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EXISTENCE OF SOLUTIONS FOR A SECOND ORDER ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATION WITH STATE-DEPENDENT DELAY

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Abstract. In this paper we study the existence of mild solutions for abstract partial functional differential equation with state-dependent delay.

1. Introduction

In this note we study the existence of mild solutions for a second order abstract Cauchy problem with state dependent delay described in the form

$$x''(t) = Ax(t) + f(t, x_{\rho(t,x_t)}), \quad t \in I = [0, a],\ x_0 = \varphi \in \mathcal{B}, \tag{1.1}$$

$$x'(0) = \zeta_0 \in X, \tag{1.2}$$

where A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operator $(C(t))_{t\in\mathbb{R}}$ defined on a Banach space $(X, \|\cdot\|)$; the function $x_s : (-\infty, 0] \to X$, $x_s(\theta) = x(s + \theta)$, belongs to some abstract phase space $\mathcal{B}$ described axiomatically and $f : I \times \mathcal{B} \to X$, $\rho : I \times \mathcal{B} \to (-\infty, a]$ are appropriate functions.

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last years. The literature devoted to this subject is concerned fundamentally with first order functional differential equations for which the state belong to some finite dimensional space, see among another works, [1, 2, 3, 4, 5, 8, 10, 11, 12, 13, 19, 24, 23]. The problem of the existence of solutions for first order partial functional differential equations with state-dependent delay have been treated in the literature recently in [14, 15, 16]. To the best of our knowledge, the existence of solutions for second order abstract partial functional differential equations with state-dependent delay is an untreated topic in the literature and this fact is the main motivation of the present work.

2. Preliminaries

In this section, we review some basic concepts, notations and properties needed to establish our results. Throughout this paper, A is the infinitesimal generator of

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a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators on the Banach space $(X, \|\cdot\|)$. We denote by $(S(t))_{t\in\mathbb{R}}$ the associated sine function which is defined by $S(t)x = \int_0^t C(s)xds$, for $x \in X$, and $t \in \mathbb{R}$. In the sequel, N and $\tilde{N}$ are positive constants such that $\|C(t)\| \le N$ and $\|S(t)\| \le \tilde{N}$, for every $t \in I$.

In this paper, $[D(A)]$ represents the domain of A endowed with the graph norm given by $\|x\|_A = \|x\| + \|Ax\|$, $x \in D(A)$, while E stands for the space formed by the vectors $x \in X$ for which $C(\cdot)x$ is of class C^1 on $\mathbb{R}$. We know from Kisińsky [18], that E endowed with the norm

$$\|x\|_E = \|x\| + \sup_{0\le t\le 1} \|AS(t)x\|, \quad x \in E, \tag{2.1}$$

is a Banach space. The operator-valued function

$$\mathcal{H}(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$$

is a strongly continuous group of bounded linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. It follows from this that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0$, as $t \to 0$, for each $x \in E$. Furthermore, if $x : [0, \infty) \to X$ is locally integrable, then $y(t) = \int_0^t S(t-s)x(s)ds$ defines an E-valued continuous function. This assertion is a consequence of the fact that

$$\int_0^t \mathcal{H}(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)\,ds & \int_0^t C(t-s)x(s)\,ds \end{bmatrix}$$

defines an $E \times X$-valued continuous function. In addition, it follows from the definition of the norm in E that a function $u : I \to E$ is continuous if, and only if, is continuous with respect to the norm in X and the set of functions $\{AS(t)u : t \in [0, 1]\}$ is an equicontinuous subset of $C(I, X)$.

The existence of solutions for the second-order abstract Cauchy problem

$$x''(t) = Ax(t) + h(t), \quad t \in I, \tag{2.2}$$

$$x(0) = w, \quad x'(0) = z, \tag{2.3}$$

where $h : I \to X$ is an integrable function, is studied in [22]. Similarly, the existence of solutions of semi-linear second-order abstract Cauchy problems has been treated in [21]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)w + S(t)z + \int_0^t S(t-s)h(s)ds, \quad t \in I, \tag{2.4}$$

is called a mild solution of (2.2)-(2.3), and that when $w \in E$ the function $x(\cdot)$ is of class C^1 on I and

$$x'(t) = AS(t)w + C(t)z + \int_0^t C(t-s)h(s)\,ds, \ t \in I. \tag{2.5}$$

For additional details on the cosine function theory, we refer the reader to [6, 22, 21].

In this work we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar at those introduced in [17]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and satisfying the following asumptions:

- (A1) If $x : (-\infty, b] \to X$, $b > 0$, is continuous on $[0, b]$ and $x_0 \in \mathcal{B}$, then for every $t \in [0, b]$ the following conditions hold:
  - (a) x_t is in $\mathcal{B}$.
  - (b) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$.
  - (c) $\|x_t\|_{\mathcal{B}} \leq M(t)\|x_0\|_{\mathcal{B}} + K(t)\sup\{\|x(s)\| : 0 \leq s \leq t\}$,

  where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.
- (A2) For the function x in (A1), x_t is a $\mathcal{B}$-valued continuous function on $[0, b]$.
- (B1) The space $\mathcal{B}$ is complete.

Example 2.1 (The phase space $\mathbf{C_r} \times \mathbf{L^p}(\mathbf{g}; \mathbf{X})$)**.** Let $g : (-\infty, -r) \to \mathbb{R}$ be a positive Lebesgue integrable function and assume that there exists a non-negative and locally bounded function γ on $(-\infty, 0]$ such that $g(\xi + \theta) \leq \gamma(\xi)g(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(g; X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable and $g\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(g : X)$ is defined by

$$\|\varphi\|_{\mathcal{B}} := \sup\{\|\varphi(\theta)\| : -r \leq \theta \leq 0\} + \Big(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p d\theta\Big)^{1/p}.$$

Assume that $g(\cdot)$ verifies the conditions (g-5), (g-6) and (g-7) in the nomenclature of [17]. In this case, $\mathcal{B} = C_r \times L^p(g; X)$ verifies assumptions (A1), (A2), (B1) see [17, Theorem 1.3.8] for details. Moreover, when $r = 0$ and $p = 2$ we have that $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + \big(\int_{-t}^0 g(\theta)\, d\theta\big)^{1/2}$ for $t \geq 0$.

Remark 2.2. Let $\varphi \in \mathcal{B}$ and $t \leq 0$. The notation φ_t represents the function defined by $\varphi_t(\theta) = \varphi(t + \theta)$. Consequently, if the function x in axiom (A1) is such that $x_0 = \varphi$, then $x_t = \varphi_t$. We observe that φ_t is well defined for every $t < 0$ since the domain of φ is $(-\infty, 0]$. We also note that, in general, $\varphi_t \notin \mathcal{B}$; consider, for example, the characteristic function $\mathcal{X}_{[\mu,0]}$, $\mu < -r < 0$, in the space $C_r \times L^p(g; X)$.

Some of our results will proved using the following well know result.

Theorem 2.3 (Leray Schauder Alternative [7, Theorem 6.5.4])**.** *Let D be a convex subset of a Banach space X and assume that $0 \in D$. Let $G : D \to D$ be a completely continuous map. Then the map G has a fixed point in D or the set $\{x \in D : x = \lambda G(x), 0 < \lambda < 1\}$ is unbounded.*

The terminology and notation are those generally used in functional analysis. In particular, for Banach spaces Z, W, the notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W and we abbreviate this notation to $\mathcal{L}(Z)$ when $Z = W$. Moreover $B_r(x, Z)$ denotes the closed ball with center at x and radius $r > 0$ in Z and, for a bounded function $x : [0, a] \to X$ and $0 \leq t \leq a$ we employ the notation $\|x\|_t$ for

$$\|x\|_t = \sup\{\|x(s)\| : s \in [0, t]\}. \tag{2.6}$$

This paper has four sections. In the next section we establish the existence of mild solutions for the abstract Cauchy problem (1.1)-(1.2). In section 4 some applications are considered.

3. Existence Results

In this section we establish the existence of mild solutions for the abstract Cauchy problem (1.1)-(1.2). To prove our results, we assume that $\rho : I \times \mathcal{B} \to (-\infty, a]$ is a continuous function and that the following conditions are verified.

- (H1) The function $f : I \times \mathcal{B} \to X$ satisfies the following properties.
  - (a) The function $f(\cdot, \psi) : I \to X$ is strongly measurable for every $\psi \in \mathcal{B}$.
  - (b) The function $f(t, \cdot) : \mathcal{B} \to X$ is continuous for each $t \in I$.
  - (c) There exist an integrable function $m : I \to [0, \infty)$ and a continuous nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that
$$\|f(t, \psi)\| \leq m(t)W(\|\psi\|_{\mathcal{B}}), \qquad (t, \psi) \in I \times \mathcal{B}. \tag{3.1}$$
- (H2) The function $t \to \varphi_t$ is well defined and continuous from the set $\mathcal{R}(\rho^-) = \{\rho(s, \psi) : (s, \psi) \in I \times \mathcal{B},\ \rho(s, \psi) \leq 0\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho) \to (0, \infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \leq J^\varphi(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho)$.

Remark 3.1. The condition (H2) is frequently verify by functions continuous and bounded. In fact, if $\mathcal{B}$ verifies axiom C_2 in the nomenclature of [17], then there exists $\mathrm{L} > 0$ such that $\|\varphi\|_{\mathcal{B}} \leq \mathrm{L} \sup_{\theta \leq 0} \|\varphi(\theta)\|$ for every $\varphi \in \mathcal{B}$ continuous and bounded, see [17, Proposition 7.1.1] for details. Consequently,
$$\|\varphi_t\|_{\mathcal{B}} \leq L \frac{\sup_{\theta \leq 0} \|\varphi(\theta)\|}{\|\varphi\|_{\mathcal{B}}} \|\varphi\|_{\mathcal{B}}$$
for every continuous and bounded function $\varphi \in \mathcal{B} \setminus \{0\}$ and every $t \leq 0$. We also observe that the space $C_r \times L^p(g; X)$ verifies axiom C_2, see [17, p.10] for details.

Motivated by (2.4) we introduce the following concept of mild solutions for the system (1.1)-(1.2).

Definition 3.2. *A function $x : (-\infty, a] \to X$ is called a mild solution of the abstract Cauchy problem* (1.1)-(1.2) *if $x_0 = \varphi$, $x_{\rho(s, x_s)} \in \mathcal{B}$ for every $s \in I$ and*
$$x(t) = C(t)\varphi(0) + S(t)\zeta_0 + \int_0^t S(t - s) f(s, x_{\rho(s, x_s)}) ds, \quad t \in I.$$

In the rest of this paper, M_a and K_a are the constants defined by $M_a = \sup_{t \in I} M(t)$ and $K_a = \sup_{t \in I} K(t)$.

Lemma 3.3 ([15, Lemma 2.1])**.** *Let $x : (-\infty, a] \to X$ be a function such that $x_0 = \varphi$ and $x|_{[0,a]} \in \mathcal{PC}$. Then*
$$\|x_s\|_{\mathcal{B}} \leq (M_a + \widetilde{J}^\varphi)\|\varphi\|_{\mathcal{B}} + K_a \sup\{\|x(\theta)\|; \theta \in [0, \max\{0, s\}]\},$$
$s \in \mathcal{R}(\rho^-) \cup I$, where $\widetilde{J}^\varphi = \sup_{t \in \mathcal{R}(\rho^-)} J^\varphi(t)$.

Now, we can prove our first existence result.

Theorem 3.4. *Let conditions (H1), (H2) hold and assume that $S(t)$ is compact for every $t \in \mathbb{R}$. If*
$$\widetilde{N} K_a \liminf_{\xi \to \infty^+} \frac{W(\xi)}{\xi} \int_0^a m(s) ds < 1,$$
then there exists a mild solution $u(\cdot)$ of (1.1)-(1.2)*. Moreover, if $\varphi(0) \in E$ then $u \in C^1(I, X)$ and condition* (1.2) *is verified.*

Proof. On the space $Y = \{u \in C(I,X) : u(0) = \varphi(0)\}$ endowed with the uniform convergence topology, we define the operator $\Gamma : Y \to Y$ by

$$\Gamma x(t) = C(t)\varphi(0) + S(t)\zeta_0 + \int_0^t S(t-s)f(s, \bar{x}_{\rho(s,\bar{x}_s)})ds, \quad t \in I, \tag{3.2}$$

where $\bar{x} : (-\infty, a] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on I. From assumption (A1) and our assumptions on φ, we infer that Γx is well defined and continuous.

Let $\bar{\varphi} : (-\infty, a] \to X$ be the extension of φ to $(-\infty, a]$ such that $\bar{\varphi}(\theta) = \varphi(0)$ on I and $\widetilde{J}^{\varphi} = \sup\{J^{\varphi}(s) : s \in \mathcal{R}(\rho^-)\}$. We claim that there exists $r > 0$ such that $\Gamma(B_r(\bar{\varphi}|_I, Y)) \subseteq B_r(\bar{\varphi}|_I, Y)$. If this property is false, then for every $r > 0$ there exist $x^r \in B_r(\bar{\varphi}|_I, Y)$ and $t^r \in I$ such that $r < \|\Gamma x^r(t^r) - \varphi(0)\|$. By using Lemma 3.3 we find that

$$\begin{aligned}
r &< \|\Gamma x^r(t^r) - \varphi(0)\| \\
&\le \|C(t^r)\varphi(0) - \varphi(0)\| + \|S(t)\zeta_0\| + \int_0^{t^r} \|S(t^r - s)\| \|f(s, \overline{x^r}_{\rho(s,(\overline{x^r})_s)})\| ds \\
&\le H(N+1)\|\varphi\|_{\mathcal{B}} + \widetilde{N}\|\zeta_0\| + \widetilde{N} \int_0^{t^r} m(s) W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}}) ds \\
&\le H(N+1)\|\varphi\|_{\mathcal{B}} + \widetilde{N}\|\zeta_0\| + \widetilde{N} \int_0^{t^r} m(s) W\Big((M_a + \widetilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a \|\overline{x^r}\|_a\Big) ds \\
&\le H(N+1)\|\varphi\|_{\mathcal{B}} + \widetilde{N}\|\zeta_0\| \\
&\quad + \widetilde{N} W\Big((M_a + \widetilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a(r + \|\varphi(0)\|)\Big) \int_0^a m(s) ds,
\end{aligned}$$

and hence

$$1 \le \widetilde{N} K_a \liminf_{\xi \to \infty} \frac{W(\xi)}{\xi} \int_0^a m(s) ds,$$

which is contrary to our assumption.

Let $r > 0$ be such that $\Gamma(B_r(\bar{\varphi}|_I, Y)) \subseteq B_r(\bar{\varphi}|_I, Y)$. Next, we will prove that Γ is completely continuous on $B_r(\bar{\varphi}|_I, Y)$. In the sequel, r^*, r^{**} are the numbers defined by $r^* := (M_a + \widetilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a(r + \|\varphi(0)\|)$ and $r^{**} := W(r^*) \int_0^a m(s)ds$.

Step 1 The set $\Gamma(B_r(\bar{\varphi}|_I, Y)(t) = \{\Gamma x(t) : x \in B_r(\bar{\varphi}|_I, Y)\}$ is relatively compact in X for all $t \in I$.

The case $t = 0$ is obvious. Let $0 < \varepsilon < t \le a$. Since the function $t \to S(t)$ is Lipschitz, we can select points $0 = t_1 < t_2 \cdots < t_n = t$ such that $\|S(s) - S(s')\| \le \varepsilon$, if $s, s' \in [t_i, t_{i+1}]$ for some $i = 1, 2, \dots, n-1$. If $x \in B_r(\bar{\varphi}|_I, Y)$, from Lemma 3.3 follows that $\|\bar{x}_{\rho(t,\bar{x}_t)}\|_{\mathcal{B}} \le r^*$ and hence

$$\| \int_0^{\tau} f(s, \bar{x}_{\rho(s,\bar{x}_s)}) ds\| \le W(r^*) \int_0^a m(s) ds = r^{**}, \quad \tau \in I. \tag{3.3}$$

Now, from (3.3) we find that

$$\begin{aligned}\Gamma x(t) &= C(t)\varphi(0) + S(t)\zeta_0 + \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}} (S(s) - S(t_i))f(t-s, \bar{x}_{\rho(t-s,\bar{x}_{t-s})})ds \\ &\quad + \sum_{i=1}^{n-1} S(t_i)\int_{t_i}^{t_{i+1}} f(t-s, \bar{x}_{\rho(t-s,\bar{x}_{t-s})})ds \\ &\in \{C(t)\varphi(0) + S(t)\zeta_0\} + \mathcal{C}_\varepsilon + \sum_{i=1}^{n-1} S(t_i)B_{r^{**}}(0,X).\end{aligned}$$

Thus,

$$\Gamma(B_r(\bar{\varphi}|_I, Y)(t) \subseteq \mathcal{C}_\varepsilon + \mathcal{K}_\varepsilon,$$

where $\mathcal{K}_\varepsilon$ is compact and $\mathrm{diam}(\mathcal{C}_\varepsilon) \leq \varepsilon r^{**}$, which permit us concluding that the set $\Gamma(B_r(\bar{\varphi}|_I, Y))(t)$ is relatively compact in X since ε is arbitrary.

Step 2 The set of functions $\Gamma(B_r(\bar{\varphi}|_I, Y))$ is equicontinuous on I.

Let $0 < \varepsilon < t < a$ and $\delta > 0$ such that $\|S(s)x - S(s')x\| < \varepsilon$, for every $s, s' \in I$ with $|s - s'| \leq \delta$. For $x \in B_r(\bar{\varphi}|_I, Y)$ and $0 < |h| < \delta$ such that $t + h \in I$ we get

$$\begin{aligned}\|\Gamma x(t+h) - \Gamma x(t)\| &\leq \|(C(t+h) - C(t))\varphi(0)\| + \varepsilon\|\zeta_0\| + \widetilde{N}W(r^*)\int_t^{t+h} m(s)ds \\ &\quad + W(r^*)\int_0^t \|(S(t+h-s) - S(t-s))\| m(s)ds \\ &\leq \|(C(t+h) - C(t))\varphi(0)\| + \varepsilon\|\zeta_0\| + \widetilde{N}W(r^*)\int_t^{t+h} m(s)ds \\ &\quad + W(r^*)\varepsilon\int_0^a m(s)ds,\end{aligned}$$

which proves that $\Gamma(B_r(\bar{\varphi}|_I, Y))$ is equicontinuous on I.

Proceeding as in the proof of [15, Theorem 2.2] we can prove that Γ is continuous. Thus, Γ is completely continuous. Now, from the Schauder Fixed Point Theorem we infer the existence of a mild solution $u(\cdot)$ for (1.1)-(1.2). The assertion concerning the regularity of $u(\cdot)$ follows directly from the properties of the space E. The proof is complete. □

Theorem 3.5. *Let conditions (H1), (H2) be satisfied. Suppose that $S(t)$ is compact for every $t \in \mathbb{R}$, $\rho(t,\psi) \leq t$ for every $(t,\psi) \in I \times \mathcal{B}$ and*

$$K_a\widetilde{N}\int_0^a m(s)ds < \int_C^\infty \frac{ds}{W(s)},$$

where $C = (K_aNH + M_a + \widetilde{J}^\varphi)\|\varphi\|_{\mathcal{B}} + K_a\widetilde{N}\|\zeta_0\|$ and $\widetilde{J}^\varphi = \sup_{t\in\mathcal{R}(\rho^-)} J^\varphi(t)$. Then there exists a mild solution of (1.1)-(1.2). If in addition, $\varphi(0) \in E$, then $u \in C^1(I,X)$ and condition (1.2) is verified.

Proof. For $u \in Y = C(I,X)$ we define Γu by (3.2). In order to use Theorem 2.3, next we will shall *a priori* estimates for the solutions of the integral equation

$z = \lambda\Gamma z$, $\lambda \in (0,1)$. If $x^{\lambda} = \lambda\Gamma x^{\lambda}$, $\lambda \in (0,1)$, from Lemma 3.3 we have that

$$\begin{aligned} \|x^{\lambda}(t)\| &\leq NH\|\varphi\|_{\mathcal{B}} + \widetilde{N}\|\zeta_0\| + \int_0^t \widetilde{N}\|f(s, \overline{x^{\lambda}}_{\rho(s,\overline{(x^{\lambda})}_s)})\| ds \\ &\leq NH\|\varphi\|_{\mathcal{B}} + \widetilde{N}\|\zeta_0\| \\ &\quad + \widetilde{N}\int_0^t m(s)W\Big((M_a + \widetilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a\|x^{\lambda}\|_{\max\{0,\rho(s,\overline{(x^{\lambda})}_s)\}}\Big) ds \\ &\leq NH\|\varphi\|_{\mathcal{B}} + \widetilde{N}\|\zeta_0\| + \widetilde{N}\int_0^t m(s)W\Big((M_a + \widetilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a\|x^{\lambda}\|_s\Big) ds, \end{aligned}$$

since $\rho(s,\overline{(x^{\lambda})}_s) \leq s$ for all $s \in I$. Defining $\xi^{\lambda}(t) = (M_a + \widetilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a\|x^{\lambda}\|_t$, we obtain

$$\xi^{\lambda}(t) \leq (K_a NH + M_a + \widetilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a\widetilde{N}\|\zeta_0\| + K_a\widetilde{N}\int_0^t m(s)W(\xi^{\lambda}(s))ds. \quad (3.4)$$

Denoting by $\beta_{\lambda}(t)$ the right-hand side of (3.4), follows that

$$\beta_{\lambda}'(t) \leq K_a\widetilde{N}m(t)W(\beta_{\lambda}(t))$$

and hence

$$\int_{\beta_{\lambda}(0)=C}^{\beta_{\lambda}(t)} \frac{ds}{W(s)} \leq K_a\widetilde{N}\int_0^a m(s)ds < \int_C^{\infty} \frac{ds}{W(s)},$$

which implies that the set of functions $\{\beta_{\lambda}(\cdot) : \lambda \in (0,1)\}$ is bounded in $C(I : \mathbb{R})$. This prove that $\{x^{\lambda}(\cdot) : \lambda \in (0,1)\}$ is also bounded in $C(I, X)$.

Arguing as in the proof of Theorem 3.4 we can prove that $\Gamma(\cdot)$ is completely continuous, and from Theorem 2.3 we conclude that there exists a mild solution $u(\cdot)$ for (1.1)-(1.2). Finally, it is clear from the preliminaries that $u(\cdot)$ is a function in $C^1(I, X)$ which verifies (1.2) when $\varphi(0) \in E$. The proof is finished. □

4. Examples

In this section we consider some applications of our abstract results.

The ordinary case. If $X = \mathbb{R}^k$, our results are easily applicable. In fact, in this case the operator A is a matrix of order $n \times n$ which generates the cosine function $C(t) = \cosh(tA^{1/2}) = \sum_{n=1}^{\infty} \frac{t^{2n}}{2n!}A^n$ with associated sine function $S(t) = A^{-\frac{1}{2}}\sinh(tA^{1/2}) = \sum_{n=1}^{\infty} \frac{t^{2n+2}}{(2n+1)!}A^n$. We note that the expressions $\cosh(tA^{1/2})$ and $\sinh(t\|A\|^{1/2})$ are purely symbolic and do not assume the existence of the square roots of A. It is easy to see that $C(t), S(t)$, $t \in \mathbb{R}$, are compact operators and that $\|C(t)\| \leq \cosh(a\|A\|^{1/2})$ and $\|S(t)\| \leq \|A\|^{1/2}\sinh(a\|A\|^{1/2})$ for all $t \in \mathbb{R}$. The next result is a consequence of Theorems 3.4 and 3.4.

Proposition 4.1. *Assume conditions (H1), (H2). If any of the following conditions is verified,*

(a) $K_a\|A\|^{1/2}\sinh(a\|A\|^{1/2}) \liminf_{\xi\to\infty^+} \frac{W(\xi)}{\xi}\int_0^a m(s)ds < 1$;

(b) $\rho(t,\psi) \leq t$ *for all* $(t,\psi) \in I \times \mathcal{B}$ *and*

$$K_a\|A\|^{1/2}\sinh(a\|A\|^{1/2})\int_0^a m(s)ds < \int_C^{\infty}\frac{ds}{W(s)},$$

where

$$C = (K_a\cosh(a\|A\|^{1/2})H + \widetilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_a\|A\|^{1/2}\sinh(a\|A\|^{1/2})\|\zeta_0\|;$$

then there exists a mild solution of (1.1)-(1.2).

A partial differential equation with state dependent delay. To complete this section, we discuss the existence of solutions for the partial differential system

$$\begin{aligned}&\frac{\partial^2 u(t,\xi)}{\partial^2 t}\\&=\frac{\partial^2 u(t,\xi)}{\partial\xi^2}+\int_{-\infty}^{t}a_1(s-t)u(s-\rho_1(t)\rho_2(\int_0^{\pi}a_2(\theta)|u(t,\theta)|^2d\theta),\xi)ds\end{aligned}\tag{4.1}$$

for $t\in I=[0,a],\xi\in[0,\pi]$, subject to the initial conditions

$$u(t,0)=u(t,\pi)=0,\quad t\geq 0,\tag{4.2}$$

$$u(\tau,\xi)=\varphi(\tau,\xi),\quad \tau\leq 0,\ 0\leq\xi\leq\pi.\tag{4.3}$$

To apply our abstract results, we consider the spaces $X=L^2([0,\pi])$; $\mathcal{B}=C_0\times L^2(g,X)$ and the operator $Af=f''$ with domain

$$D(A)=\{x\in X: x''\in X,\, x(0)=x(\pi)=0\}.$$

It is well-known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$ on X. Furthermore, A has a discrete spectrum, the eigenvalues are $-n^2$, $n\in\mathbb{N}$, with corresponding eigenvectors $z_n(\theta)=\left(\frac{2}{\pi}\right)^{1/2}\sin(n\theta)$. In addition, the following properties hold:

(a) The set $\{z_n:n\in\mathbb{N}\}$ is an orthonormal basis of X.

(b) For $x\in X$, $C(t)x=\sum_{n=1}^{\infty}\cos(nt)(x,z_n)z_n$. From this expression, it follows that $S(t)x=\sum_{n=1}^{\infty}\frac{\sin(nt)}{n}(x,z_n)z_n$, $\|C(t)\|=\|S(t)\|\leq 1$ for all $t\in\mathbb{R}$ and that $S(t)$ is compact for every $t\in\mathbb{R}$.

(c) If Φ is the group of translations on X defined by $\Phi(t)x(\xi)=\tilde{x}(\xi+t)$, where $\tilde{x}$ is the extension of x with period 2π, then $C(t)=\frac{1}{2}(\Phi(t)+\Phi(-t))$ and $A=B^2$, where B is the generator of Φ and

$$E=\{x\in H^1(0,\pi):x(0)=x(\pi)=0\},$$

see [6] for details.

Assume that $\varphi\in\mathcal{B}$, the functions $a_i:\mathbb{R}\to\mathbb{R}$, $\rho_i:[0,\infty)\to[0,\infty)$, $i=1,2$, are continuous, $a_2(t)\geq 0$ for all $t\geq 0$ and $L_1=(\int_0^{\infty}\frac{a_1^2(s)}{g(s)}ds)^{1/2}<\infty$. Under these conditions, we can define the operators $f:I\times\mathcal{B}\to X$, $\rho:I\times\mathcal{B}\to\mathbb{R}$ by

$$f(t,\psi)(\xi)=\int_{-\infty}^{0}a_1(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi)=s-\rho_1(s)\rho_2\Big(\int_0^{\pi}a_2(\theta)|\psi(0,\xi)\,|^2\,d\theta\Big),$$

and transform system (4.1)-(4.3) into the abstract Cauchy problem (1.1)-(1.2). Moreover, f is a continuous linear operator with $\|f\|\leq L_1$, ρ is continuous and $\rho(t,\psi)\leq s$ for every $s\in[0,a]$. The next results are consequence of Theorem 3.5 and Remark 3.1.

Proposition 4.2. *Assume that φ satisfies (H2). Then there exists a mild solution of* (4.1)–(4.3).

Corollary 4.3. *If φ is continuous and bounded, then there exists a mild solution of* (4.1)–(4.3).

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