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THIRD-ORDER NONLOCAL PROBLEMS WITH SIGN-CHANGING NONLINEARITY ON TIME SCALES

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Abstract. We are concerned with the existence and form of positive solutions to a nonlinear third-order three-point nonlocal boundary-value problem on general time scales. Using Green's functions, we prove the existence of at least one positive solution using the Guo-Krasnoselskii fixed point theorem. Due to the fact that the nonlinearity is allowed to change sign in our formulation, and the novelty of the boundary conditions, these results are new for discrete, continuous, quantum and arbitrary time scales.

1. Statement of the problem

We will develop an interval of λ values whereby a positive solution exists for the following nonlinear, third-order, three-point, nonlocal boundary-value problem on arbitrary time scales

$$(px^{\Delta\Delta})^{\nabla}(t) = \lambda f(t, x(t)), \quad t \in [t_1, t_3]_{\mathbb{T}}, \tag{1.1}$$

$$\alpha x(\rho(t_1)) - \beta x^{\Delta}(\rho(t_1)) = \int_{\xi_1}^{\xi_2} a(t)x(t)\nabla t,$$

$$x^{\Delta}(t_2) = 0, \quad (px^{\Delta\Delta})(t_3) = \int_{\eta_1}^{\eta_2} b(t)(px^{\Delta\Delta})(t)\nabla t, \tag{1.2}$$

where: p is a left-dense continuous, real-valued function on $\mathbb{T}$ with $p > 0$; $\lambda > 0$ is a real scalar;

(H1) the real scalars $\alpha, \beta > 0$ and the three boundary points satisfy $t_1 < t_2 < t_3 \in \mathbb{T}$ such that

$$0 < \int_{\rho(t_1)}^{\sigma^2(t_3)} \int_u^{t_2} \frac{\Delta r}{p(r)} \Delta u + \frac{\beta}{\alpha} \int_{\rho(t_1)}^{t_2} \frac{\Delta r}{p(r)} < \infty;$$

(H2) the points $\xi_i, \eta_i \in \mathbb{T}$ satisfy

$$\rho(t_1) < \xi_1 < \xi_2 < t_2, \quad \rho(t_1) \le \eta_1 < \eta_2 \le t_3;$$

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(H3) the left-dense continuous real-valued functions on $\mathbb{T}$ satisfy $a, b \geq 0$ with

$$0 < \int_{\xi_1}^{\xi_2} a(t)\nabla t < \alpha \quad \text{and} \quad 0 < \int_{\eta_1}^{\eta_2} b(t)\nabla t < 1;$$

(H4) the continuous function $f : [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}} \times [0, \infty) \to (-\infty, \infty)$ is such that

$$\lim_{x\to+\infty} \frac{f(t,x)}{x} \overset{\text{unif}}{=} +\infty, \quad t \in [\xi_1, t_2]_{\mathbb{T}};$$

(H5) there exist left-dense continuous functions $y, z : [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}} \to (0, \infty)$ and a continuous function $h : [0, \infty) \to (0, \infty)$ such that

$$-y(t) \leq f(t,x) \leq z(t)h(x), \quad 0 < \int_{\rho(t_1)}^{t_3} (z(s) + y(s))\nabla s < \infty.$$

Third-order differential equations, though less common in applications than even-order problems, nevertheless do appear, for example in the study of quantum fluids; see Gamba and Jüngel [11]. Here we approach third-order problems on general time scales, namely on any nonempty closed subset of the real line, to include the discrete, continuous, and quantum calculus as special cases. Boundary value problems on time scales that utilize both delta and nabla derivatives, such as the one here, were first introduced by Atici and Guseinov [5]. Three-point and right-focal boundary value problems, in both the continuous and discrete cases, have been addressed in [1, 2, 3, 4], by Eloe and McKelvey [9], and recently by Graef and Yang [12, 13], Sun [23], and Wong [25]. For more on existence of solutions to boundary value problems, see [7, Chapters 4 and 6-9], Davis, Erbe, and Henderson [8], Erbe and Wang [10], the text by Guo and Lakshmikantham [14], Henderson [15], Henderson and Thompson [17], Lan [19, 20], Ma and Thompson [22], and Zhang and Liu [26]. Problem (1.3), (1.4) is an extension of the continuous and discrete discussions of third-order right-focal boundary value problems to time scales, and by the addition of the nonhomogeneous nonlocal boundary conditions and the allowance of sign changes in the nonlinearity f, problem (1.1), (1.2) is introduced for the first time on any time scale, including $\mathbb{R}$, $\mathbb{Z}$, and the quantum time scale. One could also consider a third-order problem with derivatives in the order of nabla, nabla, delta, but the results would be similar; other permutations of nablas and/or deltas lead to a Green function that is less easy to calculate.

Clearly there are other approaches to the existence of positive solutions for dynamic equations on time scales than those featured in this work; for alternative approaches to the existence of solutions and multiple solutions to dynamic equations on time scales, consult, for example, Bohner and Luo [6], Henderson [16], Ma, Du, and Ge [21], and Tisdell, Drábek, and Henderson [24]. Underlying our technique, however, will be Green's function for the homogeneous, third-order, three-point boundary-value problem

$$(px^{\Delta\Delta})^{\nabla}(t) = 0, \quad t \in [t_1, t_3]_{\mathbb{T}}, \tag{1.3}$$

$$\alpha x(\rho(t_1)) - \beta x^{\Delta}(\rho(t_1)) = x^{\Delta}(t_2) = (px^{\Delta\Delta})(t_3) = 0. \tag{1.4}$$

Green's function for (1.3), (1.4) will be defined on $[\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$, nonnegative on $[\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$ contingent on the distance between boundary points, nondecreasing on $[\rho(t_1), t_2]_{\mathbb{T}}$, and nonincreasing on $[t_2, \sigma^2(t_3)]_{\mathbb{T}}$, as will be shown in the following lemmas.

Lemma 1.1. *Green's function corresponding to the problem* (1.3), (1.4) *is given by*

$$G(t,s) = \begin{cases} s\in[\rho(t_1),t_2]_{\mathbb{T}} & : \begin{cases} \int_{\rho(t_1)}^{t}\int_u^s \frac{\Delta r}{p(r)}\Delta u + \frac{\beta}{\alpha}\int_{\rho(t_1)}^{s}\frac{\Delta r}{p(r)} & : t<s \\ \int_{\rho(t_1)}^{s}\int_u^s \frac{\Delta r}{p(r)}\Delta u + \frac{\beta}{\alpha}\int_{\rho(t_1)}^{s}\frac{\Delta r}{p(r)} & : t\ge s \end{cases} \\ s\in[t_2,\sigma^2(t_3)]_{\mathbb{T}} & : \begin{cases} \int_{\rho(t_1)}^{t}\int_u^{t_2} \frac{\Delta r}{p(r)}\Delta u + \frac{\beta}{\alpha}\int_{\rho(t_1)}^{t_2}\frac{\Delta r}{p(r)} & : t<s \\ \int_{\rho(t_1)}^{t}\int_u^{t_2} \frac{\Delta r}{p(r)}\Delta u + \int_s^t\int_s^u \frac{\Delta r}{p(r)}\Delta u + \frac{\beta}{\alpha}\int_{\rho(t_1)}^{t_2}\frac{\Delta r}{p(r)} & : t\ge s. \end{cases} \end{cases} \tag{1.5}$$

Proof. We follow the approach given, for example, in Kelley and Peterson [18, Chapter 5]. As the Cauchy function $y(\cdot,s)$ satisfies the homogeneous time-scale initial-value problem

$$(py^{\Delta\Delta}(\cdot,s))^{\nabla}(t)=0,\quad y(s,s)=0,\quad y^{\Delta}(s,s)=0,\quad y^{\Delta\Delta}(\rho(s),s)=1/p(\rho(s))$$

it is easy to verify that $y(t,s)=\int_s^t\int_s^u \frac{\Delta r}{p(r)}\Delta u$. Thus the Green function takes the form

$$G(t,s) = \begin{cases} s\in[\rho(t_1),t_2]_{\mathbb{T}} & : \begin{cases} u_1(t,s) & : t<s \\ v_1(t,s) & : t\ge s \end{cases} \\ s\in[t_2,\sigma^2(t_3)]_{\mathbb{T}} & : \begin{cases} u_2(t,s) & : t<s \\ v_2(t,s) & : t\ge s, \end{cases} \end{cases}$$

where $u_i(t,s)+y(t,s)=v_i(t,s)$ and $v_i(\cdot,s)$ satisfy $(px^{\Delta\Delta})^{\nabla}(t)=0$, for $i=1,2$. Let $s\in[\rho(t_1),t_2]_{\mathbb{T}}$. Then the boundary conditions are $\alpha u_1(\rho(t_1),s)-\beta u_1^{\Delta}(\rho(t_1),s)=0$ for $t<s$ and $v_1^{\Delta}(t_2,s)=(pv_1^{\Delta\Delta}(\cdot,s))(t_3)=0$ for $s\le t$. Solving for v_1, we see that $v_1(t,s)=k(s)$ for some function k. Since $u_1=v_1-y$, $\alpha u_1(\rho(t_1),s)=\beta u_1^{\Delta}(\rho(t_1),s)$ for these s implies that

$$k(s)=\int_{\rho(t_1)}^{s}\int_u^s \frac{\Delta r}{p(r)}\Delta u+\frac{\beta}{\alpha}\int_{\rho(t_1)}^{s}\frac{\Delta r}{p(r)}.$$

Thus for $s\in[\rho(t_1),t_2]_{\mathbb{T}}$,

$$G(t,s)=\begin{cases} \int_{\rho(t_1)}^{t}\int_u^s \frac{\Delta r}{p(r)}\Delta u + \frac{\beta}{\alpha}\int_{\rho(t_1)}^{s}\frac{\Delta r}{p(r)} & : t<s \\ \int_{\rho(t_1)}^{s}\int_u^s \frac{\Delta r}{p(r)}\Delta u + \frac{\beta}{\alpha}\int_{\rho(t_1)}^{s}\frac{\Delta r}{p(r)} & : t\ge s. \end{cases}$$

Now let $s\in[t_2,\sigma^2(t_3)]_{\mathbb{T}}$, so that the boundary conditions are $\alpha u_2(\rho(t_1),s)-\beta u_2^{\Delta}(\rho(t_1),s)=u_2^{\Delta}(t_2,s)=0$ for $t<s$ and $(pv_2^{\Delta\Delta}(\cdot,s))(t_3)=0$ for $s\le t$. Clearly

$$u_2(t,s)=-q(s)\Big(\int_{\rho(t_1)}^{t}\int_u^{t_2}\frac{\Delta r}{p(r)}\Delta u+\frac{\beta}{\alpha}\int_{\rho(t_1)}^{t_2}\frac{\Delta r}{p(r)}\Big)$$

for some function q. Using the fact that $v_2=u_2+y$ and the remaining boundary condition yields $q(s)\equiv -1$ and

$$G(t,s)=\begin{cases} \int_{\rho(t_1)}^{t}\int_u^{t_2} \frac{\Delta r}{p(r)}\Delta + \frac{\beta}{\alpha}\int_{\rho(t_1)}^{t_2}\frac{\Delta r}{p(r)} & : t<s \\ \int_{\rho(t_1)}^{t}\int_u^{t_2} \frac{\Delta r}{p(r)}\Delta + \int_s^t\int_s^u \frac{\Delta r}{p(r)}\Delta u + \frac{\beta}{\alpha}\int_{\rho(t_1)}^{t_2}\frac{\Delta r}{p(r)} & : t\ge s \end{cases}$$

for $s\in[t_2,\sigma^2(t_3)]_{\mathbb{T}}$. □

Remark 1.2. As in (1.5) and the proof above, throughout the rest of the paper we take

$$u_2(t) := \int_{\rho(t_1)}^{t}\int_{u}^{t_2}\frac{\Delta r}{p(r)}\Delta u + \frac{\beta}{\alpha}\int_{\rho(t_1)}^{t_2}\frac{\Delta r}{p(r)}. \tag{1.6}$$

Lemma 1.3. *Green's function* (1.5) *corresponding to the problem* (1.3), (1.4) *satisfies*

$$0 < G(t,s) \le G(t_2,s) \le u_2(t_2)$$

for $(t,s) \in [\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}} \times (\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}$ *if and only if* (H1) *holds; that is,* $u_2(\sigma^2(t_3)) > 0$.

Proof. Fix $s \in (\rho(t_1), t_2]_{\mathbb{T}}$. Then $u_1(\rho(t_1),s) = \frac{\beta}{\alpha}\int_{\rho(t_1)}^{s}\frac{\Delta r}{p(r)} > 0$ and $u_1(\cdot,s)$ is increasing, so that $0 < u_1(t,s) \le u_1(s,s)$ for $t \in [\rho(t_1),s)_{\mathbb{T}}$. But $u_1(s,s) \equiv v_1(t,s)$ for $t \in [t_2,\sigma^2(t_3)]_{\mathbb{T}}$. It follows that $G(t_2,s) \ge G(t,s) > 0$ for $s \in (\rho(t_1),t_2]_{\mathbb{T}}$ and $t \in [\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}$. Now fix $s \in [t_2,\sigma^2(t_3)]_{\mathbb{T}}$. The branch u_2 is positive at $\rho(t_1)$, increases until t_2, and then decreases until s. We then switch to branch v_2, which continues to decrease, so that $v_2(t,s) \ge v_2(\sigma^2(t_3),s)$. As a function of s, $v_2(\sigma^2(t_3),s)$ is also decreasing, whence $v_2(t,s) \ge v_2(\sigma^2(t_3),\sigma^2(t_3)) = u_2(\sigma^2(t_3))$ for u_2 given in (1.6). Thus $G(t_2,s) \ge G(t,s)$ for $s \in [t_2,\sigma^2(t_3)]_{\mathbb{T}}$ and $t \in [\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}$ as well, and $G(t,s) > 0$ for $s \in [t_2,\sigma^2(t_3)]_{\mathbb{T}}$ and $t \in [\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}$ if and only if (H1) holds. □

Remark 1.4. If $\mathbb{T} = \mathbb{Z}$, $\alpha = 1$, $\beta = 0$, and $p(t) \equiv 1$, then the necessary and sufficient condition for the Green function to be positive is $t_2 - t_1 - 1 \ge t_3 - t_2$; see [2].

Lemma 1.5. *Assume* (H1). *For any* $(t,s) \in [\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}} \times (\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}$*, the Green function* (1.5) *corresponding to the problem* (1.3), (1.4) *satisfies, using* (1.6),

$$\frac{u_2(t)}{u_2(t_2)} \le \frac{G(t,s)}{G(t_2,s)} \le 1.$$

Proof. The right-hand inequality follows from the previous lemma. For the left-hand inequality, we proceed by analyzing branches of the Green function (1.5). For fixed $t \in [\rho(t_1),s)_{\mathbb{T}}$ and $s \in (t,t_2]_{\mathbb{T}}$,

$$\frac{G(t,s)}{G(t_2,s)} = \frac{\int_{\rho(t_1)}^{t}\int_{u}^{s}\frac{\Delta r}{p(r)}\Delta u + \frac{\beta}{\alpha}\int_{\rho(t_1)}^{s}\frac{\Delta r}{p(r)}}{\int_{\rho(t_1)}^{s}\int_{u}^{s}\frac{\Delta r}{p(r)}\Delta u + \frac{\beta}{\alpha}\int_{\rho(t_1)}^{s}\frac{\Delta r}{p(r)}} =: \phi(s).$$

Then

$$\begin{aligned}
&\phi^{\nabla}(s)\\
&= \frac{(t-s)\Big(\int_{\rho(t_1)}^{t}\int_{u}^{s}\frac{\Delta r}{p(r)}\Delta u + \frac{\beta}{\alpha}\int_{\rho(t_1)}^{s}\frac{\Delta r}{p(r)}\Big) + \Big(t-\rho(t_1)+\frac{\beta}{\alpha}\Big)\int_{t}^{s}\int_{u}^{s}\frac{\Delta r}{p(r)}\Delta u}{p^{\rho}(s)\Big(\int_{\rho(t_1)}^{s}\int_{u}^{s}\frac{\Delta r}{p(r)}\Delta u + \frac{\beta}{\alpha}\int_{\rho(t_1)}^{s}\frac{\Delta r}{p(r)}\Big)\Big(\int_{\rho(t_1)}^{\rho(s)}\int_{u}^{\rho(s)}\frac{\Delta r}{p(r)}\Delta u + \frac{\beta}{\alpha}\int_{\rho(t_1)}^{\rho(s)}\frac{\Delta r}{p(r)}\Big)}.
\end{aligned}$$

The denominator of ϕ^{∇} is clearly positive, so consider the numerator,

$$\begin{aligned}
\psi(s) &:= \psi_1(s) + \psi_2(s)\\
&:= (t-s)\int_{\rho(t_1)}^{t}\int_{u}^{s}\frac{\Delta r}{p(r)}\Delta u + (t-\rho(t_1))\int_{t}^{s}\int_{u}^{s}\frac{\Delta r}{p(r)}\Delta u\\
&\quad + \frac{\beta}{\alpha}\Big((t-s)\int_{\rho(t_1)}^{s}\frac{\Delta r}{p(r)} + \int_{t}^{s}\int_{u}^{s}\frac{\Delta r}{p(r)}\Delta u\Big).
\end{aligned}$$

Note that $\psi(t)=0$; the first part satisfies

$$\psi_1^{\nabla}(s)=\frac{(t-\rho(t_1))\nu(s)}{p^{\rho}(s)}-\int_{\rho(t_1)}^{t}\int_{u}^{s}\frac{\Delta r}{p(r)}\Delta u=-\int_{\rho(t_1)}^{t}\int_{u}^{\rho(s)}\frac{\Delta r}{p(r)}\Delta u\le 0$$

for $s\in(t,t_2]_{\mathbb{T}}$, while the second part satisfies

$$\psi_2(s)\le\frac{\beta}{\alpha}\Big((t-s)\int_{\rho(t_1)}^{s}\frac{\Delta r}{p(r)}+\int_{t}^{s}\int_{t}^{s}\frac{\Delta r}{p(r)}\Delta u\Big)=\frac{\beta}{\alpha}(t-s)\int_{\rho(t_1)}^{t}\frac{\Delta r}{p(r)}\le 0.$$

Therefore, the whole numerator satisfies $\psi(s)\le 0$, so that $\phi^{\nabla}\le 0$ and ϕ is nonincreasing as a function of s. Thus $\phi(s)\ge\phi(t_2)$ for $s\in(t,t_2]$; in other words,

$$\frac{G(t,s)}{G(t_2,s)}\ge\frac{G(t,t_2)}{G(t_2,t_2)}=\frac{u_2(t)}{u_2(t_2)}.$$

For $s\in[\rho(t_1),t_2]_{\mathbb{T}}$ and $t\in[s,\sigma^2(t_3)]_{\mathbb{T}}$,

$$\frac{G(t,s)}{G(t_2,s)}\equiv 1\ge\frac{u_2(t)}{u_2(t_2)}.$$

If $s\in[t_2,\sigma^2(t_3)]_{\mathbb{T}}$ and $t\in[\rho(t_1),s)_{\mathbb{T}}$, then

$$\frac{G(t,s)}{G(t_2,s)}=\frac{u_2(t)}{u_2(t_2)}.$$

Finally, if $s\in[t_2,\sigma^2(t_3)]_{\mathbb{T}}$ and $t\in[s,\sigma^2(t_3)]_{\mathbb{T}}$, then

$$\frac{G(t,s)}{G(t_2,s)}=\frac{u_2(t)+\int_s^t\int_s^u\frac{\Delta r}{p(r)}\Delta u}{u_2(t_2)}\ge\frac{u_2(t)}{u_2(t_2)}.$$

□

2. Exploring the nonlocal problem

In this section we turn our attention to the problem

$$(px^{\Delta\Delta})^{\nabla}(t)=\lambda y(t),\quad t\in[t_1,t_3]_{\mathbb{T}}, \tag{2.1}$$

with nonlocal boundary conditions (1.2), where y is as in (H5), and $\lambda>0$. Assume (H2) and (H3), and and use (1.6) to define

$$D:=u_2(t_2)\Big(1-\int_{\eta_1}^{\eta_2}b(t)\nabla t\Big)\Big(\int_{\xi_1}^{\xi_2}a(t)\nabla t-\alpha\Big)<0. \tag{2.2}$$

Lemma 2.1. *Assume* (H1) *through* (H5). *Then the nonhomogeneous dynamic equation* (2.1) *with boundary conditions* (1.2) *has a unique solution* x^*, *where for* $t\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}$,

$$x^*(t)=\lambda\Big(\int_{\rho(t_1)}^{t_3}G(t,s)y(s)\nabla s+A(y)u_2(t)+B(y)\left(u_2(t_2)-u_2(t)\right)\Big) \tag{2.3}$$

holds, where: $G(t,s)$ *is the Green function* (1.5) *of the boundary-value problem* (1.3), (1.4)*; and the functionals* A *and* B *are defined using* (1.6) *by*

$$A(y):=\frac{1}{D}\begin{vmatrix}\int_{\eta_1}^{\eta_2}b(t)\nabla t-1 & \int_{\xi_1}^{\xi_2}a(t)u_2(t)\nabla t+u_2(t_2)\left(\alpha-\int_{\xi_1}^{\xi_2}a(t)\nabla t\right)\\ \int_{\eta_1}^{\eta_2}b(t)\left(\int_t^{t_3}y(s)\nabla s\right)\nabla t & \int_{\xi_1}^{\xi_2}a(t)\left(\int_{\rho(t_1)}^{t_3}G(t,s)y(s)\nabla s\right)\nabla t\end{vmatrix}, \tag{2.4}$$

$$B(y) := \frac{1}{D}\begin{vmatrix} \int_{\eta_1}^{\eta_2} b(t)\nabla t - 1 & \int_{\xi_1}^{\xi_2} a(t)u_2(t)\nabla t \\ \int_{\eta_1}^{\eta_2} b(t)\left(\int_t^{t_3} y(s)\nabla s\right)\nabla t & \int_{\xi_1}^{\xi_2} a(t)\left(\int_{\rho(t_1)}^{t_3} G(t,s)y(s)\nabla s\right)\nabla t \end{vmatrix}. \tag{2.5}$$

Proof. For y as in (H5), we show that the function x^* given in (2.3) is a solution of (2.1) with conditions (1.2) only if $A(y)$ and $B(y)$ are given by (2.4) and (2.5), respectively. If x^* is a solution of (2.1), (1.2), then

$$x^*(t) = \lambda\int_{\rho(t_1)}^{t} G(t,s)y(s)\nabla s + \lambda\int_t^{t_3} G(t,s)y(s)\nabla s + Au_2(t) + B(u_2(t_2) - u_2(t))$$

for some constants A and B. Taking the delta derivative with respect to t yields

$$x^{*\Delta}(t) = \lambda\int_{\rho(t_1)}^{t} G^{\Delta}(t,s)y(s)\nabla s + \lambda\int_t^{t_3} G^{\Delta}(t,s)y(s)\nabla s + A\int_t^{t_2}\frac{\Delta r}{p(r)} - B\int_t^{t_2}\frac{\Delta r}{p(r)};$$

since p times the delta derivative of this expression is

$$(px^{*\Delta\Delta})(t) = -\lambda\int_t^{t_3} y(s)\nabla s - A + B,$$

we see that (2.1) holds. It is also clear that $x^{*\Delta}(t_2) = 0$ is satisfied. To meet the other two boundary conditions in (1.2), we must have at $\rho(t_1)$ that

$$\alpha Bu_2(t_2) = \int_{\xi_1}^{\xi_2} a(t)\Big(\lambda\int_{\rho(t_1)}^{t_3} G(t,s)y(s)\nabla s + Au_2(t) + B(u_2(t_2) - u_2(t))\Big)\nabla t, \tag{2.6}$$

while at t_3 we have

$$-A + B = \int_{\eta_1}^{\eta_2} b(t)\Big(-\lambda\int_t^{t_3} y(s)\nabla s - A + B\Big)\nabla t. \tag{2.7}$$

Combining (2.6) and (2.7), we arrive at the system of equations

$$\begin{aligned} &A\int_{\xi_1}^{\xi_2} a(t)u_2(t)\nabla t + B\Big[\int_{\xi_1}^{\xi_2} a(t)\left(u_2(t_2) - u_2(t)\right)\nabla t - \alpha u_2(t_2)\Big] \\ &= -\lambda\int_{\xi_1}^{\xi_2} a(t)\Big(\int_{\rho(t_1)}^{t_3} G(t,s)y(s)\nabla s\Big)\nabla t \end{aligned}$$

and

$$A\Big[\int_{\eta_1}^{\eta_2} b(t)\nabla t - 1\Big] + B\Big[1 - \int_{\eta_1}^{\eta_2} b(t)\nabla t\Big] = -\lambda\int_{\eta_1}^{\eta_2} b(t)\Big(\int_t^{t_3} y(s)\nabla s\Big)\nabla t.$$

The determinant of the coefficients of A and B is D, given by (2.2), which is negative, and by elementary linear algebra we verify (2.4) and (2.5) with λ factored out. Also note that $A(y) > B(y) > 0$ since $D < 0$ and

$$A(y) - B(y) = \frac{\int_{\eta_1}^{\eta_2} b(t)\Big(\int_t^{t_3} y(s)\nabla s\Big)\nabla t}{1 - \int_{\eta_1}^{\eta_2} b(t)\nabla t}.$$

□

Corollary 2.2. *Assume* (H1) *through* (H5)*. Then the unique solution x^* as in* (2.3) *of the problem* (2.1), (1.2) *satisfies $x^*(t) \ge 0$ for $t \in [\rho(t_1), \sigma^2(t_3)]$.*

Proof. From Lemma 1.3 we know that on the Green function (1.5) satisfies $G(t,s) \geq 0$. Assumption (H3) applied to (2.4) and (2.5) imply that $A(y) > B(y) > 0$. $\square$

Lemma 2.3. *Assume* (H1) *through* (H5). *Then the unique solution* x^* *as in* (2.3) *of the problem* (2.1), (1.2) *satisfies*

$$\theta\|x^*\| \leq x^*(t) \leq \lambda\theta\Theta \quad on \quad [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}},$$

where using (1.6) *we take*

$$\theta := \min\Big\{\frac{u_2(\rho(t_1))}{u_2(t_2)}, \frac{u_2(\sigma^2(t_3))}{u_2(t_2)}\Big\} \in (0,1), \quad \|x^*\| := \max_{t\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}} x^*(t) = x^*(t_2), \tag{2.8}$$

and

$$\Theta := \frac{1}{\theta} u_2(t_2)\left(1+\bar{A}\right)\int_{\rho(t_1)}^{t_3} y(s)\nabla s \tag{2.9}$$

for

$$\bar{A} := \frac{1}{D}\begin{vmatrix} \int_{\eta_1}^{\eta_2} b(t)\nabla t - 1 & \int_{\xi_1}^{\xi_2} a(t)u_2(t)\nabla t + u_2(t_2)\left(\alpha - \int_{\xi_1}^{\xi_2} a(t)\nabla t\right) \\ \int_{\eta_1}^{\eta_2} b(t)\nabla t & u_2(t_2)\int_{\xi_1}^{\xi_2} a(t)\nabla t \end{vmatrix}.$$

Proof. ¿From previous work, it is clear that for all $t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$,

$$x^*(t) \leq x^*(t_2) = \lambda\Big(\int_{\rho(t_1)}^{t_3} G(t_2,s)y(s)\nabla s + A(y)u_2(t_2)\Big).$$

For $t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$, from Lemma 1.3 and Lemma 1.5, the Green function (1.5) satisfies

$$\frac{G(t,s)}{G(t_2,s)} \geq \frac{u_2(t)}{u_2(t_2)} \geq \min\Big\{\frac{u_2(\rho(t_1))}{u_2(t_2)}, \frac{u_2(\sigma^2(t_3))}{u_2(t_2)}\Big\} = \theta \in (0,1)$$

by (H1) and (1.6), and

$$\begin{aligned} &x^*(t) \\ &= \lambda\Big(\int_{\rho(t_1)}^{t_3} \frac{G(t,s)}{G(t_2,s)} G(t_2,s)y(s)\nabla s + A(y)\frac{u_2(t)}{u_2(t_2)}u_2(t_2) + B(y)(u_2(t_2)-u_2(t))\Big) \\ &\geq \lambda\Big(\int_{\rho(t_1)}^{t_3} \theta G(t_2,s)y(s)\nabla s + A(y)\theta u_2(t_2)\Big) \\ &= \theta\|x^*\|. \end{aligned}$$

Consequently, $\theta\|x^*\| \leq x^*(t)$ for all $t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$. For D in (2.2) and $A(y)$ in (2.4),

$$\begin{aligned} &A(y) \\ &= \frac{1}{D}\begin{vmatrix} \int_{\eta_1}^{\eta_2} b(t)\nabla t - 1 & \int_{\xi_1}^{\xi_2} a(t)u_2(t)\nabla t + u_2(t_2)\left(\alpha - \int_{\xi_1}^{\xi_2} a(t)\nabla t\right) \\ \int_{\eta_1}^{\eta_2} b(t)\left(\int_t^{t_3} y(s)\nabla s\right)\nabla t & \int_{\xi_1}^{\xi_2} a(t)\left(\int_{\rho(t_1)}^{t_3} G(t,s)y(s)\nabla s\right)\nabla t \end{vmatrix} \\ &\leq \frac{1}{D}\begin{vmatrix} \int_{\eta_1}^{\eta_2} b(t)\nabla t - 1 & \int_{\xi_1}^{\xi_2} a(t)u_2(t)\nabla t + u_2(t_2)\left(\alpha - \int_{\xi_1}^{\xi_2} a(t)\nabla t\right) \\ \int_{\eta_1}^{\eta_2} b(t)\nabla t & u_2(t_2)\int_{\xi_1}^{\xi_2} a(t)\nabla t \end{vmatrix}\int_{\rho(t_1)}^{t_3} y(s)\nabla s \\ &= \bar{A}\int_{\rho(t_1)}^{t_3} y(s)\nabla s < \infty. \end{aligned}$$

As a result, for $t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$,

$$\begin{aligned} x^*(t) &\le \lambda\Big(\int_{\rho(t_1)}^{t_3} G(t_2,s)y(s)\nabla s + A(y)u_2(t_2)\Big) \\ &\le \lambda u_2(t_2)(1+\bar{A})\int_{\rho(t_1)}^{t_3} y(s)\nabla s \\ &\le \lambda\theta\Theta \end{aligned}$$

using (2.8), (2.9), and (2.10). □

3. An existence result on cones

Let $\mathcal{B}$ denote the Banach space $C[\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$ with the norm

$$\|x\| = \sup_{t\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}} |x(t)|.$$

Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{x \in \mathcal{B} : x(t) \ge \theta\|x\| \text{ on } [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}\},$$

where θ is given in (2.8). Consider the related boundary-value problem

$$\begin{gathered} (px^{\Delta\Delta})^{\nabla}(t) = \lambda f^*(t, x(t)), \quad t \in [t_1, t_3]_{\mathbb{T}}, \\ \alpha x(\rho(t_1)) - \beta x^{\Delta}(\rho(t_1)) = \int_{\xi_1}^{\xi_2} a(t)x(t)\nabla t, \\ x^{\Delta}(t_2) = 0, \quad (px^{\Delta\Delta})(t_3) = \int_{\eta_1}^{\eta_2} b(t)(px^{\Delta\Delta})(t)\nabla t, \end{gathered}$$

where

$$f^*(t, x(t)) := f(t, x^{\dagger}(t)) + y(t), \quad x^{\dagger}(t) := \max\{x(t) - x^*(t), 0\}, \tag{3.1}$$

such that x^* given in (2.3) is the solution of (2.1), (1.2), and y is from (H5).

For any fixed $x \in \mathcal{P}$, $x^{\dagger} \le x \le \|x\|$ and by (H5),

$$\begin{aligned} &\int_{\rho(t_1)}^{t_3} G(t,s)f^*(s,x(s))\nabla s \\ &\le \int_{\rho(t_1)}^{t_3} G(t_2,s)\left(z(s)h(x^{\dagger}(s)) + y(s)\right)\nabla s \\ &\le \Big(\max_{0\le\tau\le\|x\|} h(\tau) + 1\Big)\int_{\rho(t_1)}^{t_3} G(t_2,s)\left(z(s)+y(s)\right)\nabla s < \infty. \end{aligned}$$

For A in (2.4) and using (2.10), we have

$$A(z+y) \le \bar{A}\int_{\rho(t_1)}^{t_3} (z(s)+y(s))\nabla s;$$

likewise for B in (2.5) and using (H5),

$$
\begin{aligned}
&B(z+y)\\
&=\frac{1}{D}\begin{vmatrix}\int_{\eta_1}^{\eta_2} b(t)\nabla t-1 & \int_{\xi_1}^{\xi_2} a(t)u_2(t)\nabla t\\ \int_{\eta_1}^{\eta_2} b(t)\big(\int_t^{t_3}(z(s)+y(s))\nabla s\big)\nabla t & \int_{\xi_1}^{\xi_2} a(t)\big(\int_{\rho(t_1)}^{t_3} G(t,s)(z(s)+y(s))\nabla s\big)\nabla t\end{vmatrix}\\
&\le\frac{1}{D}\begin{vmatrix}\int_{\eta_1}^{\eta_2} b(t)\nabla t-1 & \int_{\xi_1}^{\xi_2} a(t)u_2(t)\nabla t\\ \int_{\eta_1}^{\eta_2} b(t)\nabla t & u_2(t_2)\int_{\xi_1}^{\xi_2} a(t)\nabla t\end{vmatrix}\int_{\rho(t_1)}^{t_3}(z(s)+y(s))\nabla s\\
&<\infty.
\end{aligned}
$$

This allows us to define for $y\in\mathcal{P}$ the operator $T:\mathcal{P}\to\mathcal{B}$ for $t\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}$ by

$$
(Tx)(t):=\lambda\Big(\int_{\rho(t_1)}^{t_3} G(t,s)f^*(s,x(s))\nabla s+A(f^*)u_2(t)+B(f^*)(u_2(t_2)-u_2(t))\Big),\tag{3.2}
$$

using (2.4), (2.5), and (3.1).

Lemma 3.1. *Assume* (H1) *through* (H5). *Then $T:\mathcal{P}\to\mathcal{P}$ is completely continuous.*

Proof. For any $x\in\mathcal{P}$, Lemmas 1.3, 1.5 and Lemma 2.3 imply that $(Tx)(t)\ge\theta\|Tx\|$ on $[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}$, so that $T(\mathcal{P})\subseteq\mathcal{P}$. By a standard application of the Arzela-Ascoli Theorem, T is completely continuous. $\square$

To establish an existence result we will employ the following fixed point theorem due to Guo and Krasnoselskii [14], and seek a fixed point of T in $\mathcal{P}$.

Theorem 3.2. *Let E be a Banach space, $P\subseteq E$ be a cone, and suppose that $\mathcal{S}_1$, $\mathcal{S}_2$ are bounded open balls of E centered at the origin with $\overline{\mathcal{S}}_1\subset\mathcal{S}_2$. Suppose further that $L:P\cap(\overline{\mathcal{S}}_2\setminus\mathcal{S}_1)\to P$ is a completely continuous operator such that either*

(i) $\|Ly\|\le\|y\|$, $y\in P\cap\partial\mathcal{S}_1$ *and* $\|Ly\|\ge\|y\|$, $y\in P\cap\partial\mathcal{S}_2$, *or*
(ii) $\|Ly\|\ge\|y\|$, $y\in P\cap\partial\mathcal{S}_1$ *and* $\|Ly\|\le\|y\|$, $y\in P\cap\partial\mathcal{S}_2$

holds. Then L has a fixed point in $P\cap(\overline{\mathcal{S}}_2\setminus\mathcal{S}_1)$.

Theorem 3.3. *Assume* (H1) *through* (H5). *Then there exists $\lambda^*>0$ such that the third-order nonlocal time scale boundary value problem* (1.1), (1.2) *has at least one positive solution in $\mathcal{P}$ for any $\lambda\in(0,\lambda^*)$.*

Proof. By Lemma 3.1, $T:\mathcal{P}\to\mathcal{P}$ given by (3.2) is completely continuous. Take $\mathcal{S}_1:=\{x\in\mathcal{B}:\|x\|<\Theta\}$ for Θ given in (2.9), and let

$$
\lambda^*:=\min\left\{1,\ \frac{\int_{\rho(t_1)}^{t_3} y(s)\nabla s}{\theta\Big(\max\limits_{0\le\tau\le\Theta} h(\tau)+1\Big)\int_{\rho(t_1)}^{t_3}(z(s)+y(s))\nabla s}\right\}.
$$

Then for any $x\in\mathcal{P}\cap\partial\mathcal{S}_1$,

$$
0\le x^\dagger(s)\le x(s)\le\|x\|=\Theta,\quad s\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}},
$$

and, for $\bar{A}$ as in the statement of Lemma 2.3,

$$\begin{aligned}(Tx)(t) &\le \lambda\Big(\int_{\rho(t_1)}^{t_3} G(t_2,s)f^*(s,x(s))\nabla s + A(f^*)u_2(t_2)\Big)\\ &\le \lambda\Big(\max_{0\le\tau\le\|x\|} h(\tau)+1\Big)\int_{\rho(t_1)}^{t_3} G(t_2,s)\,(z(s)+y(s))\,\nabla s\\ &\quad + \lambda\bar{A}\Big(\max_{0\le\tau\le\|x\|} h(\tau)+1\Big)\Big(\int_{\rho(t_1)}^{t_3}\big(z(s)+y(s)\big)\nabla s\Big)u_2(t_2)\\ &\le \lambda^* u_2(t_2)\left(1+\bar{A}\right)\Big(\max_{0\le\tau\le\|x\|} h(\tau)+1\Big)\int_{\rho(t_1)}^{t_3}(z(s)+y(s))\,\nabla s\\ &\le \Theta = \|x\|.\end{aligned}$$

Hence $\|Tx\|\le\|x\|$ for $x\in\mathcal{P}\cap\partial\mathcal{S}_1$. Pick $\Upsilon\in\mathbb{R}$ such that $\Upsilon>0$ and

$$1\le\frac{\lambda\Upsilon\theta}{\Theta+1}\int_{\xi_1}^{t_2} G(\xi_1,s)\nabla s.$$

By (H4), for any $t\in[\xi_1,t_2]_{\mathbb{T}}$, there exists a constant $K>0$ such that $f(t,y)>\Upsilon y$ for $y>K$. Pick $Q:=\max\big\{\lambda(\Theta+1),\Theta+1,\frac{K(\Theta+1)}{\theta}\big\}$. If $\mathcal{S}_2:=\{y\in\mathcal{B}:\|y\|<Q\}$, then for any $x\in\mathcal{P}\cap\partial\mathcal{S}_2$ and $t\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}$,

$$\begin{aligned}x(t)-x^*(t) &\ge x(t)-\lambda\theta\Theta\ge x(t)-\frac{\lambda\Theta}{Q}x(t)\\ &\ge\big(1-\frac{\lambda\Theta}{Q}\big)x(t)\ge\big(1-\frac{\lambda\Theta}{\lambda(\Theta+1)}\big)x(t)\\ &=\frac{x(t)}{\Theta+1}\ge 0.\end{aligned}$$

Thus

$$\min_{t\in[\xi_1,t_2]_{\mathbb{T}}}(x(t)-x^*(t))\ge\min_{t\in[\xi_1,t_2]_{\mathbb{T}}}\frac{x(t)}{\Theta+1}\ge\frac{\theta Q}{\Theta+1}\ge K,$$

so that

$$\begin{aligned}&\min_{t\in[\xi_1,t_2]_{\mathbb{T}}}(Tx)(t)\\ &=\min_{t\in[\xi_1,t_2]_{\mathbb{T}}}\lambda\Big(\int_{\rho(t_1)}^{t_3} G(t,s)f^*(s,x(s))\nabla s + A(f^*)u_2(t)+B(f^*)(u_2(t_2)-u_2(t))\Big)\\ &\ge\lambda\int_{\xi_1}^{t_2} G(\xi_1,s)f^*(s,x(s))\nabla s\\ &\ge\lambda\Upsilon\int_{\xi_1}^{t_2} G(\xi_1,s)(x(s)-x^*(s))\nabla s\ge\frac{\lambda\Upsilon\theta Q}{\Theta+1}\int_{\xi_1}^{t_2} G(\xi_1,s)\nabla s\\ &=\frac{\lambda\Upsilon\theta\|x\|}{\Theta+1}\int_{\xi_1}^{t_2} G(\xi_1,s)\nabla s\ge\|x\|.\end{aligned}$$

Hence for $x\in\mathcal{P}\cap\partial\mathcal{S}_2$ we have $\|Tx\|\ge\|x\|$. By Theorem 3.2, T has a fixed point x such that $\Theta\le\|x\|\le Q$. But then

$$x(t)-x^*(t)\ge\theta\Theta-\lambda\theta\Theta\ge(1-\lambda)\theta\Theta\ge 0.$$

As a consequence, this x solves the boundary-value problem

$$\begin{gathered}
\left(px^{\Delta\Delta}\right)^{\nabla}(t)=\lambda(f(t,x(t)-x^*(t))+y(t)),\quad t\in[t_1,t_3]_{\mathbb{T}},\\
\alpha x(\rho(t_1))-\beta x^{\Delta}(\rho(t_1))=\int_{\xi_1}^{\xi_2}a(t)x(t)\nabla t,\\
x^{\Delta}(t_2)=0,\quad (px^{\Delta\Delta})(t_3)=\int_{\eta_1}^{\eta_2}b(t)(px^{\Delta\Delta})(t)\nabla t.
\end{gathered}$$

Now set $X(t):=x(t)-x^*(t)$ for x^* given in (2.3). Then $\left(px^{\Delta\Delta}\right)^{\nabla}=\left(pX^{\Delta\Delta}\right)^{\nabla}+\left(px^{*\Delta\Delta}\right)^{\nabla}$. As x^* is the solution of (2.1), (1.2), we see that

$$\begin{gathered}
\left(pX^{\Delta\Delta}\right)^{\nabla}(t)=\lambda f(t,X(t)),\quad t\in[t_1,t_3]_{\mathbb{T}},\\
\alpha X(\rho(t_1))-\beta X^{\Delta}(\rho(t_1))=\int_{\xi_1}^{\xi_2}a(t)X(t)\nabla t,\\
X^{\Delta}(t_2)=0,\quad \left(pX^{\Delta\Delta}\right)(t_3)=\int_{\eta_1}^{\eta_2}b(t)\left(pX^{\Delta\Delta}\right)(t)\nabla t,
\end{gathered}$$

in other words, X is a positive solution of the third-order nonlocal time scale boundary value problem (1.1), (1.2). $\square$

As remarked in the Introduction, the results in this paper are new for ordinary differential equations (when $\mathbb{T}=\mathbb{R}$) and for difference equations (when $\mathbb{T}=\mathbb{Z}$).

We now provide an example to illustrate that conditions (H1)–(H5) are naturally satisfied.

Example 3.4. Consider for $\mathbb{T}=\mathbb{R}$ and the following choices: $t_1=0$, $t_2=1/2$, $t_3=1$; $p=1$; $\alpha=1=\beta$; $f(t,x)=t+x^2$; $\xi_1=1/8$, $\xi_2=1/4$; $\eta_1=5/6$, $\eta_2=7/8$; $a(t)=t=b(t)$. Then, for $h(x):=1+x^2$ with $y=1$ and $z=1$, the boundary value problem (1.1), (1.2) has at least one positive solution in $\mathcal{P}$ for any $\lambda\in(0,\lambda^*)$, where $\lambda^*\approx 0.232513$.

With these choices, (1.1), (1.2) reduces to a third-order BVP involving an ordinary differential equation. It is not difficult to verify that conditions (H1)–(H5) are satisfied. In particular, note that

$$\lambda^*=\min\left\{1,\frac{5}{8(2+\Theta^2)}\right\}=\frac{5}{8(2+\Theta^2)}\approx 0.232513.$$

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