3.13}
$$

$$
\mathbf{f}_N \in W^{1,p}(0, T; L^2(\Gamma_N)^d), \tag{3.14}
$$

$$
q_0 \in W^{1,p}(0, T; L^2(\Omega)), \tag{3.15}
$$

$$
q_b \in W^{1,p}(0, T; L^2(\Gamma_b)). \tag{3.16}
$$

Here, $1 \le p \le \infty$. Finally, we assume that the gap function, the given potential and the initial displacement satisfy

$$
g \in L^2(\Gamma_C), \quad g \ge 0 \quad \text{a.e. on } \Gamma_C, \tag{3.17}
$$

$$
\varphi_0 \in L^2(\Gamma_C), \tag{3.18}
$$

$$
\mathbf{u}_0 \in V. \tag{3.19}
$$

Next, we define the four mappings $j : V \times V \to \mathbb{R}$, $h : V \times W \to W$, $\mathbf{f} : [0, T] \to V$ and $q : [0, T] \to W$, respectively, by

$$
j(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_C} p_\nu(u_\nu - g) v_\nu \, da + \int_{\Gamma_C} p_\tau(u_\nu - g) \|\mathbf{v}_\tau\| \, da, \tag{3.20}
$$

$$
(h(\mathbf{u}, \varphi), \zeta)_W = \int_{\Gamma_C} \psi(u_\nu - g) \phi_L(\varphi - \varphi_0) \zeta \, da, \tag{3.21}
$$

$$
(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t) \cdot \mathbf{v} \, dx + \int_{\Gamma_N} \mathbf{f}_N(t) \cdot \mathbf{v} \, da, \tag{3.22}
$$

$$
(q(t), \zeta)_W = -\int_\Omega q_0(t) \zeta \, dx - \int_{\Gamma_b} q_b(t) \zeta \, da, \tag{3.23}
$$

for all $\mathbf{u}, \mathbf{v} \in V$, $\varphi, \zeta \in W$ and $t \in [0, T]$. We note that the definitions of $h$, $\mathbf{f}$ and $q$ are based on the Riesz representation theorem, moreover, it follows from assumptions (3.11)–(3.16) that the integrals in (3.20)–(3.23) are well-defined.

Using Green's formulas (3.3) and (3.4), it is easy to see that if $(\mathbf{u}, \boldsymbol{\sigma}, \varphi, \mathbf{D})$ are sufficiently regular functions which satisfy (2.3)–(2.11) then

$$
(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\mathbf{u}(t), \mathbf{v}) - j(\dot{\mathbf{u}}(t), \mathbf{v}) \ge (\mathbf{f}(t), \dot{\mathbf{u}}(t) - \mathbf{v})_V, \tag{3.24}
$$

$$
(\mathbf{D}(t), \nabla\zeta)_{\mathcal{W}} + (q(t), \zeta)_W = (h(\mathbf{u}(t), \varphi(t)), \zeta)_W, \tag{3.25}
$$

for all $\mathbf{v} \in V$, $\zeta \in W$ and $t \in [0, T]$. We substitute (2.1) in (3.24), (2.2) in (3.25), note that $\mathbf{E}(\varphi) = -\nabla\varphi$, use the initial condition (2.12) and derive a variational formulation of problem $\mathcal{P}$. It is in the terms of displacement and electric potential fields.

**Problem $\mathcal{P}_V$.** Find a displacement field $\mathbf{u} : [0, T] \to V$ and an electric potential $\varphi : [0, T] \to W$ such that

$$(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}(t)))_Q + (\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}(t)))_Q$$
$$+ (\mathcal{E}^*\nabla\varphi(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}(t)))_Q + j(\mathbf{u}(t), \mathbf{v}) - j(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \qquad (3.26)$$
$$\geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V,$$

for all $\mathbf{v} \in V$ and $t \in [0, T]$,

$$(\boldsymbol{\beta}\nabla\varphi(t), \nabla\zeta)_\mathcal{W} - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \nabla\zeta)_\mathcal{W} + (h(\mathbf{u}(t), \varphi(t)), \zeta)_W$$
$$= (q(t), \zeta)_W, \qquad (3.27)$$

for all $\zeta \in W$ and $t \in [0, T]$, and

$$\mathbf{u}(0) = \mathbf{u}_0. \qquad (3.28)$$

To study problem $\mathcal{P}_V$ we make the following smallness assumption

$$M_\psi < \frac{m_\beta}{c_0^2}, \qquad (3.29)$$

where $M_\psi$, $c_0$ and $m_\beta$ are given in (3.12), (3.2) and (3.10), respectively. We note that only the trace constant, the coercivity constant of $\boldsymbol{\beta}$ and the bound of $\psi$ are involved in (3.29); therefore, this smallness assumption involves only the geometry and the electrical part, and does not depend on the mechanical data of the problem. Moreover, it is satisfied when the obstacle is insulated, since then $\psi \equiv 0$ and so $M_\psi = 0$.

Removing this assumption remains a task for future research, since it is made for mathematical reasons, and does not seem to relate to any inherent physical constraints of the problem.

Our main existence and uniqueness result that we state now and prove in the next section is the following.

**Theorem 3.1.** *Assume that* (3.7)–(3.19) *and* (3.29) *hold. Then there exists a unique solution of Problem $\mathcal{P}_V$. Moreover, the solution satisfies*

$$\mathbf{u} \in W^{2,p}(0, T; V), \quad \varphi \in W^{1,p}(0, T; W). \qquad (3.30)$$

A quadruple of functions $(\mathbf{u}, \boldsymbol{\sigma}, \varphi, \mathbf{D})$ which satisfies (2.1), (2.2), (3.26)–(3.28) is called a *weak solution* of the piezoelectric contact problem $\mathcal{P}$. It follows from Theorem 3.1 that, under the assumptions (3.7)–(3.19), (3.29), there exists a unique weak solution of Problem $\mathcal{P}$.

To describe precisely the regularity of the weak solution, we note that the constitutive relations (2.1) and (2.2), the assumptions (3.7)–(3.10) and (3.30) show that $\boldsymbol{\sigma} \in W^{1,p}(0, T; Q)$ and $\mathbf{D} \in W^{1,p}(0, T; \mathcal{W})$. Using (2.1), (2.2), (3.26) and (3.27) implies that (3.24) and (3.25) hold for all $\mathbf{v} \in V$, $\zeta \in W$ and $t \in [0, T]$. We choose as a test function $\mathbf{v} = \dot{\mathbf{u}}(t) \pm \mathbf{z}$ where $\mathbf{z} \in C_0^\infty(\Omega)^d$ in (3.24) and $\zeta \in C_0^\infty(\Omega)$ in (3.25) and use the notation (3.20)–(3.23) to obtain

$$\mathrm{Div}\,\boldsymbol{\sigma}(t) + \mathbf{f}_0(t) = \mathbf{0}, \quad \mathrm{div}\,\mathbf{D}(t) + q_0(t) = 0,$$

for all $t \in [0, T]$. It follows now from (3.13) and (3.15) that $\mathrm{Div}\,\boldsymbol{\sigma} \in W^{1,p}(0, T; \mathcal{W})$ and $\mathrm{div}\,\mathbf{D} \in W^{1,p}(0, T; L^2(\Omega))$ and thus

$$\boldsymbol{\sigma} \in W^{1,p}(0, T; Q_1), \quad \mathbf{D} \in W^{1,p}(0, T; \mathcal{W}_1). \qquad (3.31)$$

We conclude that the weak solution $(\mathbf{u}, \boldsymbol{\sigma}, \varphi, \mathbf{D})$ of the piezoelectric contact problem $\mathcal{P}$ has the regularity implied in (3.30) and (3.31).

## 4. Proof of Theorem 3.1

The proof of Theorem 3.1 is carried out in several steps and is based on the following abstract result for evolutionary variational inequalities.

Let $X$ be a real Hilbert space with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$, and consider the problem of finding $u : [0,T] \to X$ such that

$$(A\dot{u}(t), v - \dot{u}(t))_X + (Bu(t), v - \dot{u}(t))_X + j(u(t), v) - j(u(t), \dot{u}(t))$$
$$\geq (f(t), v - \dot{u}(t))_X \quad \forall\, v \in X,\ t \in [0,T], \tag{4.1}$$

$$u(0) = u_0. \tag{4.2}$$

To study problem (4.1) and (4.2) we need the following assumptions: The operator $A : X \to X$ is strongly monotone and Lipschitz continuous, i.e.,

$$\begin{cases} \text{(a)} \ \text{There exists } m_A > 0 \text{ such that} \\ \quad (Au_1 - Au_2, u_1 - u_2)_X \geq m_A \|u_1 - u_2\|_X^2 \quad \forall\, u_1, u_2 \in X. \\ \text{(b)} \ \text{There exists } L_A > 0 \text{ such that} \\ \quad \|Au_1 - Au_2\|_X \leq L_A \|u_1 - u_2\|_X \quad \forall\, u_1, u_2 \in X. \end{cases} \tag{4.3}$$

The nonlinear operator $B : X \to X$ is Lipschitz continuous, i.e., there exists $L_B > 0$ such that

$$\|Bu_1 - Bu_2\|_X \leq L_B \|u_1 - u_2\|_X \quad \forall\, u_1, u_2 \in X. \tag{4.4}$$

The functional $j : X \times X \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a)} \ j(u, \cdot) \text{ is convex and l.s.c. on } X \text{ for all } u \in X. \\ \text{(b)} \ \text{There exists } m > 0 \text{ such that} \\ \quad j(u_1, v_2) - j(u_1, v_1) + j(u_2, v_1) - j(u_2, v_2) \\ \quad \leq m \, \|u_1 - u_2\|_X \, \|v_1 - v_2\|_X \quad \forall\, u_1, u_2, v_1, v_2 \in X. \end{cases} \tag{4.5}$$

Finally, we assume that

$$f \in C([0,T]; X), \tag{4.6}$$

and

$$u_0 \in X. \tag{4.7}$$

The following existence, uniqueness and regularity result was proved in [4] and may be found in [5, p. 230–234].

**Theorem 4.1.** *Let* (4.3)–(4.7) *hold. Then:*

(1) *There exists a unique solution $u \in C^1([0,T]; X)$ of problem* (4.1) *and* (4.2).
(2) *If $u_1$ and $u_2$ are two solutions of* (4.1) *and* (4.2) *corresponding to the data $f_1, f_2 \in C([0,T]; X)$, then there exists $c > 0$ such that*

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|_X \leq c\left(\|f_1(t) - f_2(t)\|_X + \|u_1(t) - u_2(t)\|_X\right) \quad \forall\, t \in [0,T]. \tag{4.8}$$

(3) *If, moreover, $f \in W^{1,p}(0,T; X)$, for some $p \in [1,\infty]$, then the solution satisfies $u \in W^{2,p}(0,T; X)$.*

We turn now to the proof of Theorem 3.1. To that end we assume in what follows that (3.7)–(3.19) hold and, everywhere below, we denote by $c$ various positive constants which are independent of time and whose value may change from line to line.

Let $\boldsymbol{\eta} \in C([0,T], Q)$ be given, and in the first step consider the following intermediate mechanical problem in which $\boldsymbol{\eta} = \mathcal{E}^* \nabla \varphi$ is known.

**Problem $\mathcal{P}_\eta^1$.** Find a displacement field $\mathbf{u}_\eta : [0,T] \to V$ such that

$$(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)))_Q + (\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)))_Q$$
$$+ (\boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)))_Q + j(\mathbf{u}_\eta(t), \mathbf{v}) - j(\mathbf{u}_\eta(t), \dot{\mathbf{u}}_\eta(t)) \qquad (4.9)$$
$$\geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_V \quad \forall \mathbf{v} \in V, \ t \in [0,T],$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0. \qquad (4.10)$$

We have the following result for $\mathcal{P}_\eta^1$.

**Lemma 4.2.**      (1) *There exists a unique solution $\mathbf{u}_\eta \in C^1([0,T];V)$ to the problem (4.9) and (4.10).*
   (2) *If $\mathbf{u}_1$ and $\mathbf{u}_2$ are two solutions of (4.9) and (4.10) corresponding to the data $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in C([0,T];Q)$, then there exists $c > 0$ such that*

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V \leq c\left(\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_Q + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V\right) \quad \forall t \in [0,T]. \quad (4.11)$$

   (3) *If, moreover, $\boldsymbol{\eta} \in W^{1,p}(0,T;Q)$ for some $p \in [1,\infty]$, then the solution satisfies $\mathbf{u}_\eta \in W^{2,p}(0,T;V)$.*

*Proof.* We apply Theorem 4.1 where $X = V$, with the inner product $(\cdot,\cdot)_V$ and the associated norm $\|\cdot\|_V$. We use the Riesz representation theorem to define the operators $A : V \to V$, $B : V \to V$ and the function $\mathbf{f}_\eta : [0,T] \to V$ by

$$(A\mathbf{u}, \mathbf{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q, \qquad (4.12)$$
$$(B\mathbf{u}, \mathbf{v})_V = (\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q, \qquad (4.13)$$
$$(\mathbf{f}_\eta(t), \mathbf{v})_V = (\mathbf{f}(t), \mathbf{v})_V - (\boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(\mathbf{v}))_Q, \qquad (4.14)$$

for all $\mathbf{u}, \mathbf{v} \in V$ and $t \in [0,T]$. Assumptions (3.7) and (3.8) imply that the operators $A$ and $B$ satisfy conditions (4.3) and (4.4), respectively.

It follows from (3.6) that the functional $j$, (3.20), satisfies condition (4.5)(a). We use again (3.11) and (3.6) to find

$$j(\mathbf{u}_1, \mathbf{v}_2) - j(\mathbf{u}_2, \mathbf{v}_1) + j(\mathbf{u}_2, \mathbf{v}_1) - j(\mathbf{u}_2, \mathbf{v}_2)$$
$$\leq \tilde{c}_0^2 (L_\nu + L_\tau)\|\mathbf{u}_1 - \mathbf{u}_2\|_V \|\mathbf{v}_1 - \mathbf{v}_2\|_V,$$

for all $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in V$, which shows that the functional $j$ satisfies condition (4.5)(b) on $X = V$. Moreover, using (3.13) and (3.14) it is easy to see that the function $\mathbf{f}$ defined by (3.22) satisfies $\mathbf{f} \in W^{1,p}(0,T;V)$ and, keeping in mind that $\boldsymbol{\eta} \in C([0,T];Q)$, we deduce from (4.14) that $\mathbf{f}_\eta \in C([0,T];V)$, i.e., $\mathbf{f}_\eta$ satisfies (4.6). Finally, we note that (3.19) shows that condition (4.7) is satisfied, too, and (4.14) shows that if $\boldsymbol{\eta} \in W^{1,p}(0,T;Q)$ then $\mathbf{f}_\eta \in W^{1,p}(0,T;V)$. Using now (4.12)–(4.14) we find that Lemma 4.2 is a direct consequence of Theorem 4.1. $\square$

In the next step we use the solution $\mathbf{u}_\eta \in C^1([0,T],V)$, obtained in Lemma 4.2, to construct the following variational problem for the electrical potential.

**Problem $\mathcal{P}_\eta^2$.** Find an electrical potential $\varphi_\eta : [0,T] \to W$ such that

$$(\boldsymbol{\beta}\nabla\varphi_\eta(t), \nabla\zeta)_W - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \nabla\zeta)_W + (h(\mathbf{u}_\eta(t), \varphi_\eta(t)), \zeta)_W$$
$$= (q(t), \zeta)_W, \qquad (4.15)$$

for all $\zeta \in W$, $t \in [0,T]$.

The well-posedness of problem $\mathcal{P}_\eta^2$ follows.

**Lemma 4.3.** *There exists a unique solution $\varphi_\eta \in W^{1,p}(0,T;W)$ which satisfies* (4.15).

*Moreover, if $\varphi_{\eta_1}$ and $\varphi_{\eta_2}$ are the solutions of* (4.15) *corresponding to $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in C([0,T];Q)$ then, there exists $c > 0$, such that*

$$\|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)\|_W \leq c \|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|_V \quad \forall\, t \in [0,T]. \qquad (4.16)$$

*Proof.* Let $t \in [0,T]$. We use the Riesz representation theorem to define the operator $A_\eta(t) : W \to W$ by

$$(A_\eta(t)\varphi, \zeta)_W = (\boldsymbol{\beta}\nabla\varphi, \nabla\zeta)_W - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \nabla\zeta)_W + (h(\mathbf{u}_\eta(t), \varphi), \zeta)_W, \qquad (4.17)$$

for all $\varphi, \zeta \in W$. Let $\varphi_1, \varphi_2 \in W$, then assumptions (3.10) and (3.21) imply

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \varphi_1 - \varphi_2)_W$$

$$\geq m_\beta \|\varphi_1 - \varphi_2\|_W^2 + \int_{\Gamma_C} \psi(u_{\eta\nu}(t) - g)\big(\phi_L(\varphi_1 - \varphi_0) - \phi_L(\varphi_2 - \varphi_0)\big)(\varphi_1 - \varphi_2)\, da$$

and, by (3.12)(a) combined with the monotonicity of the function $\phi_L$, we obtain

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \varphi_1 - \varphi_2)_W \geq m_\beta \|\varphi_1 - \varphi_2\|_W^2. \qquad (4.18)$$

On the other hand, using again (3.9), (3.10), (3.12) and (3.21) we have

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \zeta)_W$$

$$\leq c_\beta \|\varphi_1 - \varphi_2\|_W \|\zeta\|_W + \int_{\Gamma_C} M_\psi |\varphi_1 - \varphi_2|\,|\zeta|\, da \quad \forall\, \zeta \in W, \qquad (4.19)$$

where $c_\beta$ is a positive constant which depends on $\boldsymbol{\beta}$. It follows from (4.19) and (3.2) that

$$(A_\eta(t)\varphi_1 - A_\eta(t), \zeta)_W \leq (c_\beta + M_\psi c_0^2)\|\varphi_1 - \varphi_2\|_W \|\zeta\|_W,$$

thus,

$$\|A_\eta(t)\varphi_1 - A_\eta(t)\|_W \leq (c_\beta + M_\psi c_0^2)\|\varphi_1 - \varphi_2\|_W. \qquad (4.20)$$

Inequalities (4.18) and (4.20) show that the operator $A_\eta(t)$ is a strongly monotone Lipschitz continuous operator on $W$ and, therefore, there exists a unique element $\varphi_\eta(t) \in W$ such that

$$A_\eta(t)\varphi_\eta(t) = q(t). \qquad (4.21)$$

We combine now (4.17) and (4.21) and find that $\varphi_\eta(t) \in W$ is the unique solution of the nonlinear variational equation (4.15).

We show next that $\varphi_\eta \in W^{1,p}(0,T;W)$. To this end, let $t_1, t_2 \in [0,T]$ and, for the sake of simplicity, we write $\varphi_\eta(t_i) = \varphi_i$, $u_{\eta\nu}(t_i) = u_i$, $q_b(t_i) = q_i$, for $i = 1, 2$. Using (4.15), (3.9), (3.10) and (3.21) we find

$$m_\beta \|\varphi_1 - \varphi_2\|_W^2$$

$$\leq c_{\mathcal{E}} \|\mathbf{u}_1 - \mathbf{u}_2\|_V \|\varphi_1 - \varphi_2\|_W + \|q_1 - q_2\|_W \|\varphi_1 - \varphi_2\|_W \qquad (4.22)$$

$$+ \int_{\Gamma_C} |\psi(u_1 - g)\phi_L(\varphi_1 - \varphi_0) - \psi(u_2 - g)\phi_L(\varphi_2 - \varphi_0)|\,|\varphi_1 - \varphi_2|\, da,$$

where $c_{\mathcal{E}}$ is a positive constant which depends on the piezoelectric tensor $\mathcal{E}$.

We use the bounds $|\psi(u_i - g)| \leq M_\psi$, $|\phi_L(\varphi_1 - \varphi_0)| \leq L$, the Lipschitz continuity of the functions $\psi$ and $\phi_L$, and inequality (3.2) to obtain

$$\int_{\Gamma_C} |\psi(u_1 - g)\phi_L(\varphi_1 - \varphi_0) - \psi(u_2 - g)\phi_L(\varphi_2 - \varphi_0)|\,|\varphi_1 - \varphi_2|\, da$$

$$\leq M_\psi \int_{\Gamma_C} |\varphi_1 - \varphi_2|^2 \, da + L_\psi L \int_{\Gamma_C} |u_1 - u_2| \, |\varphi_1 - \varphi_2| \, da$$

$$\leq M_\psi \, c_0^2 \|\varphi_1 - \varphi_2\|_W^2 + L_\psi \, Lc_0\widetilde{c}_0 \|u_1 - u_2\|_V \|\varphi_1 - \varphi_2\|_W.$$

Inserting the last inequality in (4.22) yields

$$m_\beta \, \|\varphi_1 - \varphi_2\|_W$$
$$\leq (c_\mathcal{E} + L_\psi Lc_0\widetilde{c}_0) \, \|\mathbf{u}_1 - \mathbf{u}_2\|_V + \|q_1 - q_2\|_W + M_\psi \, c_0^2 \|\varphi_1 - \varphi_2\|_W. \quad (4.23)$$

It follows from inequality (4.23) and assumption (3.29) that

$$\|\varphi_1 - \varphi_2\|_W \leq c(\|\mathbf{u}_1 - \mathbf{u}_2\|_V + \|q_1 - q_2\|_W). \quad (4.24)$$

We also note that assumptions (3.15) and (3.16), combined with definition (3.23) imply that $q \in W^{1,p}(0, T; W)$. Since $\mathbf{u}_\eta \in C^1([0, T]; X)$, inequality (4.24) implies that $\varphi_\eta \in W^{1,p}(0, T; W)$.

Let $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C([0, T]; Q)$ and let $\varphi_{\eta_i} = \varphi_i$, $u_{\eta_i} = u_i$, for $i = 1, 2$. We use (4.15) and arguments similar to those used in the proof of (4.23) to obtain

$$m_\beta \, \|\varphi_1(t) - \varphi_2(t)\|_W \leq (c_\mathcal{E} + L_\psi Lc_0\widetilde{c}_0) \, \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V + M_\psi c_0^2 \|\varphi_1(t) - \varphi_2(t)\|_W$$

for all $t \in [0, T]$. This inequality, combined with assumption (3.29) leads to (4.16), which concludes the proof. $\qquad\square$

We now consider the operator $\Lambda : C([0, T]; Q) \to C([0, T]; Q)$ defined by

$$\Lambda\boldsymbol{\eta}(t) = \mathcal{E}^* \nabla\varphi_\eta(t) \qquad \forall \boldsymbol{\eta} \in C([0, T]; Q), \ t \in [0, T]. \quad (4.25)$$

We show that $\Lambda$ has a unique fixed point.

**Lemma 4.4.** *There exists a unique $\widetilde{\boldsymbol{\eta}} \in W^{1,p}(0, T; Q)$ such that $\Lambda\widetilde{\boldsymbol{\eta}} = \widetilde{\boldsymbol{\eta}}$.*

*Proof.* Let $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C([0, T]; Q)$ and denote by $\mathbf{u}_i$ and $\varphi_i$ the functions $\mathbf{u}_{\eta_i}$ and $\varphi_{\eta_i}$ obtained in Lemmas 4.2 and 4.3, for $i = 1, 2$. Let $t \in [0, T]$. Using (4.25) and (3.9) we obtain

$$\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|_Q \leq c \, \|\varphi_1(t) - \varphi_2(t)\|_W,$$

and, keeping in mind (4.16), we find

$$\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|_Q \leq c \, \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V. \quad (4.26)$$

On the other hand, since $\mathbf{u}_i(t) = \mathbf{u}_0 + \int_0^t \dot{\mathbf{u}}_i(s) \, ds$, we have

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \leq \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V \, ds, \quad (4.27)$$

and using this inequality in (4.11) yields

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V \leq c\Big(\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_Q + \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V \, ds\Big).$$

It follows now from a Gronwall-type argument that

$$\int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V \, ds \leq c \int_0^t \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_Q \, ds. \quad (4.28)$$

Combining (4.26)–(4.28) leads to

$$\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|_Q \leq c \int_0^t \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_Q \, ds.$$

Reiterating this inequality $n$ times results in

$$\|\Lambda^n \boldsymbol{\eta}_1(t) - \Lambda^n \boldsymbol{\eta}_2(t)\|_Q \le \frac{c^n}{n!} \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_{C([0,T];Q)}.$$

This inequality shows that for a sufficiently large $n$ the operator $\Lambda^n$ is a contraction on the Banach space $C([0,T];Q)$ and, therefore, there exists a unique element $\widetilde{\boldsymbol{\eta}} \in C([0,T];Q)$ such that $\Lambda \widetilde{\boldsymbol{\eta}} = \widetilde{\boldsymbol{\eta}}$. The regularity $\widetilde{\boldsymbol{\eta}} \in W^{1,p}(0,T;Q)$ follows from the fact that $\varphi_{\widetilde{\eta}} \in W^{1,p}(0,T;W)$, obtained in Lemma 4.3, combined with the definition (4.25) of the operator $\Lambda$. $\qquad\square$

We have now all the ingredient to prove the Theorem 3.1 which we complete now.

**Existence.** Let $\widetilde{\boldsymbol{\eta}} \in W^{1,p}(0,T;Q)$ be the fixed point of the operator $\Lambda$, and let $\mathbf{u}_{\widetilde{\eta}}$, $\varphi_{\widetilde{\eta}}$ be the solutions of problems $\mathcal{P}_\eta^1$ and $\mathcal{P}_\eta^2$, respectively, for $\boldsymbol{\eta} = \widetilde{\boldsymbol{\eta}}$. It follows from (4.25) that $\mathcal{E}^* \nabla \varphi_{\widetilde{\eta}} = \widetilde{\boldsymbol{\eta}}$ and, therefore, (4.9), (4.10) and (4.15) imply that $(\mathbf{u}_{\widetilde{\eta}}, \varphi_{\widetilde{\eta}})$ is a solution of problem $\mathcal{P}_V$. Property (3.30) follows from Lemmas 4.2 (3) and 4.3.

**Uniqueness.** The uniqueness of the solution follows from the uniqueness of the fixed point of the operator $\Lambda$. It can also be obtained by using arguments similar as those used in [14].

## 5. Conclusions

We presented a model for the quasistatic process of frictional contact between a deformable body made of a piezoelectric material, more precisely, an electro-viscoelastic material, and a conductive reactive foundation. The contact was modeled with the normal compliance condition and the associated Coulomb's law of dry friction. The new feature in the model was the electrical conduction of the foundation, which leads to a new boundary condition on the contact surface, (2.11), in which the normal component of the electric displacement vector is related to the penetration $u_\nu - g$ and the potential drop $\varphi - \varphi_0$. This condition provides a nonlinear coupling of the system on the contact boundary, and is a regularization of the perfect electric contact, (2.15).

The problem was set as a variational inequality for the displacements and a variational equality for the electric potential. The existence of the unique weak solution for the problem was established by using arguments from the theory of evolutionary variational inequalities involving nonlinear strongly monotone Lipschitz continuous operators, and a fixed-point theorem. It was obtained under a smallness assumption, (3.29), which involves only the electrical data of the problem and which is satisfied in the case of a contact with an insulated obstacle. This smallness assumption seems to be an artifact of the mathematical method, and in the future we plan to remove it, as it does not seem to represent any physical constraint on the system.

This work opens the way to study further problems with other conditions for electrically conductive or dielectric foundations.

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