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ON A CLASS OF NONLINEAR VARIATIONAL INEQUALITIES: HIGH CONCENTRATION OF THE GRAPH OF WEAK SOLUTION VIA ITS FRACTIONAL DIMENSION AND MINKOWSKI CONTENT

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Abstract. Weak continuous bounded solutions of a class of nonlinear variational inequalities associated to one-dimensional p-Laplacian are studied. It is shown that a kind of boundary behaviour of nonlinearity in the main problem produces a kind of high boundary concentration of the graph of solutions. It is verified by calculating lower bounds for the upper Minkowski-Bouligand dimension and Minkowski content of the graph of each solution and its derivative. Finally, the order of growth for singular behaviour of the L^p norm of derivative of solutions is given.

1. Introduction

Let $1 < p < \infty$ and $-\infty < a < b < \infty$. Let $f(t, \eta, \xi)$ be a Caratheodory function defined on $(a, b) \times \mathbb{R} \times \mathbb{R}$. We consider a class of nonlinear variational inequalities with two obstacles φ and ψ in the form:

$$\begin{gathered} u \in K(\varphi, \psi), \\ \int_a^b |u'|^{p-2}u'(v-u)'dt - \int_a^b f(t,u,u')(v-u)dt \geq 0, \\ \forall v \in K(\varphi,\psi) \text{ such that } \operatorname{supp}(v-u) \subset\subset (a,b), \end{gathered} \tag{1.1}$$

where $\varphi, \psi \in L^\infty(a, b)$, $\varphi \leq \psi$ and

$$K(\varphi, \psi) = \{v \in W^{1,p}_{\rm loc}((a, b]) \cap C([a, b]) : \varphi \leq v \leq \psi \text{ in } (a, b)\}.$$

Here the condition $v \in W^{1,p}_{\rm loc}((a, b])$ means that $v \in W^{1,p}(a + \varepsilon, b)$ for each $\varepsilon > 0$.

The main subject of the paper is the graph $G(u)$ of a continuous real function u defined on $[a, b]$, that is

$$G(u) = \{(t, u(t)) : a \leq t \leq b\}.$$

In order to describe a kind of very high boundary concentration of $G(u)$ near the point $t = a$, where u is any solution of (1.1), we associate to $G(u)$ the following two numbers:

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- The upper Minkowski-Bouligand (box-counting) dimension of $G(u)$,
$$\dim_M G(u) = \limsup_{\varepsilon\to 0} \big(2 - \frac{\log |G_\varepsilon(u)|}{\log \varepsilon}\big),$$
where $G_\varepsilon(u)$ denotes the ε-neighbourdhood of $G(u)$ and $|G_\varepsilon(u)|$ denotes the Lebesgue measure of $G_\varepsilon(u)$.
- The s-dimensional upper Minkowski content of $G(u)$,
$$M^s(G(u)) = \limsup_{\varepsilon\to 0}(2\varepsilon)^{s-2}|G_\varepsilon(u)|,$$
where $s \in (1, 2)$.

In Section 3, for arbitrarily given $s \in (1, 2)$ we will find some sufficient conditions on the obstacles φ and ψ and on the nonlinearity $f(t, \eta, \xi)$ such that each solution u of (1.1) satisfies
$$|G_\varepsilon(u)| \geq \frac{1}{2^6}(b-a)^s\varepsilon^{2-s} > 0 \quad \text{for each } \varepsilon \in (0, \varepsilon_0), \tag{1.2}$$
where $\varepsilon_0 > 0$ will be precised too. According to the definitions of $\dim_M G(u)$ and $M^s(G(u))$, the inequality (1.2) enables us to show that each solution u of (1.1) satisfies
$$\dim_M G(u) \geq s \quad \text{and} \quad M^s(G(u)) \geq \frac{1}{2^7}(b-a)^s > 0. \tag{1.3}$$
Since $\dim_M(A\cup B) = \max\{\dim_M A, \dim_M B\}$ and $u \in W^{1,p}(a+\varepsilon, b)$ for each $\varepsilon > 0$, we have that u is an absolutely continuous function on $[a+\varepsilon, b]$ which together with (1.3) gives us
$$\dim_{Mloc}(G(u); a) \geq s \quad \text{and} \quad \dim_{Mloc}(G(u); t) = 1 \quad \text{for each } t \in (a, b].$$
Here $\dim_{Mloc}(G(u); t)$ denotes the locally upper Minkowski-Bouligand dimension of $G(u)$ at a point $t \in [a, b]$, given by
$$\dim_{Mloc}(G(u); t) = \limsup_{\varepsilon\to 0} \dim_M(G(u) \cap B_\varepsilon(t, u(t))),$$
where $B_r(t_1, t_2)$ denotes a ball with radius $r > 0$ centered at the point $(t_1, t_2) \in \mathbb{R}^2$. As an easy consequence, we derive that each solution u of (1.1) satisfies:
$$u \notin W^{1,p}(a, b) \quad \text{and} \quad \text{length}(G(u)) = \infty,$$
$$u \in W^{1,p}(a+\varepsilon, b) \quad \text{and} \quad \text{length}(G(u|_{[a+\varepsilon,b]})) < \infty \quad \text{for any } \varepsilon > 0,$$
where $u|_I$ denotes the restriction of u on I. Thus, according to the previous statements, we may conclude that the graph $G(u)$ of any solution u of (1.1) is (in a sense) highly concentrated at the boundary point $t = a$. Furthermore, the statement *length*$(G(u)) = \infty$ is precised in (1.3). For arbitrarily given $s \in (1, 2)$ and under the same hypotheses on φ, ψ and $f(t, \eta, \xi)$ as in getting of (1.2)–(1.3), we will prove in Section 3 that each solution u of (1.1) satisfies
$$|G_\varepsilon(u|_{(a,c]})| \geq \frac{1}{2^6}(c-a)^s\varepsilon^{2-s} > 0 \text{ for any } c \in (a, b) \text{ and } \varepsilon \in (0, \varepsilon_c), \tag{1.4}$$
where the number ε_c will be also determined. The preceding inequality yields
$$M^s(G(u) \cap B_r(a, u(a))) \geq \frac{1}{2^7}(\frac{r}{\sqrt{5}})^s \quad \text{for any } r \in (0, b-a). \tag{1.5}$$
It completes the second inequality in (1.3).

Next, for arbitrarily given $s \in (1,2)$ and under the same hypotheses on φ, ψ and $f(t,\eta,\xi)$ as in getting of (1.2)-(1.5), we will show in Section 4 that each solution u of (1.1) such that $u \in C^1(a,b)$ satisfies

$$|G_\varepsilon(u')| \geq \frac{\sqrt{2}}{2^4}(b-a)^{s/2}\varepsilon^{1-s/2} > 0 \quad \text{for each } \varepsilon \in (0,\varepsilon_0). \tag{1.6}$$

where $\varepsilon_0 > 0$ will be also precised. Here u' denotes the derivative of u in the classical sense. According to the definitions of $\dim_M G(u')$ and $M^s(G(u'))$, from (1.6) we get that each smooth enough solution u od (1.1) satisfies:

$$\dim_M G(u') \geq 1 + \frac{s}{2} \quad \text{and} \quad M^{1+s/2}(G(u')) \geq \frac{1}{2^4}(b-a)^{s/2} > 0. \tag{1.7}$$

In (1.7) we have two estimations for singular behaviour of u' near the boundary point $t = a$. Much more information about singular behaviour of u' near the point $t = a$ can be obtained from asymptotic behaviour of $\|u'\|_{L^p(a+\varepsilon,b)}$ as $\varepsilon \approx 0$. More precisely, from above observation we have in particular that each solution u of (1.1) satisfies

$$\limsup_{\varepsilon\to 0} \|u'\|_{L^p(a+\varepsilon,b)} = \infty.$$

However, in Section 5 we will be able to precise this statement. That is, for arbitrarily given $s \in (1,2)$ and under related hypotheses on φ, ψ and $f(t,\eta,\xi)$ as in getting of (1.2)–(1.7), we will prove that each solution u of (1.1) satisfies

$$\Big(\int_{a+\varepsilon}^b |u'|^p dt\Big)^{1/p} \geq c\big(\frac{1}{\varepsilon}\big)^{s-1} \quad \text{for some } \varepsilon \in (0,\varepsilon_1), \tag{1.8}$$

where $c > 0$ and $\varepsilon_1 > 0$ will be also precised. Immediately from (1.8) we obtain the lower bound for the order of growth of the local singular behaviour of $\|u'\|_{L^p(a+\varepsilon,b)}$ as $\varepsilon \approx 0$, that is

$$\limsup_{\varepsilon\to 0} \frac{\log\big(\int_{a+\varepsilon}^b |u'|^p dt\big)^{1/p}}{\log 1/\varepsilon} \geq s-1. \tag{1.9}$$

It is worth to mention that the local regular behaviour of $\|u'\|_{L^p}$ is widely considered even in more dimensional case, where u is any solution of quasilinear elliptic equations associated to p -Laplacian. See for instance Rakotoson's paper [16] and references therein.

Preceding results were obtained in the author's paper [12] but for the case of corresponding equation:

$$\begin{gathered} -(|y'|^{p-2}y')' = f(t,y,y') \quad \text{in } (a,b), \\ y(a) = y(b) = 0, \\ y \in W^{1,p}_{\rm loc}((a,b]) \cap C([a,b]). \end{gathered} \tag{1.10}$$

In this paper we show how the methods presented in [12] permit us to obtain some new singular properties of the graph of solutions of the variational inequality (1.1). About some regular properties of solutions of quasilinear elliptic variational inequalities, we refer reader to [4, 9, 13, 17]. About the fractal dimensions and their properties we refer to [1, 3, 8, 10, 15, 18, 19].

Finally, let us remark that the existence of at least one solution y of (1.10) was discussed in [12, Appendix, p. 303-304], where the nonlinearity $f(t,\eta,\xi)$ satisfy related assumptions needed here to obtain (1.2)–(1.9) (about the existence of continuous solutions for the equations with singular nonlinearity see [11, Chapter 14]).

Moreover, if for instance $\varphi(a) = \psi(a) = 0$ and if φ is decreasing and convex on $[a, b]$ and if ψ is increasing and concave on $[a, b]$, and if $f(t, \eta, \xi)$ satisfies:

$$f(t,\eta,\xi) < 0, \quad t \in (a,b),\ \eta > \psi(t) \text{ and } \xi \in \mathbb{R},$$
$$f(t,\eta,\xi) > 0, \quad t \in (a,b),\ \eta < \varphi(t) \text{ and } \xi \in \mathbb{R},$$

then each solution y of the equation (1.10) satisfies $\varphi(t) \le y(t) \le \psi(t)$ in $[a, b]$. So in such case, each solution of (1.10) also satisfies the variational inequality (1.1) and thus, the existence of solutions of (1.1) in this case follows from the existence result of the equation (1.10).

2. Control of essential infimum and essential supremum of solutions

In this section, we present a method which plays an important role in the proofs of the main results. It is so called the control of ess inf and ess sup of solutions introduced in [5] and considered in [6] and [7] to get some qualitative properties of solutions of quasilinear elliptic equations and variational inequalities. Here, we show that this method can be applied to solutions of variational inequality (1.1) to derive some consequences needed in the proofs of the main results.

Lemma 2.1 (Control of ess sup). *Let $(a_2, b_2) \subset\subset (a, b)$ be an open interval. Let ω_2 be an arbitrarily given real number such that*

$$\operatorname*{ess\,inf}_{(a_2,b_2)} \varphi < \omega_2 < \operatorname*{ess\,inf}_{(a_2,b_2)} \psi. \tag{2.1}$$

Let J_2 be a set defined by $J_2 = (\operatorname{ess\,inf}_{(a_2,b_2)} \varphi, \omega_2)$ and let the Caratheodory function $f(t, \eta, \xi)$ satisfy:

$$f(t,\eta,\xi) \ge 0, \quad t \in (a_2,b_2),\ \eta \in J_2,\ \xi \in \mathbb{R}, \tag{2.2}$$

$$\int_{A_2} \operatorname*{ess\,inf}_{(\eta,\xi)\in J_2\times R} f(t,\eta,\xi)dt > \frac{c(p)}{(b_2-a_2)^{p-1}} \frac{(\operatorname{ess\,inf}_{(a_2,b_2)} \psi - \operatorname{ess\,inf}_{(a_2,b_2)} \varphi)^p}{\operatorname{ess\,inf}_{(a_2,b_2)} \psi - \omega_2}, \tag{2.3}$$

where $c(p) = 2[4(p-1)]^{p-1}$ and A_2 is a set defined by

$$A_2 = [a_2 + \frac{1}{4}(b_2 - a_2), b_2 - \frac{1}{4}(b_2 - a_2)].$$

Then for any solution u of (1.1) there is a $\sigma_2 \in (a_2, b_2)$ such that

$$u(\sigma_2) \ge \omega_2. \tag{2.4}$$

We will also need the dual result of Lemma 2.1.

Lemma 2.2 (control of ess inf). *Let $(a_1, b_1) \subset\subset (a, b)$ be an open interval. Let θ_1 be an arbitrarily given real number such that*

$$\operatorname*{ess\,sup}_{(a_1,b_1)} \varphi < \theta_1 < \operatorname*{ess\,sup}_{(a_1,b_1)} \psi. \tag{2.5}$$

Let J_1 be a set defined by $J_1 = (\theta_1, \operatorname{ess\,sup}_{(a_1,b_1)} \psi)$ and let the Caratheodory function $f(t, \eta, \xi)$ satisfy:

$$f(t,\eta,\xi) \le 0, \quad t \in (a_1,b_1),\ \eta \in J_1,\ \xi \in \mathbb{R}, \tag{2.6}$$

$$\int_{A_1} \operatorname*{ess\,sup}_{(\eta,\xi)\in J_1\times R} f(t,\eta,\xi)dt < -\frac{c(p)}{(b_1-a_1)^{p-1}} \frac{(\operatorname{ess\,sup}_{(a_1,b_1)} \psi - \operatorname{ess\,sup}_{(a_1,b_1)} \varphi)^p}{\theta_1 - \operatorname{ess\,sup}_{(a_1,b_1)} \varphi}, \tag{2.7}$$

where $c(p) = 2[4(p-1)]^{p-1}$ *and* A_1 *is a set defined by*

$$A_1 = [a_1 + \frac{1}{4}(b_1 - a_1), b_1 - \frac{1}{4}(b_1 - a_1)].$$

Then for any solution u *of* (1.1) *there is a* $\sigma_1 \in (a_1, b_1)$ *such that*

$$u(\sigma_1) \leq \theta_1. \tag{2.8}$$

Let us remark that the conditions (2.1) and (2.5) will be easy fulfiled in Theorem 3.3 below.

Proof of Lemma 2.1. Let t_0 and r be two real numbers defined as follows:

$$t_0 = \frac{a_2 + b_2}{2}, \quad r = \frac{1}{4}(b_2 - a_2).$$

Let $B_r = B_r(t_0)$ denote a ball with radius $r > 0$ centered at the point t_0. Then we have

$$B_{2r} = B_{2r}(t_0) = (a_2, b_2), \quad B_r = B_r(t_0) = A_2,$$

where the set A_2 is appearing in (2.3). Since $|B_{2r}| = 4r = b_2 - a_2$, where $|A|$ denotes the Lebesgue measure of a set A, and using the preceding notations, the hypotheses (2.2) and (2.3) can be rewritten in the form:

$$f(t, \eta, \xi) \geq 0, \quad t \in B_{2r},\ \eta \in J_2,\ \xi \in \mathbb{R}, \tag{2.9}$$

$$\int_{B_r} \operatorname*{ess\,inf}_{(\eta,\xi)\in J_2\times R} f(t, \eta, \xi)dt > \frac{c(p)}{4^{p-1}} \frac{1}{r^{p-1}} \frac{(\operatorname{ess\,inf}_{B_{2r}} \psi - \operatorname{ess\,inf}_{B_{2r}} \varphi)^p}{\operatorname{ess\,inf}_{B_{2r}} \psi - \omega_2}. \tag{2.10}$$

Next, let u be a solution of (1.1). Let us suppose a contrary statement to (2.4), that is

$$u(t) < \omega_2 \quad \text{for each } t \in B_{2r}. \tag{2.11}$$

Since $\varphi \leq u$ in (a, b) and because of (2.1), besides (2.11) we have also

$$\operatorname*{ess\,inf}_{B_{2r}} \varphi \leq u(t) < \omega_2 < \operatorname*{ess\,inf}_{B_{2r}} \psi \quad \text{for each } t \in B_{2r}. \tag{2.12}$$

Using $c(p) = 2[4(p-1)]^{p-1}$ and $|B_r| = 2r$, from (2.9)–2.12, we get

$$f(t, u, u') \geq 0 \quad \text{in } B_{2r}, \tag{2.13}$$

$$\int_{B_r} f(t, u, u')dt > (p-1)^{p-1} \frac{|B_r|}{r^p} \frac{(\operatorname{ess\,inf}_{B_{2r}} \psi - \operatorname{ess\,inf}_{B_{2r}} \varphi)^p}{\operatorname{ess\,inf}_{B_{2r}} \psi - \omega_2}. \tag{2.14}$$

Regarding (2.13) and (2.14) we are here in a very similar situation as in the proof [6, Theorem 5, p. 256] or [12, Theorem 4.1, p. 282]. In this direction, it is known that for any $c_0 > 1$ there exists a function $\Phi \in C_0^\infty(\mathbb{R})$, $0 \leq \Phi \leq 1$ in $\mathbb{R}$ such that the following properties are fulfilled, see [6, Lemma 5, pp. 267]:

$$\begin{gathered} \Phi(t) = 1,\ t \in B_r \quad \text{and} \quad \Phi(t) = 0,\ t \in \mathbb{R} \setminus B_{2r}, \\ \Phi(t) > 0,\ t \in B_{2r} \quad \text{and} \quad |\Phi'(t)| \leq \frac{c_0}{r},\ t \in \mathbb{R}. \end{gathered} \tag{2.15}$$

For any $c_0 > 1$, we take a test function

$$v(t) = (\operatorname*{ess\,inf}_{B_{2r}} \psi - u(t))\Phi^p(t) + u(t), \quad t \in \mathbb{R}.$$

With the help of (2.12) we have that $\operatorname{ess\,inf}_{B_{2r}} \psi - u(t) \geq 0$ in B_{2r} and so, it is easy to check that

$$v \in K(\varphi, \psi) \quad \text{and} \quad \operatorname{supp}(v - u) \subseteq B_{2r} \subset\subset (a, b),$$

where the space $K(\varphi,\psi)$ was defined in (1.1). Therefore, we may put in (1.1) this test function and we obtain

$$-\int_{B_{2r}} |u'|^p \Phi^p dt \ge -p\int_{B_{2r}} |u'|^{p-2}u'(\operatorname*{ess\,inf}_{B_{2r}} \psi - u)\Phi^{p-1}\Phi' dt + \int_{B_{2r}} f(t,u,u')(\operatorname*{ess\,inf}_{B_{2r}} \psi - u)\Phi^p dt.$$

Multiplying this inequality by -1 we get

$$\begin{aligned}\int_{B_{2r}} |u'|^p \Phi^p dt &\le p\int_{B_{2r}} |u'|^{p-1}\Phi^{p-1}(\operatorname*{ess\,inf}_{B_{2r}} \psi - u)|\Phi'| dt \\ &\quad - \int_{B_{2r}} f(t,u,u')(\operatorname*{ess\,inf}_{B_{2r}} \psi - u)\Phi^p dt.\end{aligned} \tag{2.16}$$

For the record, with the help of (2.12) we also have:

$$\begin{gathered}\operatorname*{ess\,inf}_{B_{2r}} \psi - u(t) \le \operatorname*{ess\,inf}_{B_{2r}} \psi - \operatorname*{ess\,inf}_{B_{2r}} \varphi,\ t\in B_{2r}, \\ \operatorname*{ess\,inf}_{B_{2r}} \psi - u(t) \ge \operatorname*{ess\,inf}_{B_{2r}} \psi - \omega_2,\ t\in B_{2r}.\end{gathered} \tag{2.17}$$

Using $(p-1)p' = p$ and $\delta_1(p\delta_2) \le \frac{d}{p'}\delta_1^{p'} + (\frac{p}{d})^{p-1}\delta_2^p$ especially for

$$\delta_1 = |u'|^{p-1}\Phi^{p-1}, \quad \delta_2 = (\operatorname*{ess\,inf}_{B_{2r}} \psi - u)|\Phi'|, \quad d = p',$$

with the help of (2.13), (2.16) and (2.17) we obtain

$$\begin{aligned}0 = \big[1 - \frac{d}{p'}\big]\int_{B_{2r}} |u'|^p\Phi^p dt &\le \big(\frac{p}{p'}\big)^{p-1}(\operatorname*{ess\,inf}_{B_{2r}} \psi - \operatorname*{ess\,inf}_{B_{2r}} \varphi)^p \int_{B_{2r}} |\Phi'|^p dt \\ &- (\operatorname*{ess\,inf}_{B_{2r}} \psi - \omega_2)\int_{B_r} f(t,u,u')\Phi^p dt.\end{aligned} \tag{2.18}$$

Now, by means of (2.15) we derive

$$\begin{aligned}0 &\le \big(\frac{p}{p'}\big)^{p-1}(\operatorname*{ess\,inf}_{B_{2r}} \psi - \operatorname*{ess\,inf}_{B_{2r}} \varphi)^p |B_{2r}\setminus B_r| \big(\frac{c_0}{r}\big)^p \\ &\quad - (\operatorname*{ess\,inf}_{B_{2r}} \psi - \omega_2)\int_{B_r} f(t,u,u')dt.\end{aligned}$$

Since $|B_{2r}\setminus B_r| = |B_r|$ and passing to the limit as $c_0 \to 1$ we obtain

$$\int_{B_r} f(t,y,y')dt \le (p-1)^{p-1}\frac{|B_r|}{r^p}\frac{(\operatorname{ess\,inf}_{B_{2r}} \psi - \operatorname{ess\,inf}_{B_{2r}} \varphi)^p}{\operatorname{ess\,inf}_{B_{2r}} \psi - \omega_2}.$$

But, this inequality contradicts the assumption (2.14) and so the hypothesis (2.11) is not possible. Thus, the desired statement (2.4) is proved. □

Analogously we can obtain the proof of Lemma 2.2. In Section 5, we need to use slightly different versions of preceding lemmas.

Lemma 2.3 (A version of Lemma 2.1)**.** *Let $(a_2,b_2) \subset\subset (a,b)$ be an open interval. Let $\tilde\theta_0$, $\tilde\omega_0$ and ω_2 be three arbitrarily given real numbers such that*

$$\tilde\theta_0 \le \operatorname*{ess\,inf}_{(a_2,b_2)} \varphi < \omega_2 < \operatorname*{ess\,inf}_{(a_2,b_2)} \psi \le \tilde\omega_0. \tag{2.19}$$

Let J_2 be a set defined by $J_2 = (\tilde\theta_0, \omega_2)$ and let the Caratheodory function $f(t,\eta,\xi)$ satisfy:

$$
\begin{gathered}
f(t,\eta,\xi) \geq 0, \quad t \in (a_2,b_2),\ \eta \in J_2,\ \xi \in \mathbb{R}, \\
\int_{A_2} \operatorname*{ess\,inf}_{(\eta,\xi)\in J_2\times R} f(t,\eta,\xi)dt > \frac{c(p)}{(b_2-a_2)^{p-1}} \frac{(\tilde\omega_0-\tilde\theta_0)^p}{\operatorname{ess\,inf}_{(a_2,b_2)}\psi - \omega_2},
\end{gathered}
\tag{2.20}
$$

where $c(p) = 2[4(p-1)]^{p-1}$ and A_2 is a set defined by

$$
A_2 = [a_2 + \frac{1}{4}(b_2-a_2), b_2 - \frac{1}{4}(b_2-a_2)].
$$

Then for any solution u of (1.1) there is a $\sigma_2 \in (a_2,b_2)$ such that

$$
u(\sigma_2) \geq \omega_2. \tag{2.21}
$$

We will also need the dual result of Lemma 2.3.

Lemma 2.4 (A version of Lemma 2.2)**.** *Let $(a_1,b_1) \subset\subset (a,b)$ be an open interval. Let $\tilde\theta_0$, $\tilde\omega_0$ and θ_1 be three arbitrarily given real numbers such that*

$$
\tilde\theta_0 \leq \operatorname*{ess\,sup}_{(a_1,b_1)} \varphi < \theta_1 < \operatorname*{ess\,sup}_{(a_1,b_1)} \psi \leq \tilde\omega_0.
$$

Let J_1 be a set defined by $J_1 = (\theta_1, \tilde\omega_0)$ and let the Caratheodory function $f(t,\eta,\xi)$ satisfy:

$$
\begin{gathered}
f(t,\eta,\xi) \leq 0, \quad t \in (a_1,b_1),\ \eta \in J_1,\ \xi \in \mathbb{R}, \\
\int_{A_1} \operatorname*{ess\,sup}_{(\eta,\xi)\in J_1\times R} f(t,\eta,\xi)dt < -\frac{c(p)}{(b_1-a_1)^{p-1}} \frac{(\tilde\omega_0-\tilde\theta_0)^p}{\theta_1 - \operatorname{ess\,sup}_{(a_1,b_1)}\varphi},
\end{gathered}
$$

where $c(p) = 2[4(p-1)]^{p-1}$ and A_1 is a set defined by

$$
A_1 = [a_1 + \frac{1}{4}(b_1-a_1),\ b_1 - \frac{1}{4}(b_1-a_1)].
$$

Then for any solution u of (1.1) there is a $\sigma_1 \in (a_1,b_1)$ such that

$$
u(\sigma_1) \leq \theta_1.
$$

The proof of Lemma 2.3 can be done in analogous way as we did the proof of Lemma 2.1. At the end of this section, we give an example for the nonlinearity $f(t,\eta,\xi)$ which satisfies the assumptions of Lemma 2.1 and Lemma 2.2 together.

Example 2.5. Let $a < a_2 < b_2 = a_1 < b_1 < b$. Let ω_2 and θ_1 be two numbers satisfying (2.1) and (2.5). To simplify notation, let $\tilde\theta_1$, $\tilde\omega_1$, $\tilde\theta_2$, and $\tilde\omega_2$ be defined by

$$
\begin{gathered}
\tilde\theta_1 = \operatorname*{ess\,sup}_{(a_1,b_1)} \varphi, \quad \tilde\omega_1 = \operatorname*{ess\,sup}_{(a_1,b_1)} \psi, \\
\tilde\theta_2 = \operatorname*{ess\,inf}_{(a_2,b_2)} \varphi, \quad \tilde\omega_2 = \operatorname*{ess\,inf}_{(a_2,b_2)} \psi.
\end{gathered}
$$

Next, let $f(t,\eta,\xi)$ be a Caratheodory function defined by

$$
\begin{aligned}
f(t,\eta,\xi) = \frac{\pi c(p)}{\sin\frac{\pi}{4}} \big[&(\tilde\omega_2-\tilde\theta_2)^p \frac{(\eta-\tilde\omega_2)^-}{(\tilde\omega_2-\omega_2)^2} \frac{\sin(\frac{\pi}{b_2-a_2}(t-a_2))}{(b_2-a_2)^p} K_{[a_2,b_2]}(t) \\
&- (\tilde\omega_1-\tilde\theta_1)^p \frac{(\eta-\tilde\theta_1)^+}{(\theta_1-\tilde\theta_1)^2} \frac{\sin(\frac{\pi}{b_1-a_1}(t-a_1))}{(b_1-a_1)^p} K_{[a_1,b_1]}(t) \big],
\end{aligned}
$$

where $c(p)$ is the same as in Lemma 2.1 and Lemma 2.2, and where $K_A(t)$ denotes as usually the characteristic function of a set A. Also, $\eta^- = \max\{0, -\eta\}$ and $\eta^+ = \max\{0, \eta\}$. Such defined $f(t, \eta, \xi)$ satisfies the assumptions of Lemma 2.1. and Lemma 2.2 together.

3. Lower bounds for $\dim_M G(u)$ and $M^s(G(u))$

In this section, the statements (1.2)-(5) will be verified. It will be made by using the following two lemmas. The first one is a version of [12, Lemma 2.1, p. 271] which gives us some useful metric properties of the graph of rapidly oscillating continuous functions. The second one deals with some sufficient conditions on the nonlinearity $f(t, \eta, \xi)$ such that each solution of (1.1) is rapid oscillating in the sense of the first lemma. It is a consequence of the results obtained in previous section.

Lemma 3.1. *Let a_k be a decreasing sequence of real numbers from interval (a, b) satisfying*

$$\begin{aligned} &a_k \searrow a \text{ and there is an } \varepsilon_0 > 0 \text{ such that for each } \varepsilon \in (0, \varepsilon_0) \\ &\text{there is a } k(\varepsilon) \in \mathbf{N} \text{ such that } a_{j-1} - a_j \leq \varepsilon/2 \text{ for each } j \geq k(\varepsilon). \end{aligned} \tag{3.1}$$

Let $\theta(t)$ and $\omega(t)$ be two measurable and bounded real functions on $[a, b]$, $\theta(t) \leq \omega(t)$, $t \in [a, b]$, such that

$$\begin{gathered} \operatorname*{ess\,inf}_{(a_{2k+2}, a_{2k+1})} \theta \geq \operatorname*{ess\,inf}_{(a_{2k+1}, a_{2k})} \theta, \\ \operatorname*{ess\,sup}_{(a_{2k+1}, a_{2k})} \omega \leq \operatorname*{ess\,sup}_{(a_{2k}, a_{2k-1})} \omega, \quad k \geq 1. \end{gathered} \tag{3.2}$$

Let u be a continuous function on $(a, b]$ such that there is a sequence $\sigma_k \in (a_k, a_{k-1})$ satisfying

$$u(\sigma_{2k}) \geq \operatorname*{ess\,sup}_{(a_{2k}, a_{2k-1})} \omega \quad \text{and} \quad u(\sigma_{2k+1}) \leq \operatorname*{ess\,inf}_{(a_{2k+1}, a_{2k})} \theta, \ k \geq 1.$$

Then

$$|G_\varepsilon(u)| \geq \int_a^{a_{k(\varepsilon)}} (\omega(t) - \theta(t))dt \quad \text{for each } \varepsilon \in (0, \varepsilon_0), \tag{3.3}$$

where $k(\varepsilon)$ and ε_0 are appearing in (3.1). *Moreover, if for a real number $c \in (a, b)$ there is an $\varepsilon_c \in (0, \varepsilon_0)$ such that $a_{k(\varepsilon)-1} \in (a, c)$ for each $\varepsilon \in (0, \varepsilon_c)$ then we have*

$$|G_\varepsilon(u|_{[a,c]})| \geq \int_a^{a_{k(\varepsilon)}} (\omega(t) - \theta(t))dt \quad \text{for each } \varepsilon \in (0, \varepsilon_c). \tag{3.4}$$

Let us remark that the condition (3.2) can be easy satisfied if for instance $\theta(t)$ is decreasing and $\omega(t)$ is increasing on $[a, b]$. The proof of Lemma 3.1 is omitted because it is very similar to the proof of [12, Lemma 2.1, p. 271].

Next, we want to find some conditions on $f(t, \eta, \xi)$ such that each solution u of (1.1) admits rapid oscillations in the sense of Lemma 3.1.

Lemma 3.2. *Let a_k be a decreasing sequence of real numbers from interval (a, b) satisfying* (3.1). *Let for each $k \geq 1$ the obstacles $\varphi(t)$ and $\psi(t)$ satisfy:*

$$\begin{gathered} \operatorname*{ess\,inf}_{(a_{2k}, a_{2k-1})} \varphi < \operatorname*{ess\,sup}_{(a_{2k}, a_{2k-1})} \psi/2 < \operatorname*{ess\,inf}_{(a_{2k}, a_{2k-1})} \psi, \\ \operatorname*{ess\,sup}_{(a_{2k+1}, a_{2k})} \psi > \operatorname*{ess\,inf}_{(a_{2k+1}, a_{2k})} \varphi/2 > \operatorname*{ess\,sup}_{(a_{2k+1}, a_{2k})} \varphi. \end{gathered} \tag{3.5}$$

Let for each $k \geq 1$ the sets J_k be defined by:

$$J_{2k} = (\operatorname*{ess\,inf}_{(a_{2k},a_{2k-1})} \varphi, \operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \psi/2),$$
$$J_{2k+1} = (\operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \varphi/2, \operatorname*{ess\,sup}_{(a_{2k+1},a_{2k})} \psi).$$

Next, let for each $k \geq 1$ the Caratheodory function $f(t,\eta,\xi)$ satisfy:

$$f(t,\eta,\xi) \geq 0, \quad t \in (a_{2k}, a_{2k-1}),\ \eta \in J_{2k},\ \xi \in \mathbb{R}, \tag{3.6}$$

$$\begin{aligned} &\int_{A_{2k}} \operatorname*{ess\,inf}_{(\eta,\xi)\in J_{2k}\times R} f(t,\eta,\xi)dt \\ &> \frac{c(p)}{(a_{2k-1}-a_{2k})^{p-1}} \frac{(\operatorname{ess\,inf}_{(a_{2k},a_{2k-1})} \psi - \operatorname{ess\,inf}_{(a_{2k},a_{2k-1})} \varphi)^p}{\operatorname{ess\,inf}_{(a_{2k},a_{2k-1})} \psi - \operatorname{ess\,sup}_{(a_{2k},a_{2k-1})} \psi/2}, \end{aligned} \tag{3.7}$$

and

$$f(t,\eta,\xi) \leq 0, \quad t \in (a_{2k+1}, a_{2k}),\ \eta \in J_{2k+1},\ \xi \in \mathbb{R}, \tag{3.8}$$

$$\begin{aligned} &\int_{A_{2k+1}} \operatorname*{ess\,sup}_{(\eta,\xi)\in J_{2k+1}\times R} f(t,\eta,\xi)dt \\ &< -\frac{c(p)}{(a_{2k}-a_{2k+1})^{p-1}} \frac{(\operatorname{ess\,sup}_{(a_{2k+1},a_{2k})} \psi - \operatorname{ess\,sup}_{(a_{2k+1},a_{2k})} \varphi)^p}{\operatorname{ess\,inf}_{(a_{2k+1},a_{2k})} \varphi/2 - \operatorname{ess\,sup}_{(a_{2k+1},a_{2k})} \varphi}, \end{aligned} \tag{3.9}$$

where $c(p) = 2[4(p-1)]^{p-1}$ and A_k is a family of sets defined by

$$A_k = [a_k + \frac{1}{4}(a_{k-1}-a_k),\ a_{k-1} - \frac{1}{4}(a_{k-1}-a_k)],\ k \geq 1.$$

Then for any solution u of (1.1) there is a sequence $\sigma_k \in (a_k, a_{k-1})$ which satisfies

$$u(\sigma_{2k}) \geq \operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \psi/2 \quad \text{and} \quad u(\sigma_{2k+1}) \leq \operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \varphi/2,\ k \geq 1. \tag{3.10}$$

Proof. Let k be a fixed natural number, $k \geq 1$, and let

$$\omega_2 = \operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \psi/2 \quad \text{and} \quad \theta_1 = \operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \varphi/2.$$

Regarding to the hypotheses (3.5) and (3.6)–(3.9), it is clear that the assumptions of Lemma 2.1 and Lemma 2.2 are satisfied on the intervals $[a_{2k}, a_{2k-1}]$ and $[a_{2k+1}, a_{2k}]$ respectively. Therefore, we may use these two lemmas and so, there is a $\sigma_2 = \sigma_{2k} \in (a_{2k}, a_{2k-1})$ and $\sigma_1 = \sigma_{2k+1} \in (a_{2k+1}, a_{2k})$ satisfying (2.4) and (2.8) respectively. Since k is arbitrarily fixed, it implies the existence of a sequence $\sigma_k \in (a_{k+1}, a_k)$ which satisfies the desired condition (3.10). □

Combining the preceding two lemmas we derive some new metric properties for solutions of (1.1). It is the subject of the following result.

Theorem 3.3. *For arbitrarily given real number $s \in (1,2)$, let the sequence a_k and the obstacles φ and ψ be given by:*

$$\begin{aligned} a_k = a + \frac{b-a}{2}(\frac{1}{k})^{1/\beta}, \quad k \geq 1, \\ \varphi(t) = -2(t-a) \quad \text{and} \quad \psi(t) = 2(t-a),\ t \in (a,b), \end{aligned} \tag{3.11}$$

where β satisfies $1 < \beta < \infty$ and $\beta = \frac{s}{2-s}$. If the Caratheodory function $f(t,\eta,\xi)$ satisfies (3.6)–(3.9) *in respect to such (φ,ψ,a_k), then each solution u of* (1.1) *satisfies:*

$$|G_\varepsilon(u)| \geq \frac{1}{2^6}(b-a)^s\varepsilon^{2-s} \quad \text{for each } \varepsilon \in (0, \varepsilon_0 = \frac{b-a}{\beta}), \tag{3.12}$$

$$|G_\varepsilon(u|_{[a,c]})| \geq \frac{1}{2^6}(c-a)^s\varepsilon^{2-s} \quad \text{for each } \varepsilon \in (0, \varepsilon_c), \tag{3.13}$$

$$\dim_M G(u) \geq s \quad \text{and} \quad M^s(G((y)) \geq \frac{1}{2^7}(b-a)^s, \tag{3.14}$$

$$M^s(G(u) \cap B_r(a,u(a))) \geq \frac{1}{2^7}(\frac{r}{\sqrt{5}})^s \quad \text{for any } r \in (0, b-a), \tag{3.15}$$

where $\varepsilon_c = \min\{\varepsilon_0, \frac{(c-a)^{\beta+1}}{\beta(b-a)^\beta}\}$.

Proof. The proof is done in a few steps.

Proof of (3.12). It is not difficult to check see the proof of [12, Corollary 5.2, p. 289], that the sequence a_k given in (3.11) satisfies the hypothesis (3.1) in respect to ε_0 and $k(\varepsilon)$ determined by

$$c_0 \left(\frac{1}{\varepsilon}\right)^{\frac{\beta}{\beta+1}} \leq k(\varepsilon) \leq 2c_0 \left(\frac{1}{\varepsilon}\right)^{\frac{\beta}{\beta+1}} \quad \text{for each } \varepsilon \in (0, \varepsilon_0), \tag{3.16}$$

where $c_0 = 2\left(\frac{b-a}{\beta}\right)^{\frac{\beta}{\beta+1}}$ and $\varepsilon_0 = \frac{b-a}{\beta}$.

Let us remark that double inequalities in (3.16) is needed to ensure $k(\varepsilon) \in \mathbf{N}$. Also, it is clear that the obstacles φ and ψ given in (3.11) satisfy the hypothesis (3.5). Thus, the assumptions of Lemma 3.2 are fulfilled and therefore, we have that each solution u of (1.1) has rapid oscillations in the sense of (3.10). Moreover, it implies that each solution u of (1.1) satisfies the main assumption of Lemma 3.1, where $\omega = \psi/2$ and $\theta = \varphi/2$. So, we obtain

$$|G_\varepsilon(u)| \geq \frac{1}{2}\int_a^{a_{k(\varepsilon)}} (\psi(t) - \varphi(t))dt \quad \text{for each } \varepsilon \in (0, \varepsilon_0).$$

Putting the data from (3.11) in the right hand side of the preceding inequality, we get

$$|G_\varepsilon(u)| \geq \int_a^{a_{k(\varepsilon)}} 2(t-a)dt = \big(\frac{b-a}{2}\big)^2\big(\frac{1}{k(\varepsilon)}\big)^{2/\beta} \quad \text{for each } \varepsilon \in (0, \varepsilon_0). \tag{3.17}$$

Let us remark that from the left inequality in (3.16) we have in particular

$$\frac{1}{k(\varepsilon)} \geq \frac{1}{4}\big(\frac{\beta}{b-a}\big)^{\frac{\beta}{\beta+1}}\varepsilon^{\frac{\beta}{\beta+1}} \quad \text{for each } \varepsilon \in (0, \varepsilon_0).$$

Putting this inequality in (3.17), for any $\varepsilon \in (0, \varepsilon_0)$, we get

$$|G_\varepsilon(u)| \geq \frac{(b-a)^2}{4}\left(\frac{1}{4}\right)^{\frac{2}{\beta}} \frac{\beta^{\frac{2}{\beta+1}}}{(b-a)^{\frac{2}{\beta+1}}}\varepsilon^{\frac{2}{\beta+1}} \geq \frac{1}{2^6}(b-a)^s\varepsilon^{2-s},$$

where we have used that $\beta > 1$ and $2\beta/(\beta+1) = s$. Thus, we have proved the inequality (3.12).

Proof of (3.13). For $c \in (a,b)$, let

$$\varepsilon_c = \min\{\varepsilon_0 = \frac{b-a}{\beta}, \frac{1}{\beta}\frac{(c-a)^{\beta+1}}{(b-a)^{\beta}}\}. \tag{3.18}$$

It is easy to check that for any $c \in (a,b)$ the number ε_c given by (3.18) satisfies

$$a_{k(\varepsilon)-1} \in (a,c) \quad \text{for each } \varepsilon \in (0,\varepsilon_c),$$

where the sequence a_k is given in (3.11) and the number $k(\varepsilon)$ is given in (3.16). Therefore, we may apply Lemma 3.1 again and so, for each $c \in (a,b)$ and for any solution u of (1.1) we have

$$|G_\varepsilon(u|_{[a,c]})| \geq \frac{1}{2}\int_a^{a_{k(\varepsilon)}} (\psi(t)-\varphi(t))dt \quad \text{for each } \varepsilon \in (0,\varepsilon_c).$$

Putting the data from (3.11) in this inequality and using the same calculation as in the proof of (3.12) we prove (3.13).

Proof of (3.14). According to the definition of $\dim_M G(u)$, from (3.12) immediately follows that

$$\begin{aligned}
\dim_M G(u) &= \limsup_{\varepsilon\to 0}\big(2 - \frac{\log|G_\varepsilon(u)|}{\log\varepsilon}\big)\\
&\geq \limsup_{\varepsilon\to 0}\big(2 - \frac{\log[\varepsilon^{2-s}(b-a)^s/2^6]}{\log\varepsilon}]\big)\\
&= \limsup_{\varepsilon\to 0}\big(2-(2-s)\frac{\log\varepsilon}{\log\varepsilon} - \frac{\log[(b-a)^s/2^6]}{\log\varepsilon}\big) = s.
\end{aligned}$$

It proves the first inequality in (3.14). Also, according to the definition of $M^s(G(u))$, from (3.12) we get:

$$\begin{aligned}
M^s(G(u)) &= \limsup_{\varepsilon\to 0}(2\varepsilon)^{s-2}|G_\varepsilon(u)| \geq \limsup_{\varepsilon\to 0}(2\varepsilon)^{s-2}[\frac{(b-a)^s}{2^6}\varepsilon^{2-s}]\\
&= 2^{s-2}\frac{(b-a)^s}{2^6}\limsup_{\varepsilon\to 0}(\varepsilon^{s-2}\varepsilon^{2-s}) \geq \frac{1}{2^7}(b-a)^s,
\end{aligned}$$

which proves the second inequality in (3.14).

Proof of (3.15). At the first, since $u \in K(\varphi,\psi)$ we have in particular that

$$\varphi(t) \leq u(t) \leq \psi(t), t \in [a,b] \quad \text{and} \quad u(a) = 0.$$

Making intersections of $\varphi(t) = -2(t-a)$ and $\psi(t) = 2(t-a)$ with $B_r(a,0)$, it is easy to see that

$$G(u|_{[a,a+\frac{r}{\sqrt{5}}]}) \subseteq G(u) \cap B_r(a,0) \quad \text{for any } r \in (0,\sqrt{5}(b-a)),$$

and so, we have

$$M^s(G(u|_{[a,a+\frac{r}{\sqrt{5}}]})) \leq M^s(G(u)\cap B_r(a,0)) \quad \text{for any } r \in (0,(b-a)). \tag{3.19}$$

On the other hand, using (3.13) for $c = a + \frac{r}{\sqrt{5}}$, we get

$$M^s(G(u|_{[a,a+\frac{r}{\sqrt{5}}]})) \geq \frac{1}{2^7}(\frac{r}{\sqrt{5}})^s \quad \text{for any } r \in (0,b-a).$$

Combining this inequality with (3.19) we get the proof of (3.15). Thus, we have proved all statements of Theorem 3.3. □

At the end of this section, we give an example of such a class of the nonlinearity $f(t,\eta,\xi)$ which satisfies the assumptions of Theorem 3.3.

Example 3.4. In order to simplify the notation, let $\tilde\theta_{2k+1}$, $\tilde\omega_{2k+1}$, $\tilde\theta_{2k}$, $\tilde\omega_{2k}$, θ_{2k+1}, and ω_{2k} be defined by:

$$\tilde\theta_{2k}=\operatorname*{ess\,inf}_{(a_{2k},a_{2k-1})}\varphi,\quad \tilde\omega_{2k}=\operatorname*{ess\,inf}_{(a_{2k},a_{2k-1})}\psi,\quad \omega_{2k}=\operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})}\psi/2,$$
$$\tilde\theta_{2k+1}=\operatorname*{ess\,sup}_{(a_{2k+1},a_{2k})}\varphi,\quad \tilde\omega_{2k+1}=\operatorname*{ess\,sup}_{(a_{2k+1},a_{2k})}\psi,\quad \theta_{2k+1}=\operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})}\varphi/2.$$

Let $f=f(t,\eta,\xi)$ be a Caratheodory function

$$\begin{aligned} f=\frac{\pi c(p)}{\sin\frac{\pi}{4}}\sum_{k=1}^{\infty}\Big[&(\tilde\omega_{2k}-\tilde\theta_{2k})^p\frac{(\eta-\tilde\omega_{2k})^-}{(\tilde\omega_{2k}-\omega_{2k})^2}\frac{\sin(\frac{\pi}{a_{2k-1}-a_{2k}}(t-a_{2k}))}{(a_{2k-1}-a_{2k})^p}K_{[a_{2k},a_{2k-1}]}(t)\\ &-(\tilde\omega_{2k+1}-\tilde\theta_{2k+1})^p\frac{(\eta-\tilde\theta_{2k+1})^+}{(\theta_{2k+1}-\tilde\theta_{2k+1})^2}\frac{\sin(\frac{\pi}{a_{2k}-a_{2k+1}}(t-a_{2k-1}))}{(a_{2k}-a_{2k+1})^p}K_{[a_{2k+1},a_{2k}]}(t)\Big],\end{aligned}$$

where $c(p)$ is appearing in (3.7) and (3.9) , and where $K_A(t)$ denotes as usually the characteristic function of a set A. Also, $\eta^-=\max\{0,-\eta\}$ and $\eta^+=\max\{0,\eta\}$. It is not difficult to check that $f(t,\eta,\xi)$ is continuous in all its variables and that $f(t,\eta,\xi)$ satisfies the hypotheses (3.6)–(3.9).

4. Lower bounds for $\dim_M G(u')$ and $M^s(G(u'))$

In this section, the inequalities (1.6) and (1.7) will be verified. As the first, we give a discrete version of Lemma 3.1, which is a modification of [12, Lemma 6.3, p. 291].

Lemma 4.1. *Let σ_k be a decreasing sequence of real numbers from interval (a,b) satisfying*

$$\begin{gathered}\sigma_k\searrow a \text{ and there is an } \varepsilon_0>0 \text{ such that for each } \varepsilon\in(0,\varepsilon_0)\\ \text{there is a } k(\varepsilon)\in\mathbf{N} \text{ such that } \sigma_{j-1}-\sigma_j\le\varepsilon/2 \text{ for each } j\ge k(\varepsilon).\end{gathered}\tag{4.1}$$

Let δ_k be a sequence of real numbers such that

$$\delta_{2k+1}>0 \text{ and } \delta_{2k}<0,\ k\ge1.$$

Let z be a continuous function on $(a,b]$ for which there is a sequence $s_k\in(\sigma_k,\sigma_{k-1})$ such that

$$z(s_{2k+1})\ge\delta_{2k+1}\quad\text{and}\quad z(s_{2k})\le\delta_{2k},\quad k\ge1.$$

Then there holds true

$$|G_\varepsilon(z)|\ge\sum_{k=k(\varepsilon)}^{\infty}\delta_{2k+1}(\sigma_{2k}-\sigma_{2k+1})\quad\text{for each}\quad\varepsilon\in(0,\varepsilon_0),$$

where $k(\varepsilon)$ and ε_0 are appearing in (4.1).

The proof of the lemma above is exactly the same as the proof of [12, Lemma 2.1, p. 271].

As a basic result, we need the following lemma on the asymptotic behaviour of $|G_\varepsilon(u')|$ as $\varepsilon\approx0$, where u' is the derivative in the classical sense of any smooth enough real function u.

Lemma 4.2. *Let a_k be a decreasing sequence of real numbers from interval (a,b) satisfying* (3.1). *Let ω_{2k} and θ_{2k+1} be two sequences of real numbers satisfying*

$$\omega_{2k} > \min\{\theta_{2k+1}, \theta_{2k-1}\}, \quad k \geq 1. \tag{4.2}$$

Let u be a real function, $u \in C((a,b]) \cap C^1(a,b)$, for which there is a sequence $\sigma_k \in (a_k, a_{k-1})$ such that

$$u(\sigma_{2k}) \geq \omega_{2k} \quad \text{and} \quad u(\sigma_{2k+1}) \leq \theta_{2k+1}, \quad k \geq 1. \tag{4.3}$$

Then

$$|G_\varepsilon(u')| \geq \sum_{k=k(\varepsilon/2)}^{\infty} (\omega_{2k} - \theta_{2k+1}) \quad \text{for each } \varepsilon \in (0, \varepsilon_0), \tag{4.4}$$

where $k(\varepsilon)$ and ε_0 are defined in (3.1).

Proof. Lagrange's mean value theorem, applied on the interval (σ_k, σ_{k-1}), where the sequence σ_k is defined in (4.3), we get the existence of a sequence $s_k \in (\sigma_k, \sigma_{k-1})$, $k \geq 1$ such that

$$u'(s_{2k+1}) = \frac{u(\sigma_{2k}) - u(\sigma_{2k+1})}{\sigma_{2k} - \sigma_{2k+1}}, \quad u'(s_{2k}) = \frac{u(\sigma_{2k-1}) - u(\sigma_{2k})}{\sigma_{2k-1} - \sigma_{2k}}. \tag{4.5}$$

Using (4.3) and the notation:

$$z(t) = u'(t),\ t \in (a,b), \quad \delta_{2k+1} = \frac{\omega_{2k} - \theta_{2k+1}}{\sigma_{2k} - \sigma_{2k+1}}, \quad \delta_{2k} = \frac{\theta_{2k-1} - \omega_{2k}}{\sigma_{2k-1} - \sigma_{2k}},$$

the statement (4.5) can be rewritten in the form: there is $s_k \in (\sigma_k, \sigma_{k-1})$, $k \geq 1$, such that

$$z(s_{2k+1}) \geq \delta_{2k+1} > 0 \quad \text{and} \quad z(s_{2k}) \leq \delta_{2k} < 0, \quad k \geq 1. \tag{4.6}$$

On the other hand, it is easy to see that the sequence σ_k just like a_k satisfies a very similar condition to (3.1); that is,

$$\sigma_k \searrow a \quad \text{and} \quad \sigma_{j-1} - \sigma_j \leq \varepsilon/2 \quad \text{for each } j \geq k(\frac{\varepsilon}{2}),\ \varepsilon \in (0, \varepsilon_0), \tag{4.7}$$

where $k(\varepsilon)$ and ε_0 are exactly the same as in (3.1). Now, by means of (4.6) and (4.7), we have that the function z satisfies the assumptions of Lemma 4.1 and so, we get:

$$\begin{aligned}
|G_\varepsilon(u')| = |G_\varepsilon(z)| &\geq \sum_{k=k(\varepsilon/2)}^{\infty} \delta_{2k+1}(\sigma_{2k} - \sigma_{2k+1}) \\
&= \sum_{k=k(\varepsilon/2)}^{\infty} \frac{(\omega_{2k} - \theta_{2k+1})}{\sigma_{2k} - \sigma_{2k+1}}(\sigma_{2k} - \sigma_{2k+1}) \\
&= \sum_{k=k(\varepsilon/2)}^{\infty} (\omega_{2k} - \theta_{2k+1}) \quad \text{for each } \varepsilon \in (0, \varepsilon_0).
\end{aligned}$$

Thus, Lemma 4.2 is proved. □

Next, we give the main result of this section.

Theorem 4.3. *Let the hypotheses of Theorem 3.3 be still assumed; that is: for arbitrarily given real number $s \in (1,2)$, let the sequence a_k and the obstacles φ and ψ be given by* (3.11)*, and let the Caratheodory function $f(t,\eta,\xi)$ satisfy* (3.6)–(3.9) *in respect to such (φ,ψ,a_k). Then each solution u of* (1.1) *satisfies:*

$$|G_\varepsilon(u')| \geq \frac{\sqrt{2}}{2^4}(b-a)^{s/2}\varepsilon^{1-s/2} \quad \text{for each } \varepsilon \in (0,\varepsilon_0 = \frac{b-a}{\beta}), \tag{4.8}$$

$$\dim_M G(u') \geq 1 + \frac{s}{2} \quad \text{and} \quad M^{1+s/2}(G(u')) \geq \frac{1}{2^4}(b-a)^{s/2}. \tag{4.9}$$

The proof of the above theorem can be done with similar arguments as in [14, Theorem 3.4 and Corollary 3.5].

5. Full control of ess inf and ess sup of solutions

In contrast to the method of control of ess inf and ess sup of solutions of (1.1) which was presented in Section 2, here we involve on the nonlinearity $f(t,\eta,\xi)$ slightly stronger conditions than (2.3) and (2.7) to obtain some stronger conclusions than (2.4) and (2.8). More precisely, for any solution u of (1.1) we need to estimate from below the measure of sets where $\operatorname{ess\,inf} u$ and $\operatorname{ess\,sup} u$ are exceeded. It will play an important role in the following section, where the inequality (1.8) and (1.9) will be proved. The so called full control of ess inf and ess sup of solutions of corresponding equation (1.10) was considered in [12, Section 4]. Here, it is the subject of the following two lemmas.

Lemma 5.1. *Let $(a_2,b_2) \subset\subset (a,b)$ be an open interval. Let ω_2 be an arbitrarily given real number such that*

$$\operatorname*{ess\,sup}_{(a_2,b_2)} \varphi < \omega_2 < \operatorname*{ess\,inf}_{(a_2,b_2)} \psi. \tag{5.1}$$

Let J_2 be a set defined by $J_2 = (\operatorname{ess\,inf}_{(a_2,b_2)} \varphi, \omega_2)$ and let the Caratheodory function $f(t,\eta,\xi)$ satisfy:

$$f(t,\eta,\xi) \geq 0, \quad t \in (a_2,b_2),\ \eta \in J_2,\ \xi \in \mathbb{R}, \tag{5.2}$$

$$\operatorname*{ess\,inf}_{t\in A_2} f(t,\eta,\xi) > \frac{c(p)}{(b_2-a_2)^p}\frac{(\operatorname{ess\,sup}_{(a_2,b_2)}\psi - \operatorname{ess\,inf}_{(a_2,b_2)}\varphi)^p}{\operatorname{ess\,inf}_{(a_2,b_2)}\psi - \omega_2}, \quad \eta \in J_2,\ \xi \in \mathbb{R}, \tag{5.3}$$

where $c(p) = 2(16^p)(p-1)^{p-1}$ and A_2 is a set defined by

$$A_2 = [a_2 + \frac{1}{16}(b_2-a_2),\, b_2 - \frac{1}{16}(b_2-a_2)].$$

Then for any solution u of (1.1) *we have*

$$u(t) \geq \omega_2 \quad \text{for each } t \in [a_2 + \frac{1}{4}(b_2-a_2), b_2 - \frac{1}{4}(b_2-a_2)]. \tag{5.4}$$

The dual result of Lemma 5.1 is the following.

Lemma 5.2. *Let $(a_1,b_1) \subset\subset (a,b)$ be an open interval. Let θ_1 be an arbitrarily given real number such that*

$$\operatorname*{ess\,sup}_{(a_1,b_1)} \varphi < \theta_1 < \operatorname*{ess\,inf}_{(a_1,b_1)} \psi.$$

Let J_1 *be a set defined by* $J_1 = (\theta_1, \operatorname{ess\,sup}_{(a_1,b_1)} \psi)$ *and let the Caratheodory function* $f(t,\eta,\xi)$ *satisfy:*

$$f(t,\eta,\xi) \le 0, \quad t \in (a_1,b_1),\ \eta \in J_1,\ \xi \in \mathbb{R},$$

$$\operatorname*{ess\,sup}_{t\in A_1} f(t,\eta,\xi) < -\frac{c(p)}{(b_1-a_1)^p} \frac{(\operatorname{ess\,sup}_{(a_1,b_1)} \psi - \operatorname{ess\,inf}_{(a_1,b_1)} \varphi)^p}{\theta_1 - \operatorname{ess\,sup}_{(a_1,b_1)} \varphi}, \quad \eta \in J_1,\ \xi \in \mathbb{R},$$

where $c(p) = 2(16^p)(p-1)^{p-1}$ *and* A_1 *is a set defined by*

$$A_1 = [a_1 + \frac{1}{16}(b_1-a_1),\, b_1 - \frac{1}{16}(b_1-a_1)].$$

Then for any solution u *of* (1.1) *we have*

$$u(t) \le \theta_1 \quad \text{for each } t \in [a_1 + \frac{1}{4}(b_1-a_1), b_1 - \frac{1}{4}(b_1-a_1)]. \tag{5.5}$$

The above lemma can be proved analogously as in the proof of Lemma 5.1 to be shown below. For the proof we use the following two propositions that will be shown later.

Proposition 5.3. *Let* $(c,d) \subseteq (a_2,b_2)$ *be an open interval. Let* ω_2 *be an arbitrarily given real number such that*

$$\operatorname*{ess\,sup}_{(c,d)} \varphi < \omega_2 < \operatorname*{ess\,inf}_{(c,d)} \psi. \tag{5.6}$$

Let J_2 *be a set defined by* $J_2 = (\operatorname{ess\,inf}_{(c,d)} \varphi, \omega_2)$ *and let the Caratheodory function* $f(t,\eta,\xi)$ *satisfy*

$$f(t,\eta,\xi) \ge 0, \quad t \in (c,d),\ \eta \in J_2,\ \xi \in \mathbb{R}. \tag{5.7}$$

Then for any solution u *of* (1.1) *such that* $u(c) = u(d) = \omega_2$ *there is a* $t^* \in (c,d)$ *satisfying*

$$u(t^*) \ge \omega_2. \tag{5.8}$$

The condition $u(c) = u(d) = \omega_2$ can be avoided as follows.

Proposition 5.4. *Let* $(c,d) \subseteq (a_2,b_2)$ *be an open interval such that*

$$N(c,d) \subseteq (a_2,b_2), \quad \text{where } N(c,d) = (c - \frac{d-c}{2}, d + \frac{d-c}{2}).$$

Let ω_2 *be an arbitrarily given real number such that*

$$\operatorname*{ess\,inf}_{N(c,d)} \varphi < \omega_2 < \operatorname*{ess\,inf}_{N(c,d)} \psi. \tag{5.9}$$

Let J_2 *be a set defined by* $J_2 = (\operatorname{ess\,inf}_{(a_2,b_2)} \varphi, \omega_2)$ *and let the Caratheodory function* $f(t,\eta,\xi)$ *satisfy:*

$$f(t,\eta,\xi) \ge 0, \quad t \in N(c,d),\ \eta \in J_2,\ \xi \in \mathbb{R}, \tag{5.10}$$

$$\operatorname*{ess\,inf}_{t\in(c,d)} f(t,\eta,\xi) > 2^{p+1} \frac{(p-1)^{p-1}}{(d-c)^p} \frac{(\operatorname{ess\,sup}_{(a_2,b_2)} \psi - \operatorname{ess\,inf}_{(a_2,b_2)} \varphi)^p}{\operatorname{ess\,inf}_{N(c,d)} \psi - \omega_2}, \tag{5.11}$$

for $\eta \in J_2$ *and* $\xi \in \mathbb{R}$. *Then for any solution* u *of* (1.1) *there is a* $t^* \in N(c,d)$ *satisfying* $u(t^*) \ge \omega_2$.

The proof of these two propositions will be presented later; meanwhile we proceed with the proof of Lemma 5.1.

Proof of Lemma 5.1. Since for any $(c,d) \subseteq (a_2,b_2)$ and a function $g = g(t)$ we have

$$\operatorname*{ess\,inf}_{(a_2,b_2)} g \le \operatorname*{ess\,inf}_{(c,d)} g \quad \text{and} \quad \operatorname*{ess\,sup}_{(c,d)} g \le \operatorname*{ess\,sup}_{(a_2,b_2)} g,$$

one can show that the main hypotheses (5.1)–(5.2) guarantee that the conditions (5.6)–(5.7) and (5.9)–(5.10) are satisfied, where $(c,d) \subseteq a_2, b_2)$ such that $N(c,d) \subseteq (a_2,b_2)$. Thus, Proposition 5.3 may be used here as well as Proposition 5.4 provided the hypothesis (5.11) is satisfied too.

Next, we claim that:

$$\begin{gathered} \text{for any } (c,d) \subseteq A_2 \text{ such that } d-c = (b_2-a_2)/8 \\ \text{there is } t^* \in (c - \frac{b_2-a_2}{16}, d + \frac{b_2-a_2}{16}) \text{ such that } u(t^*) \ge \omega_2, \end{gathered} \tag{5.12}$$

where

$$A_2 = [a_2 + \frac{b_2-a_2}{16},\, b_2 - \frac{b_2-a_2}{16}].$$

To prove (5.12), let (c,d) be an open interval such that $(c,d) \subseteq A_2$ and $d - c = (b_2 - a_2)/8$. It is clear that

$$N(c,d) = (c - \frac{b_2-a_2}{16},\, d + \frac{b_2-a_2}{16}) \subseteq (a_2,b_2),$$

where $N(c,d) = (c - \frac{d-c}{2}, d + \frac{d-c}{2})$. Putting $b_2 - a_2 = 8(d-c)$ in (5.3) and using $c(p) = 2(16^p)(p-1)^{p-1}$ we get

$$\begin{aligned} &\operatorname*{ess\,inf}_{t\in(c,d)} f(t,\eta,\xi) \ge \operatorname*{ess\,inf}_{t\in A_2} f(t,\eta,\xi) \\ &> 2^{p+1} \frac{(p-1)^{p-1}}{(d-c)^p} \frac{(\operatorname{ess\,sup}_{(a_2,b_2)} \psi - \operatorname{ess\,inf}_{(a_2,b_2)} \varphi)^p}{\operatorname{ess\,inf}_{(a_2,b_2)} \psi - \omega_2} \\ &\ge 2^{p+1} \frac{(p-1)^{p-1}}{(d-c)^p} \frac{(\operatorname{ess\,sup}_{(a_2,b_2)} \psi - \operatorname{ess\,inf}_{(a_2,b_2)} \varphi)^p}{\operatorname{ess\,inf}_{N(c,d)} \psi - \omega_2}, \quad \eta \in J_2,\ \xi \in \mathbb{R}. \end{aligned}$$

Therefore, the assumption (5.11) is satisfied too and so, by Proposition 5.4 there is $t^* \in N(c,d)$ such that $u(t^*) \ge \omega_2$. Thus, the assertion (5.12) is verified.

Next, we define two intervals (c_1,d_1) and (c_2,d_2) by

$$\begin{aligned} (c_1,d_1) &= (a_2 + \frac{1}{16}(b_2-a_2), a_2 + \frac{3}{16}(b_2-a_2)) \\ (c_2,d_2) &= (b_2 - \frac{3}{16}(b_2-a_2), b_2 - \frac{1}{16}(b_2-a_2)). \end{aligned}$$

It is easy to check that

$$(c_i,d_i) \subseteq A_2 \quad \text{and} \quad d_i - c_i = (b_2-a_2)/8, \text{ for } i = 1,2.$$

So, applying (5.12) to both interval $[c_1,d_1]$ and $[c_2,d_2]$ we get two points t_1^* and t_2^* such that

$$t_i^* \in (c_i - \frac{1}{16}(b_2-a_2), d_i + \frac{1}{16}(b_2-a_2)) \quad \text{and} \quad u(t_i^*) \ge \omega_2, \quad \text{for } i = 1,2. \tag{5.13}$$

It is clear that

$$\big[a_2 + \frac{1}{4}(b_2-a_2), b_2 - \frac{1}{4}(b_2-a_2)\big] \subseteq [t_1^*, t_2^*] \subseteq (a_2,b_2). \tag{5.14}$$

Next, we claim that

$$u(t) \ge \omega_2 \quad \text{for each } t \in [t_1^*, t_2^*]. \tag{5.15}$$

On the contrary, if there is a point $t_0 \in [t_1^*, t_2^*]$ satisfying $u(t_0) < \omega_2$ then by means of (5.13) we can construct an open interval $(c,d) \subseteq (t_1^*, t_2^*)$ such that $u(c) = u(d) = \omega_2$ and $u(t) < \omega_2$ in (c,d). For example, we can choose c and d by

$$c = \max\{t \in [t_1^*, t_0] : u(t) = \omega_2\} \quad \text{and} \quad d = \min\{t \in [t_0, t_2^*] : u(t) = \omega_2\}.$$

But, by Proposition 5.3 it is not possible and so, the assertion (5.15) holds true. Because of (5.14), it gives us the desired conclusion (5.4). Thus, Lemma 5.1 is proved. □

Proof of Proposition 5.3. Let us suppose the opposite claim to (5.8); that is,

$$u(c) = u(d) = \omega_2 \quad \text{and} \quad u(t) < \omega_2 \quad \text{for each } t \in (c,d). \tag{5.16}$$

We are going to prove that (5.16) is not possible. In this direction, let v be a test function defined by

$$v(t) = \begin{cases} \omega_2 & \text{in } (c,d), \\ u(t) & \text{otherwise.} \end{cases}$$

Since $u \in K(\varphi, \psi)$ and because of (5.6) and (5.16), we have also that $v \in K(\varphi, \psi)$ and

$$v(t) - u(t) = \begin{cases} \omega_2 - u(t) > 0 & \text{in } (c,d), \\ 0 & \text{otherwise.} \end{cases}$$

Hence, this test function can be applied in (1.1) and so, we obtain

$$0 \le \int_c^d |u'|^p dt \le -\int_c^d f(t,u,u')(\omega_2 - u(t))dt \le 0,$$

where the main assumption (5.7) is used. So, we get $u' = 0$ in (c,d). But, it contradicts (5.16). Thus, (5.16) is not possible and the desired conclusion (5.8) is proved. □

Proof of Proposition 5.4. Let $(c,d) \subseteq (a_2, b_2)$ be an interval such that $N(c,d) \subseteq (a_2, b_2)$, where $N(c,d) = (c - \frac{d-c}{2}, d + \frac{d-c}{2})$. Let ω_2 be an arbitrarily given real number satisfying (5.9) and let the Caratheodory function $f(t,\eta,\xi)$ satisfy (5.10) and (5.11). Immediately from (5.11) we get

$$\int_c^d \operatorname*{ess\,inf}_{(\eta,\xi)\in J_2 \times R} f(t,\eta,\xi)dt > 2^p \frac{(p-1)^{p-1}}{(d-c)^{p-1}} \frac{(\operatorname{ess\,sup}_{(a_2,b_2)} \psi - \operatorname{ess\,inf}_{(a_2,b_2)} \varphi)^p}{\operatorname{ess\,inf}_{N(c,d)} \psi - \omega_2}. \tag{5.17}$$

Let the numbers c_2 and d_2 and the set A_2 be defined by

$$c_2 = c - \frac{d-c}{2}, \quad d_2 = d + \frac{d-c}{2}, \quad A_2 = [c,d].$$

Then

$$\begin{gathered} N(c,d) = (c_2, d_2), \\ A_2 = [c_2 + \frac{1}{4}(d_2 - c_2), d_2 - \frac{1}{4}(d_2 - c_2)], \\ 2(d-c) = d_2 - c_2. \end{gathered}$$

Therefore, from the inequalities (5.9), (5.10) and (5.17), we get

$$\tilde{\theta}_0 \le \operatorname*{ess\,inf}_{(c_2,d_2)} \varphi < \omega_2 < \operatorname*{ess\,inf}_{(c_2,d_2)} \psi \le \tilde{\omega}_0,$$
$$f(t,\eta,\xi) \ge 0, \quad t \in (c_2,d_2),\ \eta \in J_2,\ \xi \in \mathbb{R},$$
$$\int_{A_2} \operatorname*{ess\,inf}_{(\eta,\xi)\in J_2\times R} f(t,\eta,\xi)dt > \frac{c(p)}{(d_2-c_2)^{p-1}} \frac{(\tilde{\omega}_0-\tilde{\theta}_0)^p}{\operatorname{ess\,inf}_{(c_2,d_2)} \psi - \omega_2},$$

where $J_2 = (\tilde{\theta}_0, \omega_2)$, $\tilde{\theta}_0 = \operatorname{ess\,inf}_{(a_2,b_2)} \varphi$, $\tilde{\omega}_0 = \operatorname{ess\,sup}_{(a_2,b_2)} \psi$, and $c(p) = 2[4(p-1)]^{p-1}$. Hence, the assumptions of Lemma 2.3 are satisfied especially on the open interval $(c_2,d_2) \subset\subset (a,b)$, it implies the existence of a $t^* \in (c_2,d_2)$ such that $u(t^*) \ge \omega_2$. Thus, Proposition 5.4 is shown. □

6. The asymptotic behaviour of $\|u'\|_{L^p}$ as $\varepsilon \approx 0$

In this section, we will study the asymptotic behaviour of $\|u'\|_{L^p}$ as $\varepsilon \approx 0$ which was presented by the inequalities (1.8) and (1.9). It will be made for such continuous functions which satisfy a "jumping" condition in the sense of (5.4) and (5.5), as follows.

Lemma 6.1. *Let a_k be a decreasing sequence of real numbers from interval (a,b) satisfying*

$$\begin{gathered} a_k \searrow a \text{ and } a_k - a_{k+1} \le a_{k-1} - a_k,\ k \ge 1 \text{ and} \\ \text{there is an } \varepsilon_2 > 0 \text{ such that for each } \varepsilon \in (0,\varepsilon_2) \\ \text{there is a } j(\varepsilon) \in \mathbf{N} \text{ such that } a_{j(\varepsilon)} > a + \varepsilon. \end{gathered} \tag{6.1}$$

Let u be a real function defined on $[a,b]$ such that $u \in W^{1,p}_{\rm loc}((a,b]) \cap C([a,b])$ and

$$\begin{gathered} u(t) > 0 \quad \text{for each } t \in \Lambda_{2k}, \\ u(t) < 0 \quad \text{for each } t \in \Lambda_{2k+1},\ k \ge 1, \end{gathered} \tag{6.2}$$

where

$$\Lambda_k = [a_k + \frac{1}{4}(a_{k-1}-a_k), a_{k-1} - \frac{1}{4}(a_{k-1}-a_k)], \quad k \ge 1.$$

Then there is a sequence $x_k \in (a,b)$, $k \in \mathbf{N}$ and a constants c only depending on given data such that each solution u of (1.1) satisfies

$$\int_{a+\varepsilon}^b |u'(t)|^p dt \ge c \sum_{k=3}^{j(\varepsilon)} \frac{(\max_{\Lambda_k} |u|)^p}{(a_{k-2}-a_{k-1})^{p-1}} \quad \text{for each } \varepsilon \in (0,\varepsilon_2), \tag{6.3}$$

where $j(\varepsilon)$ is appearing in (6.1).

Proof. First, it is well known (see for instance in [2, Theorem 9.12 pp.166]) that in the space $W^{1,p}_0(\Omega)$, $\Omega \subseteq \mathbb{R}^N$, there is a constant $c_p > 0$ such that for $u \in W^{1,p}_0(\Omega)$ and $p > N$,

$$\sup_\Omega |u| \le c_p |\Omega|^{1/N-1/p} \|\nabla u\|_p. \tag{6.4}$$

Next, let u be a real function satisfying (6.2). Then there is a sequence x_k of the zero-points of u such that:

$$\begin{gathered} u(x_k) = 0, \quad x_k \in (a_k - \frac{1}{4}(a_k - a_{k+1}), a_k + \frac{1}{4}(a_{k-1}-a_k)), \\ \Lambda_k \subseteq (x_k, x_{k-1}),\ k \ge 2 \quad \text{and} \quad |x_k - x_{k-1}| \le \frac{3}{2}(a_{k-2}-a_{k-1}), \quad k \ge 3. \end{gathered} \tag{6.5}$$

In particular for $N = 1$ and $\Omega = (x_k, x_{k-1})$ we have $u \in W_0^{1,p}(x_k, x_{k-1})$ and so, from (6.4) follows

$$\sup_{(x_k,x_{k-1})} |u| \le c_p |x_k - x_{k-1}|^{1-1/p} \|u'\|_{L^p(x_k,x_{k-1})};$$

that is to say

$$\|u'\|^p_{L^p(x_k,x_{k-1})} \ge c^p \frac{1}{|x_k - x_{k-1}|^{p-1}} \big(\sup_{(x_k,x_{k-1})} |u|\big)^p, \quad k \ge 2, \tag{6.6}$$

where the constant $c > 0$ does not depend on k, only on p. Now, according to (6.2), (6.5) and (6.6) we calculate that

$$\begin{aligned} \|u'\|^p_{L^p(a+\varepsilon,b)} &\ge \sum_{k=2}^{j(\varepsilon)} \|u'\|^p_{L^p(x_k,x_{k-1})} \ge c^p \sum_{k=2}^{j(\varepsilon)} \frac{1}{|x_k - x_{k-1}|^{p-1}} \big(\sup_{(x_k,x_{k-1})} |u|\big)^p \\ &\ge c^p (\frac{2}{3})^p \sum_{k=3}^{j(\varepsilon)} \frac{(\max_{\Lambda_k} |u|)^p}{(a_{k-2} - a_{k-1})^{p-1}} \quad \text{for each } \varepsilon \in (0, \varepsilon_2). \end{aligned}$$

Thus, Lemma 6.1 is proved. □

Combining Lemmas 5.1 and 5.2, we are able to derive a kind of rapid oscillations for solutions of (1.1) in the sense of (5.4) and (5.5).

Lemma 6.2. *Let a_k be a decreasing sequence of real numbers from interval (a, b) satisfying (3.1). Let for each $k \ge 1$ the obstacles $\varphi(t)$ and $\psi(t)$ satisfy:*

$$\begin{gathered} \operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \varphi < \operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \psi/2 < \operatorname*{ess\,inf}_{(a_{2k},a_{2k-1})} \psi, \\ \operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \psi > \operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \varphi/2 > \operatorname*{ess\,sup}_{(a_{2k+1},a_{2k})} \varphi. \end{gathered} \tag{6.7}$$

Let the sets J_k be defined by:

$$\begin{gathered} J_{2k} = (\operatorname*{ess\,inf}_{(a_{2k},a_{2k-1})} \varphi, \operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \psi/2), \\ J_{2k+1} = (\operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \varphi/2, \operatorname*{ess\,sup}_{(a_{2k+1},a_{2k})} \psi), \quad k \ge 1. \end{gathered}$$

Next, let for each $k \ge 1$ the Caratheodory function $f(t, \eta, \xi)$ satisfy

$$f(t, \eta, \xi) \ge 0, \quad t \in (a_{2k}, a_{2k-1}),\ \eta \in J_{2k},\ \xi \in \mathbb{R}, \tag{6.8}$$

$$\operatorname*{ess\,inf}_{t\in A_{2k}} f(t, \eta, \xi) > \frac{c(p)}{(a_{2k-1} - a_{2k})^p} \frac{(\operatorname{ess\,sup}_{(a_{2k},a_{2k-1})} \psi - \operatorname{ess\,inf}_{(a_{2k},a_{2k-1})} \varphi)^p}{\operatorname{ess\,inf}_{(a_{2k},a_{2k-1})} \psi - \operatorname{ess\,sup}_{(a_{2k},a_{2k-1})} \psi/2}, \tag{6.9}$$

where $\eta \in J_{2k}$, $\xi \in \mathbb{R}$ and:

$$f(t, \eta, \xi) \le 0, \quad t \in (a_{2k+1}, a_{2k}),\ \eta \in J_{2k+1},\ \xi \in \mathbb{R}, \tag{6.10}$$

$$\operatorname*{ess\,sup}_{t\in A_{2k+1}} f(t, \eta, \xi) dt < -\frac{c(p)}{(a_{2k} - a_{2k+1})^p} \frac{(\operatorname{ess\,sup}_{(a_{2k+1},a_{2k})} \psi - \operatorname{ess\,inf}_{(a_{2k+1},a_{2k})} \varphi)^p}{\operatorname{ess\,inf}_{(a_{2k+1},a_{2k})} \varphi/2 - \operatorname{ess\,sup}_{(a_{2k+1},a_{2k})} \varphi}, \tag{6.11}$$

where $\eta \in J_{2k+1}$, $\xi \in \mathbb{R}$ and $c(p) = 2(16^p)(p-1)^{p-1}$ and A_k is a family of sets defined by

$$A_k = [a_k + \frac{1}{16}(a_{k-1} - a_k), a_{k-1} - \frac{1}{16}(a_{k-1} - a_k)],\ k \ge 1.$$

Then for any solution u of (1.1) we have:

$$u(t) \geq \operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \psi/2 \quad \text{for each } t \in \Lambda_{2k}, \tag{6.12}$$

$$u(t) \leq \operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \varphi/2 \quad \text{for each } t \in \Lambda_{2k+1},\ k \geq 1, \tag{6.13}$$

where Λ_k is a family of sets defined by

$$\Lambda_k = [a_k + \frac{1}{4}(a_{k-1} - a_k), a_{k-1} - \frac{1}{4}(a_{k-1} - a_k)], \quad k \geq 1.$$

Proof. It is clear that the assumptions of Lemmas 5.1 and 5.2 are fulfilled on the intervals $[a_2, b_2] = [a_{2k}, a_{2k-1}]$ and $[a_1, b_1] = [a_{2k+1}, a_{2k}]$ respectively, where $\omega_2 = \operatorname{ess\,sup}_{(a_{2k},a_{2k-1})} \psi/2$ and $\theta_1 = \operatorname{ess\,inf}_{(a_{2k+1},a_{2k})} \varphi/2$. Therefore, from (5.4) and (5.5) immediately follows (6.12) and (6.13)). □

Regarding Example 3.4 above, it is easy to construct a class of Caratheodory functions $f(t, \eta, \xi)$ which satisfies the assumptions of Lemma 6.2.

Next, we give the main result of the section.

Theorem 6.3. *For arbitrarily given real number $s \in (1, 2)$, let the sequence a_k and the obstacles φ and ψ be given by (3.11). If the Caratheodory function $f(t, \eta, \xi)$ satisfies (6.8)–(6.11) in respect to such (φ, ψ, a_k) then there are two positive constants c and ε_2 depending only on given data such that each solution u of (1.1) satisfies*

$$\Big(\int_{a+\varepsilon}^{b} |u'|^p dt\Big)^{1/p} \geq c\Big(\frac{1}{\varepsilon}\Big)^{s-1} \quad \text{for each } \varepsilon \in (0, \min\{\varepsilon_2, 1\}),$$

$$\limsup_{\varepsilon \to 0} \frac{\log\big(\int_{a+\varepsilon}^{b} |u'|^p dt\big)^{1/p}}{\log 1/\varepsilon} \geq s - 1.$$

Proof. It is easy to see that φ, ψ and a_k given by (3.11) satisfy the assumptions of Lemma 6.2. It implies that each solution u of (1.1) satisfies the assumptions of Lemma 6.1, where $j(\varepsilon) = k(\varepsilon)$, and $k(\varepsilon)$ is given in (3.16), and

$$\varepsilon_2 = \min\Big\{\frac{b-a}{\beta}, \Big(\frac{b-a}{2}\big(\frac{1}{2c_0}\big)^{\frac{1}{\beta}}\Big)^{\frac{\beta+1}{\beta}}\Big\},$$

where c_0 is appearing in (3.16). For the record, in order to prove that a_k given in (3.11) satisfies (6.1) in respect to ε_2, it is used the following elementary inequalities

$$\frac{1}{\beta}\Big(\frac{1}{k}\Big)^{1+1/\beta} \leq \Big(\frac{1}{k-1}\Big)^{1/\beta} - \Big(\frac{1}{k}\Big)^{1/\beta} \leq \frac{1}{\beta}\Big(\frac{1}{k-1}\Big)^{1+1/\beta} \leq \frac{2^{1+1/\beta}}{\beta}\Big(\frac{1}{k}\Big)^{1+1/\beta},$$

where $k \geq 2$ and $\beta > 0$. Putting such (φ, ψ, a_k) into (6.3), we obtain

$$\int_{a+\varepsilon}^{b} |u'(t)|^p dt \geq c \sum_{k=3}^{k(\varepsilon)} \Big(\frac{a_k + a_{k-1}}{2} - a\Big)^p \frac{1}{(a_{k-2} - a_{k-1})^{p-1}} \quad \text{for each } \varepsilon \in (0, \varepsilon_2).$$

Now, with the help of the same technical details as in the proof of [12, Theorem 8.1, p. 298-299], from (3.11) and previous inequality easy follows that

$$\|u'\|^p_{L^p(a+\varepsilon,b)} \geq c_1 \sum_{k=3}^{k(\varepsilon)} k^{(1+\frac{1}{\beta})(p-1)-\frac{p}{\beta}} \geq c_1 (k(\varepsilon))^{(1+\frac{1}{\beta})(p-1)-\frac{p}{\beta}+1}, \quad \varepsilon \in (0, \varepsilon_2).$$

Taking the p-root in the preceding inequality and using (3.16), we obtain

$$\begin{aligned}\|u'\|_{L^p(a+\varepsilon,b)} \geq c_1(k(\varepsilon))^{(1+\frac{1}{\beta})(1-\frac{1}{p})-\frac{1}{\beta}+\frac{1}{p}} &\geq c_1\Big(\frac{1}{\varepsilon}\Big)^{\frac{2\beta}{\beta+1}-\frac{1}{p}}\Big(\frac{1}{\varepsilon}\Big)^{(\frac{1}{p}-1)\frac{\beta}{\beta+1}}\\ &\geq c_1\Big(\frac{1}{\varepsilon}\Big)^{s-1},\ \varepsilon\in(0,\min\{\varepsilon_2,1\}).\end{aligned}$$

It proves Theorem 6.3 □

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