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APPLICATION OF PETTIS INTEGRATION TO DIFFERENTIAL INCLUSIONS WITH THREE-POINT BOUNDARY CONDITIONS IN BANACH SPACES

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Abstract. This paper provide some applications of Pettis integration to differential inclusions in Banach spaces with three point boundary conditions of the form

$$\ddot{u}(t) \in F(t, u(t), \dot{u}(t)) + H(t, u(t), \dot{u}(t)), \quad \text{a.e. } t \in [0, 1],$$

where F is a convex valued multifunction upper semicontinuous on $E \times E$ and H is a lower semicontinuous multifunction. The existence of solutions is obtained under the non convexity condition for the multifunction H, and the assumption that $\mathrm{F}(t, x, y) \subset \Gamma_1(t)$, $\mathrm{H}(t, x, y) \subset \Gamma_2(t)$, where the multifunctions $\Gamma_1, \Gamma_2 : [0, 1] \rightrightarrows E$ are uniformly Pettis integrable.

1. Introduction

In the theory of integration in infinite-dimensional spaces, Pettis integrability is a more general concept than that of Bochner integrability. Indeed, it is known that a Banach space E is infinite dimensional if and only if there exists a Pettis integrable E-valued function, which is not Bochner integrable. There is a rich literature dealing with the Pettis integral. For acquit extensive account, we refer the reader to the monographe by Musial [15], where further references can be found. On the other hand, the set-valued integration has shown to be useful tool for modeling a lot of situations in several fields ranging from mathematical economics to optimization and optimal control. Recently, special attention has been paid to the Pettis integral of multifunctions. For example, let us mention the recent contributions of Amrani and Castaing [1], Amrani, Castaing and Valdier [2], and Castaing [7] which deal with the Pettis integral of bounded, especially weakly compact, convex valued multifunctions. See also [9], [11], [13], [14], [16] and the references therein.

Existence of solutions for second order differential inclusions of the form $\ddot{u}(t) \in F(t, u(t), \dot{u}(t))$ with three-point boundary conditions, where $F : [0, 1] \times E \times E \rightrightarrows E$ is a convex compact valued multifunction, Lebesgue-measurable on $[0, 1]$, and upper semicontinuous on $E \times E$, under the assumption that $F(t, x, y) \subset \Gamma(t)$ in the case where Γ is integrably bounded and the case where Γ is uniformly Pettis integrable, has been studied by Azzam-Laouir, Castaing and Thibault [5].

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Let θ be a given number in $]0,1[$; the aim of our article is to provide existence results for the general problem of three point boundary conditions associated with the differential inclusion

$$\begin{gathered} \ddot{u}(t) \in F(t,u(t),\dot{u}(t)) + H(t,u(t),\dot{u}(t)), \quad \text{a.e. } t\in[0,1], \\ u(0)=0; \quad u(\theta)=u(1). \end{gathered} \tag{1.1}$$

We suppose that $F:[0,1]\times E\times E \rightrightarrows E$ is upper semicontinuous on $E\times E$ and measurable on $[0,1]$. We take $H:[0,1]\times E\times E \rightrightarrows E$ as a measurable multifunction lower semicontinuous on $E\times E$. Furthermore we suppose that $F(t,x,y)\subset\Gamma_1(t)$, $H(t,x,y)\subset\Gamma_2(t)$ for all $(t,x,y)\in[0,1]\times E\times E$ for some convex $\|\cdot\|$-compact valued, and measurable multifunctions $\Gamma_1,\Gamma_2:[0,1]\rightrightarrows E$ which are uniformly Pettis integrable. Then we show that the differential inclusion (1.1) has at least a solution $u\in\mathbf{W}^{2,1}_{P,E}([0,1])$.

2. Notation and Preliminaries

Throughout, $(E,\|\cdot\|)$ is a separable Banach space and E' is its Topological dual, $\overline{\mathbf{B}}_E$ is the unit closed ball of E, $\mathcal{L}([0,1])$ is the σ-algebra of Lebesgue-measurable sets of $[0,1]$, $\lambda=dt$ is the Lebesgue measure on $[0,1]$, and $\mathcal{B}(E)$ is the σ-algebra of Borel subsets of E. By $\mathbf{L}^1_E([0,1])$ we denote the space of all Lebesgue-Bochner integrable E valued mappings defined on $[0,1]$. We denote the topology of uniform convergence on weakly compact convex sets by $\mathcal{T}^w_{co}$. Restricted to E', this is the Mackey topology, which is the strongest locally convex topology on E' and we denote it by $\mathcal{T}(E',E)$. We recall some preliminary results. Let $f:[0,1]\to E$ be a scalarly integrable mapping, that is, for every $x'\in E'$, the scalar function $t\mapsto\langle x',f(t)\rangle$ is Lebesgue-integrable on $[0,1]$. A scalarly integrable mapping $f:[0,1]\to E$ is Pettis integrable if, for every Lebesgue measurable set A in $[0,1]$, the weak integral $\int_A f(t)dt$ defined by $\langle x',\int_A f(t)dt\rangle=\int_A\langle x',f(t)\rangle dt$ for all $x'\in E'$, belongs to E. We denote by $\mathbf{P}^1_E([0,1])$ the space of all Pettis-integrable E-valued mappings defined on $[0,1]$. The Pettis norm of any element $f\in\mathbf{P}^1_E([0,1])$ is defined by $\|f\|_{Pe}=\sup_{x'\in\overline{\mathbf{B}}_{E'}}\int_{[0,1]}|\langle x',f(t)\rangle|dt$. The space $\mathbf{P}^1_E([0,1])$ endowed with $\|\cdot\|_{Pe}$ is a normed space. A subset $\mathcal{K}\subset\mathbf{P}^1_E([0,1])$ is Pettis uniformly integrable (PUI for short) if, for every $\varepsilon>0$, there exists $\delta>0$ such that

$$\lambda(A)\le\delta \Rightarrow \sup_{f\in\mathcal{K}}\|\mathbf{1}_A f\|_{Pe}\le\varepsilon,$$

where $\mathbf{1}_A$ stands for the characteristic function of A. If $f\in\mathbf{P}^1_E([0,1])$, the singleton $\{f\}$ is PUI since the set $\{\langle x',f\rangle:\|x'\|\le1\}$ is uniformly integrable.

Let $\mathbf{C}_E([0,1])$ be the Banach space of all continuous mappings $u:[0,1]\to E$, endowed with the sup-norm, and let $\mathbf{C}^1_E([0,1])$ be the Banach space of all continuous mappings $u:[0,1]\to E$ with continuous derivative, equipped with the norm

$$\|u\|_{\mathbf{C}^1}=\max\{\max_{t\in[0,1]}\|u(t)\|,\max_{t\in[0,1]}\|\dot{u}(t)\|\}.$$

Recall that a mapping $v:[0,1]\to E$ is said to be scalarly derivable when there exists some mapping $\dot{v}:[0,1]\to E$ (called the weak derivative of v) such that, for every $x'\in E'$, the scalar function $\langle x',v(.)\rangle$ is a.e derivable and its derivative is equal to $\langle x',\dot{v}(.)\rangle$. The weak derivative $\ddot{v}$ of $\dot{v}$ when it exists is the weak second derivative.

By $\mathbf{W}^{2,1}_{P,E}([0,1])$ we denote the space of all continuous mappings in $\mathbf{C}_E([0,1])$ such that their first weak derivatives are continuous and their second weak derivatives belong to $\mathbf{P}^1_E([0,1])$.

For closed subsets A and B of E, the excess of A over B is defined by

$$e(A,B)=\sup_{a\in A} d(a,B)=\sup_{a\in A}(\inf_{b\in B}\|a-b\|),$$

and the support function $\delta^*(\cdot,A)$ associated with A is defined on E' by

$$\delta^*(x',A)=\sup_{a\in A}\langle x',a\rangle.$$

Recall that we have

$$d(x,A)=\sup_{x'\in\overline{\mathbf{B}}_{E'}}[\langle x',x\rangle-\delta^*(x',A)],\ \forall x\in E. \tag{2.1}$$

For a set $A\subset E$, $\overline{co}A$ is its closed convex hull.

Recall also that a set $K\subset \mathbf{P}^1_E([0,1])$ is said to be decomposable if and only if for every $u,v\in K$ and any $A\in\mathcal{L}([0,1])$ we have $u.\mathbf{1}_A+v.(1-\mathbf{1}_A)\in K$.

3. The main result

We begin with a lemma which summarizes some properties of some Green type function (see [3], [5]). It will be used full in the study of our boundary problems.

Lemma 3.1. *Let E be a separable Banach space and let $G:[0,1]\times[0,1]\to\mathbb{R}$ be the function defined by*

$$G(t,s)=\begin{cases}-s & \text{if } 0\le s\le t,\\ -t & \text{if } t<s\le\theta,\\ t(s-1)/(1-\theta) & \text{if } \theta<s\le 1,\end{cases} \tag{3.1}$$

if $0\le t<\theta$, and

$$G(t,s)=\begin{cases}-s & \text{if } 0\le s<\theta,\\ (\theta(s-t)+s(t-1))/(1-\theta) & \text{if } \theta\le s\le t,\\ t(s-1)/(1-\theta) & \text{if } t<s\le 1,\end{cases} \tag{3.2}$$

if $\theta\le t\le 1$. Then the following assertions hold.

(1) $G(.,s)$ is differentiable on $[0,1]$, for every $s\in[0,1]$, and its derivative is

$$\frac{\partial G}{\partial t}(t,s)=\begin{cases}0 & \text{if } 0\le s\le t,\\ -1 & \text{if } t<s\le\theta,\\ (s-1)/(1-\theta) & \text{if } \theta<s\le 1,\end{cases}$$

if $0\le t<\theta$, and

$$\frac{\partial G}{\partial t}(t,s)=\begin{cases}0 & \text{if } 0\le s<\theta,\\ (s-\theta)/(1-\theta) & \text{if } \theta\le s\le t,\\ (s-1)/(1-\theta) & \text{if } t<s\le 1,\end{cases}$$

if $\theta\le t\le 1$.

(2) $G(.,.)$ and $\frac{\partial G}{\partial t}(.,.)$ satisfies

$$\sup_{t,s\in[0,1]}|G(t,s)|\le 1,\qquad \sup_{t,s\in[0,1]}|\frac{\partial G}{\partial t}(t,s)|\le 1.$$

(3) For $f \in \mathbf{P}^1_E([0,1])$ and for the mapping $u_f : [0,1] \to E$ defined by

$$u_f(t) = \int_0^1 G(t,s)f(s)ds, \quad \forall t \in [0,1],$$

one has: (3i) $u_f(0) = 0$ and $u_f(\theta) = u_f(1)$.
(3ii) The mapping $t \mapsto u_f(t)$ is continuous from [0,1] into E, i.e., $u_f \in \mathbf{C}_E([0,1])$.
(3iii) The mapping u_f is scalarly derivable, that is, for every $x' \in E'$, the scalar function $\langle x', u_f(.)\rangle$ is a.e derivable, and its weak derivative $\dot{u}_f$ satisfies

$$\begin{aligned}
\lim_{h\to 0}\langle x', \frac{u_f(t+h)-u_f(t)}{h}\rangle &= \langle x', \dot{u}_f(t)\rangle \\
&= \int_0^1 \frac{\partial G}{\partial t}(t,s)\langle x', f(s)\rangle ds \\
&= \langle x', \int_0^1 \frac{\partial G}{\partial t}(t,s), f(s)ds\rangle
\end{aligned}$$

for all $t \in [0,1]$ and for all $x' \in E'$. Consequently

$$\dot{u}_f(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)ds, \quad \forall t \in [0,1],$$

and $\dot{u}_f$ is a continuous mapping from $[0,1]$ into E. (3vi) The mapping $\dot{u}_f$ is scalarly derivable, that is, there exists a mapping $\ddot{u}_f : [0,1] \to E$ such that, for every $x' \in E'$, the scalar function $\langle x', \dot{u}_f(\cdot)\rangle$ is a.e derivable with $\frac{d}{dt}\langle x', \dot{u}_f(t)\rangle = \langle x', \ddot{u}_f(t)\rangle$; furthermore

$$\ddot{u}_f = f \quad \text{a.e. on } [0,1].$$

Let us mention a useful consequence of Lemma 3.1.

Proposition 3.2. *Let E be a separable Banach space and let $f : [0,1] \to E$ be a continuous mapping (respectively a mapping in $\mathbf{P}^1_E([0,1])$). Then the mapping*

$$u_f(t) = \int_0^1 G(t,s)f(s)ds, \quad \forall t \in [0,1],$$

is the unique $\mathbf{C}^2_E([0,1])$-solution (respectively $\mathbf{W}^{2,1}_{P,E}([0,1])$-solution) to the differential equation

$$\begin{gathered}
\ddot{u}(t) = f(t) \quad \forall t \in [0,1], \\
u(0) = 0; \quad u(\theta) = u(1).
\end{gathered}$$

The following proposition is an analogous version of the continuous selection theorem of Bressan and Colombo [6] and Fryszkowski [10], in the case where the multifunction has values in $\mathbf{P}^1_E([0,1])$. For the proof of this result we refer the reader to [4].

Proposition 3.3. *Let $M : [0,1] \rightrightarrows \mathbf{P}^1_E([0,1])$ be a lower semicontinuous multifunction with closed and decomposable values. Then M has a continuous selection.*

For the proof of our Theorem, we need the following Lemma due to Grothendieck [12]. See also [7] for a more general result concerning the Mackey topology for bounded sequences in $\mathbf{L}^\infty_{E'}$.

Lemma 3.4. *Let (g_n) be a sequence of uniformly bounded mappings in $\mathbf{L}^{\infty}_{\mathbb{R}}([0,T])$, which converges pointwise to 0. Then for all uniformly integrable subset K of $\mathbf{L}^1_{\mathbb{R}}([0,T])$, the sequence $(\langle g_n, h\rangle) = (\int_0^1 g_n(t)h(t)dt)$ converges uniformly to 0, for all $h \in K$.*

Now we are able to give our main result.

Theorem 3.5. *Let E be a separable Banach space and let $F : [0,1] \times E \times E \rightrightarrows E$ be a convex compact valued multifunction, Lebesgue-measurable on $[0,1]$, and upper semicontinuous on $E \times E$. Let $H : [0,1] \times E \times E \rightrightarrows E$ be a multifunction with nonempty closed values such that H is $\mathcal{L}([0,1]) \otimes \mathcal{B}(E) \otimes \mathcal{B}(E)$-measurable and lower semicontinuous on $E \times E$. Assume that for $i = 1, 2$ there are some convex $\|\cdot\|$-compact valued, and measurable multifunctions $\Gamma_i : [0,1] \rightrightarrows E$ which are Pettis uniformly integrable, such that $F(t,x,y) \subset \Gamma_1(t)$ and $H(t,x,y) \subset \Gamma_2(t)$ for all $(t,x,y) \in [0,1] \times E \times E$. Then the differential inclusion*

$$\begin{gathered}\ddot{u}(t) \in \mathrm{F}(t, u(t), \dot{u}(t)) + \mathrm{H}(t, u(t), \dot{u}(t)), \quad \text{a.e. } t \in [0,1], \\ u(0) = 0; \quad u(\theta) = u(1).\end{gathered}$$

has at least one solution $u \in \mathbf{W}^{2,1}_{P,E}([0,1])$.

Proof. Step 1. Taking $\overline{co}(\{0\} \cup \Gamma_i(t))$ if necessary, we may suppose that $0 \in \Gamma_i(t)$ for all $t \in [0,1]$ and $i = 1, 2$.
For $t \in [0,1]$, let $\Gamma(t) = \Gamma_1(t) + \Gamma_2(t)$ and observe that the multifunction Γ inherits all the properties of Γ_1 and Γ_2. Let us consider the differential inclusion

$$\begin{gathered}\ddot{u}(t) \in \Gamma(t), \quad \text{a.e. } t \in [0,1], \\ u(0) = 0; \quad u(\theta) = u(1).\end{gathered} \tag{3.3}$$

We wish to show that the $\mathbf{W}^{2,1}_{P,E}([0,1])$-solutions set $\mathbf{X}_\Gamma$ of (3.3) is nonempty and convex compact in the Banach space $\mathbf{C}^1_E([0,1])$ endowed with the norm $\|.\|_{\mathbf{C}^1}$. Furthermore, if a sequence (u_n) of $\mathbf{X}_\Gamma$ $\|.\|_{\mathbf{C}^1}$-converges to u, then $(\dot{u}_n)$ converges pointwise to $\dot{u}$ and $(\ddot{u}_n)$ converges $\sigma(\mathbf{P}^1_E, \mathbf{L}^{\infty}_E \otimes E')$ to $\ddot{u}$. The proof of this last assertion is similar of the one in [5, Lemma 5]; we include it here for the convenience of the reader.

Let us recall that the set $\mathbf{S}^{Pe}_\Gamma$ of all Pettis integrable selections of Γ is nonempty and sequentially compact for the topology of pointwise convergence on $\mathbf{L}^\infty \otimes E'$ and that the multivalued integral

$$\int_0^1 \Gamma(t)dt = \{ \int_0^1 f(t)dt; f \in \mathbf{S}^{Pe}_\Gamma \}$$

is convex and norm compact in E (see [1], [2], [7]). In view of Lemma 3.1 and Proposition 3.2, the solutions set $\mathbf{X}_\Gamma$ of (3.3) is characterized by

$$\mathbf{X}_\Gamma = \{u_f : [0,1] \to E : u_f(t) = \int_0^1 G(t,s)f(s)ds, \forall t \in [0,1]; f \in \mathbf{S}^{Pe}_\Gamma\}.$$

Clearly $\mathbf{X}_\Gamma$ is convex. Furthermore, if (t_n) is a sequence in $[0,1]$, which converges to $t \in [0,1]$ we have, by Lemma 3.1,

$$\begin{aligned} \|u_f(t_n) - u_f(t)\| &= \sup_{x' \in \overline{\mathbf{B}}_{E'}} |\langle x', u_f(t_n) - u_f(t)\rangle| \\ &= \sup_{x' \in \overline{\mathbf{B}}_{E'}} |\langle x', \int_0^1 G(t_n, s) f(s) ds - \int_0^1 G(t,s) f(s) ds\rangle| \\ &\le \sup_{x' \in \overline{\mathbf{B}}_{E'}} \int_0^1 |G(t_n, s) - G(t,s)||\langle x', f(s)\rangle| ds \\ &\le \sup_{x' \in \overline{\mathbf{B}}_{E'}} \int_0^1 |G(t_n, s) - G(t,s)||\delta^*(x', \Gamma(s))| ds \end{aligned} \tag{3.4}$$

and

$$\|\dot{u}_f(t_n) - \dot{u}_f(t)\| \le \sup_{x' \in \overline{\mathbf{B}}_{E'}} \int_0^1 |\frac{\partial G}{\partial t}(t_n, s) - \frac{\partial G}{\partial t}(t,s)||\delta^*(x', \Gamma(s))| ds \tag{3.5}$$

for all $f \in \mathbf{S}_\Gamma^{Pe}$. As the sequences $(v_n(.)) := (|G(t_n, .) - G(t, .)|)$ and $(w_n(.)) := (|\frac{\partial G}{\partial t}(t_n, .) - \frac{\partial G}{\partial t}(t, .)|)$ are uniformly bounded and converge pointwise to 0 and as the set $\{|\delta^*(x', \Gamma(.))| : x' \in \overline{\mathbf{B}}_{E'}\}$ is uniformly integrable in $\mathbf{L}^1_{\mathbb{R}}([0,1])$, by Lemma 3.4 we conclude that $(v_n(.))$ and $(w_n(.))$ converge uniformly to 0 on this set in the duality $\langle \mathbf{L}^\infty_{\mathbb{R}}, \mathbf{L}^1_{\mathbb{R}}\rangle$. Hence the second member of (3.4) and (3.5) tends to 0. This says that $\mathbf{X}_\Gamma$ and $\{\dot{u}_f : u_f \in \mathbf{X}_\Gamma\}$ are equicontinuous in $\mathbf{C}_E([0,1])$. Furthermore, the sets $\mathbf{X}_\Gamma(t) = \{u_f(t) : \ u_f \in \mathbf{X}_\Gamma\}$ and $\{\dot{u}_f(t) : \ u_f \in \mathbf{X}_\Gamma\}$ are relatively compact in E because they are included in the norm compact sets $\int_0^1 G(t,s)\Gamma(s) ds$ and $\int_0^1 \frac{\partial G}{\partial t}(t,s)\Gamma(s) ds$ respectively. The Ascoli-Arzelà theorem yields that $\mathbf{X}_\Gamma$ is relatively compact in $\mathbf{C}^1_E([0,1])$ with respect to $\|.\|_{\mathbf{C}^1}$. We claim that $\mathbf{X}_\Gamma$ is closed in $(\mathbf{C}^1_E([0,1]), \|.\|_{\mathbf{C}^1})$. Let (u_{f_n}) be a sequence in $\mathbf{X}_\Gamma$ converging to $\xi \in \mathbf{C}^1_E([0,1])$ with respect to $\|.\|_{\mathbf{C}^1}$. As $\mathbf{S}_\Gamma^{Pe}$ is sequentially compact for the topology of pointwize convergence on $\mathbf{L}^\infty_E \otimes E'$, we extract from (f_n) a subsequence that we do not relabel and which converges $\sigma(\mathbf{P}^1_E, \mathbf{L}^\infty_E \otimes E')$ to a mapping $f \in \mathbf{S}_\Gamma^{Pe}$. In particular

$$\begin{aligned} \lim_{n\to\infty} \langle x', \int_0^1 G(t,s) f_n(s) ds\rangle &= \lim_{n\to\infty} \int_0^1 \langle G(t,s) x', f_n(s)\rangle ds \\ &= \int_0^1 \langle G(t,s) x', f(s)\rangle ds \\ &= \langle x', \int_0^1 G(t,s) f(s) ds\rangle. \end{aligned} \tag{3.6}$$

As the set valued integral $\int_0^1 G(t,s)\Gamma(s) ds$ $(t \in [0,1])$ is norm-compact, (3.6) shows that the sequence $(u_{f_n}(\cdot)) = (\int_0^1 G(.,s) f_n(s) ds)$ converges pointwise to $u_f(.)$ for E endowed with the strong topology. Thus we get $\xi = u_f$. This shows the compactness of $\mathbf{X}_\Gamma$ in $\mathbf{C}^1_E([0,1])$.

Step 2. Let us observe that, for any Lebesgue-measurable mappings $v, w : [0,1] \to E$, there is a Pettis integrable selection $s \in \mathbf{S}_{\Gamma_1}^{Pe}$ such that $s(t) \in F(t, v(t), w(t))$ a.e. Indeed, there exist two sequences (v_n) and (w_n) of simple E-valued mappings converging to v and w respectively, for E endowed with the norm topology. Notice that the multifunctions $F(.,v_n(.),w_n(.))$ are measurable. Let s_n be a Lebesgue-measurable selection of $F(.,v_n(.),w_n(.))$. As $s_n(t)\in F(t,v_n(t),w_n(t))\subset\Gamma_1(t)$; for all $t\in[0,1]$ and $\mathbf{S}^{Pe}_{\Gamma_1}$ is sequentially $\sigma(\mathbf{P}^1_E,\mathbf{L}^\infty_E\otimes E')$-compact, we may extract from (s_n) a subsequence (s'_n) which converges $\sigma(\mathbf{P}^1_E,\mathbf{L}^\infty_E\otimes E')$ to a mapping $s\in\mathbf{S}^{Pe}_{\Gamma_1}$. Let $(e^*_k)_{k\in\mathbb{N}}$ be a dense sequence for the Mackey topology $\mathcal{T}(E',E)$. Let $k\in\mathbb{N}$ be fixed. Applying the Mazur's trick to $(\langle e^*_k,s'_n(.)\rangle)_n$ provides a sequence (z_n) with $z_n\in co\{\langle e^*_k,s'_m(.)\rangle : m\geq n\}$ such that (z_n) converges pointwise a.e. to $\langle e^*_k,s(.)\rangle$. Using this fact and the pointwise convergence of the sequences (v_n) and (w_n) and the upper semicontinuity of $F(t,\cdot,\cdot)$, it is not difficult to check that $\langle e^*_k,s(t)\rangle\leq\delta^*(e^*_k,F(t,v(t),w(t))$ a.e. Indeed, Let A be a measurable set of $[0,1]$,

$$\begin{aligned}\int_A\langle e^*_k,s(t)\rangle dt&=\lim_{n\to\infty}\int_A\langle e^*_k,s'_n(t)\rangle dt\\&\leq\limsup_{n\to\infty}\int_A\delta^*(e^*_k,F(t,v_n(t),w_n(t)))dt\\&\leq\int_A\limsup_{n\to\infty}\delta^*(e^*_k,F(t,v_n(t),w_n(t)))dt\\&=\int_A\delta^*(e^*_k,F(t,v(t),w(t)))dt.\end{aligned}$$

Then, for all $k\in\mathbb{N}$

$$\langle e^*_k,s(t)\rangle\leq\delta^*(e^*_k,F(t,v(t),w(t))),\text{a.e. } t\in[0,1]. \tag{3.7}$$

On the other hand, in view of [8, Lemma III.33 and Corollary I.15], we have

$$\sup_{x'\in E'}[\langle x',s(t)\rangle-\delta^*(x',F(t,v(t),w(t)))]=\sup_{k\in\mathbf{N}}[\langle e^*_k,s(t)\rangle-\delta^*(e^*_k,F(t,v(t),w(t)))]. \tag{3.8}$$

Using relation (2.1) given in Section 2 we get by (3.7) and (3.8),

$$d(s(t),F(t,v(t),w(t))\leq\sup_{k\in\mathbf{N}}[\langle e^*_k,s(t)\rangle-\delta^*(e^*_k,F(t,v(t),w(t)))]\leq 0.$$

Consequently $s(t)\in F(t,v(t),w(t))$ a.e. $t\in[0,1]$.

Step 3. Let $\Phi:\mathbf{X}_\Gamma\rightrightarrows\mathbf{P}^1_E([0,1])$ be the multifunction given by

$$\Phi(u_f)=\{v\in\mathbf{P}^1_E([0,1]):v(t)\in H(t,u_f(t),\dot u_f(t)),\ \text{a.e. on } [0,1]\}.$$

We will prove that, for $\mathbf{X}_\Gamma$ endowed with the norm $\|\cdot\|_{\mathbf{C}^1}$, the multifunction Φ admits a continuous selection. It is clear that Φ has nonempty closed decomposable values. According to Proposition 3.3, it sufficient to prove that Φ is lower semicontinuous. Let $u_{f_0}\in\mathbf{X}_\Gamma$, $v_0\in\Phi(u_{f_0})$ and let (u_{f_n}) be a sequence in $\mathbf{X}_\Gamma$ converging to u_{f_0} in $(\mathbf{C}^1_E([0,1]),\|\cdot\|_{\mathbf{C}^1})$. For any $n\in\mathbb{N}$, $H(.,u_{f_n}(.),\dot u_{f_n}(.))$ is measurable with nonempty closed values, so the multifunction Λ_n defined from $[0,1]$ into E by

$$\Lambda_n(t)=\{w\in H(t,u_{f_n}(t),\dot u_{f_n}(t)):\|w-v_0(t)\|=d(v_0(t),H(t,u_{f_n}(t),\dot u_{f_n}(t)))\}$$

is also measurable with nonempty closed values. In view of the existence theorem of measurable selections (see [8]), there is a measurable mapping $v_n:[0,1]\to E$ such that $v_n(t)\in\Lambda_n(t)$, for all $t\in[0,1]$. This yields $v_n(t)\in H(t,u_{f_n}(t),\dot u_{f_n}(t))$ and

$$\begin{aligned}\lim_{n\to\infty}\|v_n(t)-v_0(t)\|&=\lim_{n\to\infty}d(v_0(t),H(t,u_{f_n}(t),\dot u_{f_n}(t)))\\&\leq\lim_{n\to\infty}e(H(t,u_{f_0}(t),\dot u_{f_0}(t)),H(t,u_{f_n}(t),\dot u_{f_n}(t)))=0.\end{aligned}$$

This says that (v_n) converges pointwise to v_0 and since $H(t,x,y) \subset \Gamma_2(t)$ for all $(t,x,y) \in [0,1] \times E \times E$, the convergence also holds strongly in $\mathbf{P}^1_E([1,0])$. Indeed,

$$\begin{aligned}\lim_{n\to\infty} \|v_n - v_0\|_{Pe} &= \lim_{n\to\infty} \sup_{x' \in \overline{\mathbf{B}}_{E'}} \int_0^1 |\langle x', v_n(t) - v_0(t)\rangle| dt \\ &= \lim_{n\to\infty} \sup_{x' \in \overline{\mathbf{B}}_{E'}} \int_0^1 |\langle x', v_n(t)\rangle - \langle x', v_0(t)\rangle| dt.\end{aligned}$$

As $v_n(t) \in \Gamma_2(t)$ for all $n \in \mathbb{N}$ and as Γ_2 is scalarly uniformly integrable and hence the set $\{\langle x', v_n(.)\rangle : \|x'\| \le 1\}$ is uniformly integrable in $\mathbf{L}^1_E([0,1])$, we get

$$\lim_{n\to\infty} \|v_n - v_0\|_{Pe} = \lim_{n\to\infty} \sup_{x' \in \overline{\mathbf{B}}_{E'}} \int_0^1 \lim_{n\to\infty} |\langle x', v_n(t)\rangle - \langle x', v_0(t)\rangle| dt = 0.$$

Therefore Φ is lower semicontinous. An application of Proposition 3.3 implies that, for $\mathbf{X}_\Gamma$ endowed with the norm $\|\cdot\|_{\mathbf{C}^1}$, there exists a continuous mapping $\varphi : \mathbf{X}_\Gamma \to \mathbf{P}^1_E([0,1])$ such that $\varphi(u) \in \Phi(u)$ for all $u \in \mathbf{X}_\Gamma$, or equivalently for each $u \in \mathbf{X}_\Gamma$ the inclusion $\varphi(u)(t) \in H(t,u(t),\dot{u}(t))$ holds for a.e. $t \in [0,1]$.

Step 4. For all $u \in \mathbf{X}_\Gamma$, let us define the multifunction Ψ by

$$\Psi(u) = \{v \in \mathbf{X}_\Gamma :\ \ddot{v}(t) \in F(t,u(t),\dot{u}(t)) + \varphi(u)(t),\ \text{a.e.}\}.$$

In view of Step 2, and since $\varphi(u) \in \mathbf{S}^{Pe}_{\Gamma_2}$ for all $u \in \mathbf{X}_\Gamma$, for any measurable selection s of $F(.,u(.),\dot{u}(.))$ the mapping $g := s + \varphi(u)$ is in $\mathbf{S}^{Pe}_\Gamma$ (because $g(t) = s(t) + \varphi(u)(t) \in \Gamma_1(t) + \Gamma_2(t) = \Gamma(t)$) and the mapping v defined by $v(t) = \int_0^1 G(t,s)g(s)ds$ is in $\Psi(u)$, and hence $\Psi(u)$ is a nonempty set. It clear that $\Psi(u)$ is a convex subset of $\mathbf{X}_\Gamma$. We need to check that $\Psi : \mathbf{X}_\Gamma \rightrightarrows \mathbf{X}_\Gamma$ is upper semicontinuous on the convex compact set $\mathbf{X}_\Gamma$. Equivalently we need to check that the graph of Ψ, $\mathrm{gph}(\Psi) = \{(u,v) \in \mathbf{X}_\Gamma \times \mathbf{X}_\Gamma : v \in \Psi(u)\}$, is sequentially closed in $\mathbf{X}_\Gamma \times \mathbf{X}_\Gamma$. Let (u_n, v_n) be a sequence in $\mathrm{gph}(\Psi)$ converging to $(u,v) \in \mathbf{X}_\Gamma \times \mathbf{X}_\Gamma$. By repeating the arguments given in Step 1, we obtain that (u_n, v_n) converges uniformly to (u,v) in $(\mathbf{C}^1_E([0,1]), \|\cdot\|_{\mathbf{C}^1})$, and that $(\ddot{u}_n,\ \ddot{v}_n)$ converges $\sigma(\mathbf{P}^1_E, \mathbf{L}^\infty_E \otimes E')$ to $(\ddot{u}, \ddot{v})$. As $\ddot{v}_n(t) - \varphi(u_n)(t) \in F(t,u_n(t),\dot{u}_n(t))$, a.e., repeating the arguments given at the end of Step 2, we get $\ddot{v}(t) - \varphi(u)(t) \in F(t,u(t),\dot{u}(t))$, a.e. This shows that $\mathrm{gph}(\Psi)$ is closed in $\mathbf{X}_\Gamma \times \mathbf{X}_\Gamma$ and hence we get the upper semicontinuity of Ψ. An application of the Kakutani fixed point theorem gives some $u \in \mathbf{X}_\Gamma$ such that $u \in \Psi(u)$. This means $\ddot{u}(t) \in F(t,u(t),\dot{u}(t)) + \varphi(u)(t)$, a.e. Since $\varphi(u)(t) \in H(t,u(t),\dot{u}(t))$, we get

$$\begin{gathered}\ddot{u}(t) \in F(t,u(t),\dot{u}(t)) + H(t,u(t),\dot{u}(t)), \quad \text{a.e. } t \in [0,1], \\ u(0) = 0; \quad u(\theta) = u(1).\end{gathered}$$

This completes the proof of the theorem. □

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Addendum posted by the editor on September 15, 2016

A reader informed us that Proposition 3.3 and Theorem 3.5 are incorrect. The same proposition (stated as Proposition 2.2) was also used in

D. Azzam-Laouir, I. Boutana, A. Makhlouf; Application of Pettis integration to delay second order differential inclusions, Electronic Journal of Qualitative Theory of Differential Equations 2012, 88 pp 1–15.

A reader pointed out the mistake and the authors posted a corrigendum that says

> In the above article, Proposition 2.2 is not true since the normed space $P_E^1([0,1])$ is not complete. Consequently, to correct Theorem 3.1 we have to assume that Γ_1 is Pettis uniformly integrable and that Γ_2 is integrably bounded. Then in the proof we can use Proposition 2.2 with $L_E^1([0,1])$ instead of $P_E^1([0,1])$ to conclude the result. This version of Proposition 2.2 can be found in
>
> A. Fryszkowski, Continuous selections for a class of nonconvex mul- tivalued maps, Studia Math., 76, (1983), pp. 163-174.

We asked the authors to post a similar addendum to the EJDE article, but the authors did not reply. So we attached this note.

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