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NONLOCAL CAUCHY PROBLEM FOR QUASILINEAR INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. The aim of this paper is to prove the existence of mild solutions of the nonlocal Cauchy problem for a nonlinear integrodifferential equation. The results are established by using the method of semigroup and the Schaefer theorem.

1. Introduction

The purpose of this paper is to study the existence of mild solution of the following nonlinear integrodifferential equation with nonlocal condition

$$\frac{dy(t)}{dt} = A(t,y)y + \int_0^t f(t,s,y(s))ds, \quad t \in J = [0,b], \tag{1.1}$$

$$y(0) + g(y) = y_0, \tag{1.2}$$

where $f : \Delta \times E \to E$ and $A : J \times E \to E$ are continuous functions, $g : C(J,E) \to E$, $y_0 \in E$ and E is a real Banach space with the norm $\| \cdot \|$. Here $\Delta = \{(t,s) : 0 \leq s \leq t \leq b\}$.

Such problems with the classical initial condition or nonlinear boundary conditions have been studied by Conti [8], Conti and Iannaccci [9], Kartsatos [11], Anichini [1], Anichini and Conti [2] and Marino and Pietramala [12].

The nonlocal condition, which is a generalization of the classical initial condition, was motivated by physical problems. The pioneering work on nonlocal conditions is due to Byszewski [6]. In the few past years several papers have been devoted to studying the existence of solutions of differential equations with nonlocal conditions. Among others, we refer to the papers of Balachandran and Chandrasekarn [4], Balachandran and Illamaran [3], Byszewski [6, 7] and Ntouyas and Tsamatos [13]. The results generalise [5, Theorem 3.1].

In this paper we study the existence of solutions for the problem (1.1)-(1.2)) by using the classical fixed point theorem for compact maps due to Schaefer [14].

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2. Preliminaries and basic hypothesis

In the remainder of the paper $C(J,E)$ is the Banach space of continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{\|y(t)\| : t \in J\}$$

and $B(E)$ denotes the Banach space of bounded linear operators from E into E with the norm

$$\|N\|_{B(E)} := \sup\{\|Ny\| : \|y\| = 1\}.$$

The following lemmas are crucial in the proof of our main theorem.

Lemma 2.1 ([10, p. 36])**.** *Suppose that $\phi_1, \phi_2 \in C(J,R)$, $\phi_3 \in L^1(J,R)$, $\phi_3(t) \geq 0$ a.e. on J and $\phi_1(t) \leq \phi_2(t) + \int_0^t \phi_3(s)\phi_1(s)ds$. Then*

$$\phi_1(t) \leq \phi_2(t) + \int_0^t \phi_3(s)\phi_2(s) \times \exp(\int_s^t \phi_3(\tau)d\tau)ds.$$

Lemma 2.2 ([14] and [15, p. 29])**.** *Let X be a Banach space and let $N : X \to X$ be a continuous compact map. If the set*

$$\Omega := \{y \in X : \lambda y = N(y) \text{ for some } \lambda \geq 1\}$$

is bounded, then N has a fixed point.

Let us list the following hypotheseses:

- (H1) $A : J \times E \to B(E)$ is a continuous function such that for all $r > 0$, there exists $r_1 = r_1(r) > 0$ such that $\|v\| \leq r$ implies $\|A(t,v)\|_{B(E)} \leq r_1$, for all $v \in E$.
- (H2) $f : J \times E \to E$, $(t,u) \to f(t,v)$ is a continuous function.
- (H3) There exists a constant $L > 0$ such that $\|g(y)\| \leq L$ for each $y \in E$.
- (H4) $\|f(t,s,y)\| \leq p(t)\psi(\|y\|)$ for almost all $t \in J$ and all $y \in E$, where $p \in L^1(J,R_+)$ and $\psi : R_+ \to (0,\infty)$ is continuous and increasing with

$$M\int_0^b p(s)ds < \int_c^\infty \frac{du}{\psi(u)},$$

where $c = M\|y_0\| + ML$ and

$$M = \sup\{U_y(t,s)\|_{B(E)} : (t,s) \in J \times J\}.$$

- (H5) For each bounded $B \subset C(J,E), y \in B$, and $t \in J$ the set

$$\big\{U_y(t,0)y_0 - U_y(t,0)g(y) + \int_0^t U_y(t,s)\int_0^s f(s,\tau,y(\tau))d\tau ds\big\}$$

is relatively compact.

Remark 2.3. From (H1) we are able to claim the existence for any fixed $u \in C(J,E)$ of a unique function $U_n : J \times J \to B(E)$ defined and continuous on $J \times J$ such that

$$U_u(t,s) = I + \int_s^t A_u(w)U_u(w,s)dw \tag{2.1}$$

where I stands for the identity operator on E and $A_u(t) := A(t, u(t))$. From (2.1) one has

$$U_u(t,t) = I, \quad U_u(t,s)U_u(s,r) = U_u(t,r), \quad (t,s,r) \in J \times J \times J\,;$$
$$\frac{\partial U_u(t,s)}{\partial t} = A_u(t)U_u(t,s) \quad \text{for almost all } t \in J, \quad \forall s \in J.$$

Remark 2.4. From (H1) it follows that $u \in C(J,E)$ implies $A_u \in C(J, B(E))$ and $\|u_n - u^\bullet\|_\infty \to 0$ implies

$$\|A_{u_n} - A_{u^\bullet}\|_\infty := \max\{\|\|A_{u_n}(t) - A_{u^\bullet}(t)\|_{B(E)} : t \in J\} \to 0, \quad \text{as } n \to \infty.$$

A continuous solution $y(t)$ of the integral equation

$$y(t) = U_y(t,0)y_0 - U_y(t,0)g(y) + \int_0^t U_y(t,s) \int_0^s f(s,\tau,y(\tau))d\tau ds$$

is called a mild solution of (1.1)–(1.2).

3. An existence theorem

Theorem 3.1. *Let $g : C(J,E) \to E$ be a continuous function and assume that (H1)–(H5) are satisfied. Then problem* (1.1)-(1.2) *has at least one mild solution on J.*

Proof. We transform the problem (1.1)–(1.2) into a fixed point problem. Consider the map $N : C(J,E) \to C(J,E)$ defined by

$$(Ny)(t) := U_y(t,0)y_0 - U_y(t,0)g(y) + \int_0^t U_y(t,s) \int_0^s f(s,\tau,y(\tau))d\tau ds, \quad t \in J$$

We remark that the fixed points of N are mild solutions to (1.1)-(1.2). We shall show that N is a continuous compact map. The proof will be given in several steps.
Step 1. $U_u(t,s)$ is continuous with respect to u; i.e., $\|u_n - u^\bullet\|_\infty \to 0$ implies

$$\|U_{u_n} - U_{u^\bullet}\|_\infty := \sup_{(t,s)\in J\times J} \{\|U_{u_n}(t,s) - U_{u^\bullet}(t,s)\|_{B(E)}\} \to 0 \quad \text{as } n \to \infty.$$

Indeed, let $\|u_n - u^\bullet\|_\infty \to 0$. Then there exists $r > 0$ such that $\|u_n\|_\infty, \|u^\bullet\|_\infty \le r$. Moreover, if $s \le t$ we have

$$\begin{aligned}
\|U_{u_n} - U_{u^*}\|_\infty &\le \int_s^t \|U_{u_n}(w,s)\|_{B(E)}\|A_{u_n}(w) - A_{u^*}(w)\|_{B(E)}dw \\
&\quad + \int_s^t \|A_{u^*}\|_{B(E)}\|U_{u_n}(w,s) - U_{u^*}(w,s)\|_{B(E)}dw \\
&\le M \int_s^t \|A_{u_n}(w) - A_{u^*}(w)\|_{B(E)}dw \\
&\quad + \int_s^t \|A_{u^*}\|_{B(E)}\|U_{u_n}(w,s) - U_{u^*}(w,s)\|_{B(E)}dw\,.
\end{aligned}$$

Using Lemma 2.1 we obtain

$$
\begin{aligned}
&\|U_{u_n} - U_{u^*}\|_\infty \\
&\le M\int_s^t \|A_{u_n}(w) - A_{u^*}(w)\|_{B(E)} dw + M\int_s^t \|A_{u^*}(w)\|_{B(E)} \\
&\quad \times \big[\int_s^t \|A_{u_n}(\tau) - A_{u^*}(\tau)\|_{B(E)} d\tau\big] \exp(\int_w^t \|A_{u^*}(z)\|_{B(E)} dz) dw \\
&\le bM\|A_{u_n} - A_{u^*}\|_\infty + b^2 M\|A_{u^*}\|_\infty \|A_{u_n} - A_{u^*}\|_\infty \exp(b\|A_{u^*}\|_\infty) \\
&\le \|A_{u_n} - A_{u^*}\|_\infty Mb(1 + br_1 \exp(br_1)).
\end{aligned}
$$

The conclusion follows from Remark 2.4.

Step 2. N maps bounded sets into relatively compact sets; i.e. N is a compact map. Let $B_r = \{y \in C(J, E) : \|y\|_\infty \le r\}$. Then for each $t \in J$ we have

$$
(Ny)(t) = U_y(t, 0)y_0 - U_y(t, 0)g(y) + \int_0^t U_y(t, s) \int_0^s f(s, \tau, y(\tau)) d\tau ds, \quad t \in J.
$$

By (H3) and (H4), for each $t \in J$, we have

$$
\begin{aligned}
\|Ny\| &\le \|U_y(t, 0)\|_{B(E)} \|y_0\| + \|U_y(t, 0)\| \|g(y)\| \\
&\quad + \int_0^t \|U_y(t, s) \int_0^s f(s, \tau, y(\tau)) d\tau\| ds \\
&\le M\|y_0\| + ML + Mb \sup_{y\in[0,r]} \psi(y)(\int_0^t p(s) ds)
\end{aligned}
$$

or

$$
\|Ny\|_\infty \le M\|y_0\| + ML + Mb \sup_{t\in J}(\int_0^t p(s) ds) \max_{y\in B} \sup_{y\in[0,r]} \psi(y) := l.
$$

Now let $t_1, t_2, \in J, t_1 < t_2$ and $y \in B_r$. Then

$$
\begin{aligned}
&\|(Ny)(t_2) - (Ny)(t_1)\| \\
&\le \|U_y(t_2, 0) - U_y(t_1, 0)\|_{B(E)} \|y_0\| + \|U_y(t_2, 0) - U_y(t_1, 0)\|_{B(E)} L \\
&\quad + \|\int_0^{t_1} [U_y(t_2, s) - U_y(t_1, s)] \int_0^s f(s, \tau, y(\tau)) d\tau ds\| \\
&\quad + \|\int_{t_1}^{t_2} U_y(t_2, s) \int_0^s f(s, \tau, y(\tau)) d\tau ds\| \\
&\le \|U_y(t_2, 0) - U_y(t_1, 0)\|_{B(E)} \|y_0\| + \|U_y(t_2, 0) - U_y(t_1, 0)\|_{B(E)} L \\
&\quad + \|\int_0^{t_1} [U_y(t_2, s) - U_y(t_1, s)] \int_0^s f(s, \tau, y(\tau)) d\tau ds\| \\
&\quad + \|\int_{t_1}^{t_2} U_y(t_2, s) \int_0^s f(s, \tau, y(\tau)) d\tau ds\| \\
&\le \|U_y(t_2, 0) - U_y(t_1, 0)\|_{B(E)} \|y_0\| + \|U_y(t_2, 0) - U_y(t_1, 0)\|_{B(E)} L \\
&\quad + \int_0^{t_1} \|U_y(t_2, s) - U_y(t_1, s)\| p(s)\psi(\|y(s)\|) ds + M\int_{t_1}^{t_2} p(s)\psi(\|y(s)\|) ds
\end{aligned}
$$

This string of inequalities is bounded by $K(t_2 - t_1)$ for some $K > 0$; hence $N(B_r)$ is an equicontinuous family of functions. Therefore by the Ascoli Arzela theorem $N(B_r)$ is relatively compact.

Step 3. The set $\Omega = \{y \in C(J,E) : \lambda y = N(y), \lambda > 1\}$ is bounded. Let $y \in \Omega$. Then $\lambda y = N(y)$ for some $\lambda > 1$. Then

$$y(t) = \lambda^{-1}U_y(t,0)y_0 - \lambda^{-1}U_y(t,0)g(y) + \lambda^{-1}\int_0^t U_y(t,s)\int_0^s f(s,\tau,y(\tau))d\tau ds \quad t \in J.$$

By (H3) and (H4) this implies that for each $t \in J$ we have

$$\|y(t)\| \le M\|y_0\| + ML + M\int_0^t p(s)\psi(\|y(t)\|)ds.$$

Let us take the right-hand side of the above inequality as $v(t)$; then we have $v(0) = M\|y_0\| + ML$ and $\|y(t)\| \le v(t)$, $t \in J$. Using the nondecresing character of ψ we get

$$v' \le Mp(t)\psi(v(t)), \quad t \in J.$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \le M\int_0^b p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}\,.$$

This inequality implies that there exists a constant d such that $v(t) \le d, t \in J$, and hence $\|y\|_\infty \le d$ where d depends only on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J,E)$. As a consequence of Lemma 2.2 we deduce that N has a fixed point which is mild solution of (1.1)-(1.2). □

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