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A NEW PROOF OF HARNACK'S INEQUALITY FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS IN DIVERGENCE FORM

RAQUEL CRESCIMBENI, LILIANA FORZANI, ALEJANDRA PERINI

Abstract. In this paper we give a new proof of Harnack's inequality for elliptic operator in divergence form. We imitate the proof given by Caffarelli for operators in nondivergence form.

1. Introduction

At the end of the 1950's De Giorgi [4] showed that weak solutions of the second order elliptic partial differential equations in divergence form

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) = 0, \tag{1.1}$$

satisfy pointwise estimations, which allowed him to prove that all weak solutions of (1.1) are locally Hölder continuous. In 1961, Moser [9] proved that nonnegative weak solutions of (1.1) satisfy the so called Harnack's inequality: Let $\Omega \subset \mathbb{R}^n$ be an open set, for all Q' and Q open cubes in $\mathbb{R}^n$ such that $Q' \subset Q \subset \Omega$, $Q' = \frac{1}{4}Q$, there exists a constant $C > 1$, which depends on Q, Q' and the uniform ellipticity of (1.1), such that

$$\sup_{Q'} u \le C \inf_{Q'} u \quad \text{(Harnack inequality)}$$

for any nonnegative weak solution u of (1.1) in Q. As a consequence of Harnack inequality, Moser obtained Hölder regularity for all weak solutions of (1.1), and so Moser's method became the classical method for proving the regularity of weak solutions. The next big step in the study of Hölder regularity was given by Krylov and Safonov [7] in 1980. They proved the Harnack inequality for the case of strong solutions of parabolic equations with elliptic part in nondivergence form. In 1986,

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Caffarelli [2] gave a proof of the Harnack inequality for nonnegative smooth solutions of second order elliptic partial differential equations in nondivergence form

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = 0. \tag{1.2}$$

In this proof, and as a consequence of the Maximum Principle of Alexandroff-Bakelman-Pucci (see [11]), Caffarelli used two properties of the nonnegative solutions of the equation (1.2) on cubes of $\mathbb{R}^n$; namely:

Property 1.1. *There exist constants $\gamma_0 > 0$ and $0 < C_1 < 1$ such that if u is a nonnegative solution of $Lu = 0$ in Q_{2r,x_0} and*

$$|\{x \in Q_{r,x_0} : u(x) > 1\}| > \gamma_0 r^n,$$

then $\inf_{Q_{\frac{r}{2},x_0}} u(x) > C_1$, where $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^n$ and Q_{r,x_0} an open cube of size r and center x_0, (i.e $Q_{r,x_0} = \{x \in \mathbb{R}^n : \|x - x_0\|_\infty < r/2\}$ where $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$) .

Property 1.2. *Let $M > 2$, there exists $C_2 > 0$ such that if u is a nonnegative solution of $Lu = 0$ in Q_{Mr,x_0} and $\inf_{Q_{r,x_0}} u \ge 1$, then $\inf_{Q_{\frac{Mr}{2},x_0}} u(x) > C_2$.*

These two properties and the Calderon-Zygmund decomposition are the main tools that Caffarelli used to prove the weak Harnack inequality for nonnegative solutions of (1.2). As a consequence of this inequality, Caffarelli obtained an oscillation property for all solutions, which together with Property 1.2 allowed him to prove the Harnack inequality.

Again, as a consequence of the Harnack inequality, Caffarelli proved the Hölder continuity for all solutions in nondivergence form (see [2]).

To prove the Harnack inequality for nonnegative weak solutions of (1.1), Moser used an iterative argument for the functions given by

$$\Phi(p,h) = \Big(\frac{1}{|Q_{h,0}|} \int_{Q_{h,0}} u^p dx \Big)^{1/p}$$

with $p \in \mathbb{R}$, $0 < h < 1$, where for fix h, $\Phi(p,h)$ tends to $\sup_{Q_{h,0}} u$ and to $\inf_{Q_{h,0}} u$ when p tends to $+\infty$ and $-\infty$, respectively. Moreover, Moser used the Caccioppoli inequality for subsolutions and supersolutions, the Poincar and Sobolev inequalities to estimate the supremum and the infimum of u. Finally, to connect these estimates he used the John-Niremberg inequality for the bounded mean oscillation functions.

The proof of the Harnack inequality, that Moser and Caffarelli obtained, are completely different, because when the coefficients are not differentiable, these operators must be treated in different forms. The reason of this follows from the theory of equations in divergence form that is based on integral (energy) estimates, while all the theory of equations in nondivergence form is based on pointwise estimates, since when the coefficients are just measurable functions, the equation (1.2) provides only pointwise information.

The purpose of this work is to present a Harnack inequality proof for operator in divergence form, imitating the techniques applied by Caffarelli [2] for the operators in nondivergence form. To arrive at our objective we use the Aimar, Forzani and Toledano results [13] and [1], where they proved the weak Harnack inequality, in more general spaces, as a consequence of the Property 1.1 and the Property 1.2. The scheme of the proof that Aimar, Forzani and Toledano follow, is the same as

Caffarellis scheme for the Harnack inequaltiy for operator in nondivergence form. For this reason, to give a new proof of the Harnack inequality for nonnegative weak solutions for operators in divergence form, our principal objective in this paper is to prove the validity of Properties 1.1 and 1.2 for these operators.

In Section 2, we present some definitions, notations and general results of the elliptics differential equations in the divergence form. In Section 3, we state the main result of this work and we present the scheme to obtain the Harnack inequality for operator in divergence form, using the Aimar, Forzani and Toledano results. In Section 4, we prove some previous results of Sobolev Spaces and differential equations. Finally in Section 5, we prove the main result of this paper, that is the validity of the Property 1.1 and the Property 1.2 for operator in divergence form.

2. Definitions and Notations

We are interested in studying the operators given by (1.1) where the coefficients $a_{ij}(x)$ are measurable functions in $\Omega \subset \mathbb{R}^n$ (Ω bounded domain) and $A = (a_{ij})$ is the coefficient matrix which is symmetric. Therefore, throughout this work, we will assume that all the eigenvalues of A are bounded for positive constants, that is, there exist positive constants λ and Λ such that they satisfy the inequality

$$0 < \lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \tag{2.1}$$

for all $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n - \{0\}$ and $x \in \Omega$. An operator L with a matrix $A = (a_{ij})$ satisfying (2.1) is called uniformly elliptic in Ω and the constants λ and Λ will be the ellipticity constants.

We shall define the concept of solution that we are going to use in this work, that is, the solution of the operator in divergence form. A function u is called a weak solution in Ω of the operator (1.1) if $u \in W^{1,2}(\Omega)$ and satisfies

$$\int_\Omega \sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial u}{\partial x_j}\frac{\partial \Phi}{\partial x_i}dx = 0, \tag{2.2}$$

for all $\Phi \in C_0^1(\Omega)$. In the same way u is called a weak subsolution ($Lu \geq 0$) (weak supersolution ($Lu \leq 0$)) in Ω of the same equation if $u \in W^{1,2}(\Omega)$ and satisfies

$$\int_\Omega \sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial u}{\partial x_j}\frac{\partial \Phi}{\partial x_i}dx \leq 0 \quad (\geq 0),$$

for all nonnegative Φ such that $\Phi \in C_0^1(\Omega)$.

In this work we use some well known results of the weak solutions of (1.1), the proofs of which are not difficult (see [5] or [8]). They are:

(1)
$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial \Phi}{\partial x_i}dx = 0$$
for all $\phi \in C_0^1(\Omega)$ if and only if
$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial \Phi}{\partial x_i}dx = 0$$
for all $\phi \in W_0^{1,2}(\Omega)$.

(2) Let v be a positive weak subsolution of $Lu = 0$ in Ω then v^k with $k \geq 1$ is a weak subsolution of $Lu = 0$ in Ω.

(3) **Scale argument:** If u is a solution in Ω of the equation $Lu = 0$ then $\tilde{u}(y) = u(ry)$ is a solution in $\tilde{\Omega} = \{y = r^{-1}x, x \in \Omega\}$ of the equation $\tilde{L}\tilde{u} = 0$ where $\tilde{L}$ is the operator (1.1) with the coefficients $\tilde{a}_{ij}(y) = a_{ij}(ry)$, $y \in \tilde{\Omega}$ and a_{ij} the coefficients of L. Moreover L and $\tilde{L}$ have the same ellipticity constants.

To use the Aimar, Forzani and Toledano result [1] for functions of the vectorial space U, we need the following definitions:

A vectorial space of functions U satisfies **Property 1.1** if there exist constants $\gamma_0 > 0$ and $0 < C_1(\lambda, \Lambda, n) < 1$ such that if $u \in U$ is nonnegative in Q_{2r,x_0} and $|\{x \in Q_{r,x_0} : u(x) > 1\}| > \gamma_0 r^n$ then $\inf_{Q_{\frac{r}{2},x_0}} u > C_1$.

A vectorial space U, of functions, satisfies **Property 1.2** if, given $M > 2$, there exists $C_2 = C_2(\lambda, \Lambda, M) > 0$ such that if $u \in U$ is nonnegative in Q_{Mr,x_0} and $\inf_{Q_{r,x_0}} u \geq 1$, then $\inf_{Q_{\frac{Mr}{2},x_0}} u > C_2$.

A vectorial space U, of functions, satisfies the **weak Harnack inequality** if there exist $p > 0$ and $C = C(\lambda, \Lambda, p) > 0$ such that

$$\Big(\frac{1}{|Q_{2r,x_0}|}\int_{Q_{2r,x_0}} u^p dx\Big)^{1/p} \leq C \inf_{Q_{r,x_0}} u \tag{2.3}$$

for all $u \in U$, nonnegative in $Q_{4r,x_0} \subset \Omega$.

A set U is **locally bounded** if $\sup_Q |u| < \infty$ for all $u \in U$ and for each cube Q in Ω, that is, if all $u \in U$ belong to $L^\infty_{\rm loc}(\Omega)$. We will refer to this property by saying that $U \in L^\infty_{\rm loc}(\Omega)$.

A vectorial space $U \in L^\infty_{\rm loc}(\Omega)$ satisfies the **oscillation property** if there exists $0 < \theta < 1$ such that

$$\operatorname{osc}_{Q_{r,x_0}} u \leq \theta \operatorname{osc}_{Q_{4r,x_0}} u, \tag{2.4}$$

for all $u \in U$ and Q_{r,x_0} such that $Q_{4r,x_0} \subset \Omega$, where $\operatorname{osc}_{Q_{r,x_0}} u = \sup_{Q_{r,x_0}} u - \inf_{Q_{r,x_0}} u$.

A vectorial space U, of functions, satisfies the **Hölder continuity property** if there exist positive constants C and α such that $|u(x) - u(y)| \leq C|x - y|^\alpha$ for all $u \in U$ and for all $Q_{4r,x_0} \subset \Omega$, with $x, y \in Q_{r,x_0}$.

A vectorial space U, of functions, satisfies the **Harnack inequality** if there exist $\beta = \beta(\lambda, \Lambda, n) > 0$ such that

$$\sup_{Q_{r,x_0}} u \leq \beta \inf_{Q_{r,x_0}} u \tag{2.5}$$

for all $u \in U$, nonnegative in $Q_{4r,x_0} \subset \Omega$.

3. Statement of the main result

In [1] the authors proved, that in the general setting of spaces of homogeneous type, the Properties 1.1 and 1.2 mentioned above, are sufficient conditions to establish the weak Harnack inequality.

The technique used by Aimar, Forzani and Toledano for the weak Harnack inequality proof is like the Caffarelli's steps to prove the weak Harnack inequality for nonnegative solutions of the elliptic operator in nondivergence form given by (1.2). More precisely they obtained the following theorem.

Theorem 3.1. *For U a vectorial space of functions, we have*

1. *if U satisfies the Properties 1.1 and 1.2 then U satisfy the weak Harnack inequality.*
2. *if U satisfies the weak Harnack inequality then U satisfy the oscillation property. Moreover if $U \in L^\infty_{\rm loc}(\Omega)$ then U satisfies the Hölder α- continuity.*
3. *if U satisfies the Property 1.2 and the oscillation property then U satisfies the Harnack inequality.*

Now we present the main result of this work, where the vectorial space U in $\mathbb{R}^n$ is

$$U = \{u \in W^{1,2}(\Omega) \text{ such that } u \text{ is a weak solution of } Lu = 0\}, \tag{3.1}$$

where L is given by (1.1).

Theorem 3.2. *The vectorial space of functions U given by (3.1) satisfies the Property 1.1 and the Property 1.2 defined in Section 2.*

The proof of this Theorem will be given in Section 5. Using this result and the Theorem 3.1 for our particular case of the vectorial space of functions U in $\mathbb{R}^n$ given by (3.1), we obtain a new proof of the Harnack inequality for nonnegative weak solutions of the operator in divergence form, that follows the lines of Caffarelli's proof for nonnegative smooth solutions of the operator in nondivergence form.

4. Previous Results

First of all, we present some classic results about Sobolev Spaces and differential equations.

Theorem 4.1 (Sobolev Inequality)**.** *Let $u \in {W_0}^{1,2}(\Omega)$. Then there exists a constant $\beta = \beta(n)$ such that*

$$\Big(\int_\Omega |u|^{2^*} dx\Big)^{1/2^*} \le \beta\Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2}, \tag{4.1}$$

where $2^* = 2n/(n-2)$.

For a proof the above theorem, see for example [5].

Theorem 4.2 (Caccioppoli Inequality)**.** *Let $M > 1$ and u a positive weak subsolution of $Lu = 0$ in Q_{Mr,x_0} and $\Phi \in W_0^{1,2}(Q_{Mr,x_0})$. Then*

$$\int_{Q_{Mr,x_0}} |\nabla u|^2 \Phi^2 dx \le C \int_{Q_{Mr,x_0}} |\nabla \Phi|^2 u^2 dx, \tag{4.2}$$

where $C = C(\lambda, \Lambda, n)$.

For a proof of the above theorem, see for example [9]. Caccioppoli estimates will permit us to prove other results such as the boundedness of the norm L^∞ of the solutions.

Theorem 4.3. *Let u be a positive weak subsolution of $Lu = 0$ in Q_{4r,x_0}. Then*

$$\|u\|_{L^\infty(Q_{r,x_0})} \le \frac{C}{r^{\frac{n}{2}}} \|u\|_{L^2(Q_{2r,x_0})}, \tag{4.3}$$

where $C = C(\lambda, \Lambda)$.

Proof. By the scale argument it is sufficient to prove

$$\|u\|_{L^\infty(Q_{1,\frac{x_0}{r}})} \le C\|u\|_{L^2(Q_{2,\frac{x_0}{r}})}, \tag{4.4}$$

where $C = C(\Lambda, n)$. For $k = \frac{n}{n-2}$ and for all $j \in \mathbb{N}$ we define the number

$$N_j = \Big(\int_{Q_{r_j,\frac{x_0}{r}}} u^{2k^j}\,dx\Big)^{1/(2k^j)}, \tag{4.5}$$

where the r_j are such that the succession of cubes $Q_{r_j,\frac{x_0}{r}}$ satisfying

$$Q_{2,\frac{x_0}{r}} \supset Q_{r_1,\frac{x_0}{r}} \supset Q_{r_2,\frac{x_0}{r}} \supset \cdots \supset Q_{r_{j-1},\frac{x_0}{r}} \supset Q_{r_j,\frac{x_0}{r}} \supset \cdots \supset Q_{1,\frac{x_0}{r}}, \tag{4.6}$$

with $\operatorname{dist}(\partial Q_{r_j,\frac{x_0}{r}}, \partial Q_{r_{j-1},\frac{x_0}{r}}) \sim j^{-2}$, where $\sim$ denote equivalent.

First we have to see that

$$\|u\|_{L^\infty(Q_{1,\frac{x_0}{r}})} \le \limsup_{j\to\infty} N_j. \tag{4.7}$$

In fact, let us suppose that $\|u\|_{L^\infty(Q_{1,\frac{x_0}{r}})} = M$ and let $M' < M$. We define

$$A = \{x \in Q_{1,\frac{x_0}{r}} : |u(x)| > M'\}.$$

Then $|A| > 0$. By definitions of N_j and A we obtain

$$N_j = \Big(\int_{Q_{r_j,\frac{x_0}{r}}} u^{2k^j}\,dx\Big)^{1/(2k^j)} \ge \Big(\int_A u^{2k^j}\,dx\Big)^{1/(2k^j)} \ge M'|A|^{1/(2k^j)}.$$

Since $\lim_{j\to\infty} |A|^{1/(2k^j)} = 1$, then $\liminf_{j\to\infty} N_j \ge M$ and (4.7) follows.

Let $\Phi \in C_0^1(Q_{r_{j-1},\frac{x_0}{r}})$ such that $\Phi \equiv 1$ in $Q_{r_j,\frac{x_0}{r}}$ and $|\nabla\Phi| \le \frac{c}{r_{j-1}-r_j}$ in $Q_{r_{j-1},\frac{x_0}{r}}$. Since $u \in W^{1,2}(Q_{4,\frac{x_0}{r}})$ then $v = \Phi u \in W_0^{1,2}(Q_{r_{j-1},\frac{x_0}{r}})$. By the Sobolev inequality (4.1) and the Caccioppoli inequality (4.2) we have

$$\begin{aligned}
\int_{Q_{r_j,\frac{x_0}{r}}} u^{2k} &\le \int_{Q_{r_{j-1},\frac{x_0}{r}}} (\Phi u)^{2k}\,dx \\
&\le \beta\Big(\int_{Q_{r_{j-1},\frac{x_0}{r}}} |\nabla(\Phi u)|^2 dx\Big)^k \\
&\le \beta\Big(\int_{Q_{r_{j-1},\frac{x_0}{r}}} |\nabla\Phi|^2 u^2 dx + \int_{Q_{r_{j-1},\frac{x_0}{r}}} |\Phi|^2|\nabla u|^2 dx\Big)^k \\
&\le \beta\Big((1 + C(n,\lambda,\Lambda))\int_{Q_{r_{j-1},\frac{x_0}{r}}} |\nabla\Phi|^2 u^2 dx\Big)^k \\
&\le \beta\Big(\frac{c^2(1 + C(n,\lambda,\Lambda))}{(r_{j-1}-r_j)^2}\int_{Q_{r_{j-1},\frac{x_0}{r}}} u^2 dx\Big)^k.
\end{aligned} \tag{4.8}$$

By item 2) in Section 2 we have that $u^{k^{j-1}}$ is a positive subsolution. Applying (4.8) to $u^{k^{j-1}}$ we have

$$\begin{aligned} N_j^{2k^{j-1}} &= \Big(\int_{Q_{r_j,\frac{x_0}{r}}} \big(u^{k^{j-1}}\big)^{2k} dx\Big)^{1/k} \\ &\le \frac{\beta^{1/k} c^2(1+C(n,\lambda,\Lambda))}{(r_{j-1}-r_j)^2} \int_{Q_{r_{j-1},\frac{x_0}{r}}} \big(u^{k^{j-1}}\big)^2 dx \\ &\le Cj^4 N_{j-1}^{2k^{j-1}}. \end{aligned}$$

Then,

$$N_j \le \big(Cj^4\big)^{1/(2k^{j-1})} N_{j-1}.$$

Iterating this last inequality we obtain

$$N_j \le N_0 \prod_{i=1}^{\infty} \big(Ci^4\big)^{1/(2k^{i-1})};$$

so that

$$\ln N_j \le \ln N_0 + \sum_{i=1}^{\infty} \frac{1}{2k^{i-1}} \ln\big(Ci^4\big);$$

that is,

$$N_j \le e^{\sum_{i=1}^{\infty} \frac{1}{2k^{i-1}} \ln\left(Ci^4\right)} N_0 \le e^C N_0.$$

By (4.7) we have

$$\begin{aligned} \|u\|_{L^\infty(Q_{1,\frac{x_0}{r}})} &\le \limsup_{j\to\infty} N_j \le e^C N_0 \\ &= e^C \Big(\int_{Q_{r_0,\frac{x_0}{r}}} u^{2k^0}\Big)^{1/(2k^0)} \\ &= c\|u\|_{L^2(Q_{2,\frac{x_0}{r}})}; \end{aligned}$$

so we obtain (4.4). □

Our second step is to give another result which will provide us that the logarithm of a weak solution of (1.1) is a weak subsolution. Furthermore, we will obtain an estimate in the L^2 norm of the $\nabla(-\log(u+\epsilon))$, with $\epsilon \in (0,1)$ and u is a nonnegative weak solutions of (1.1). The statement of this result is as follows.

Lemma 4.4. *Let u be a nonnegative weak solution of $Lu = 0$ in Q_{2Mr,x_0} and f is defined for $x \in \mathbb{R}_0^+$ by $f(x) = \max\{-\log(x+\epsilon), 0\}$ with $\epsilon \in (0,1)$, then*

(1) *$v = f(u)$ is a nonnegative weak subsolution of (1.1) in Q_{2Mr,x_0}, $M \ge 1$.*

(2)
$$\frac{1}{|Q_{Mr,x_0}|} \int_{Q_{Mr,x_0}} |\nabla v|^2 dx \le C(Mr)^{-2}, \tag{4.9}$$
where $C = C(\lambda,\Lambda,n) > 0$.

Proof. The function f is differentiable in $\mathbb{R}^+ \cup \{0\}$, except in $x = 1-\epsilon$. The first derivative is

$$f'(x) = \begin{cases} -\frac{1}{x+\epsilon} & \text{if } 0 \le x < 1-\epsilon \\ 0 & \text{if } x > 1-\epsilon \end{cases}$$

then for a fix ϵ we have $f' \in L^\infty(\mathbb{R}_0^+)$. Moreover the second derivative is

$$f''(x) = \begin{cases} \frac{1}{(x+\epsilon)^2} & \text{if } 0 \le x < 1-\epsilon \\ 0 & \text{if } x > 1-\epsilon \end{cases}$$

then $f''(x) = [f'(x)]^2$ for $x + \epsilon \neq 1$. For $\Psi \in C_0^1(Q_{2Mr,x_0})$, $\Psi \equiv 1$ onto Q_{Mr,x_0} and $|\nabla\Psi| \le \frac{c(n)}{Mr}$ in Q_{2Mr,x_0}, we consider $w(x) = \Psi^2(x) f'(u(x))$ if $u(x) \neq 1-\epsilon$.

Since f is a piecewise smooth function with $f' \in L^\infty(\mathbb{R}_0^+)$, we can deduce that $f'(u) \in W^{1,2}(Q_{2Mr,x_0})$ and $\nabla(f'(u)) = f''(u)\nabla u$ at almost every point of Q_{2Mr,x_0}, then $w = \Psi^2 f'(u) \in W_0^{1,2}(Q_{2Mr,x_0})$ and

$$\begin{aligned} 0 &= -\langle Lu, w\rangle = -\big\langle Lu, \Psi^2 f'(u)\big\rangle \\ &= \int \sum_{i=1}^n a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial(\Psi^2 f'(u))}{\partial x_i}dx \\ &= \int \sum_{i=1}^n a_{ij}(x)\frac{\partial u}{\partial x_j}\Big(2\Psi\frac{\partial\Psi}{\partial x_i}f'(u) + \Psi^2 f''(u)\frac{\partial u}{\partial x_i}\Big)dx \\ &= \int \sum_{i=1}^n a_{ij}(x)\frac{\partial u}{\partial x_j}2\Psi\frac{\partial\Psi}{\partial x_i}f'(u)dx + \int \sum_{i=1}^n a_{ij}(x)\frac{\partial u}{\partial x_j}\Psi^2\frac{\partial u}{\partial x_i}f''(u)dx. \end{aligned} \tag{4.10}$$

By the ellipticity property given by (2.1), the previous identity and the inequality $2ab \le \delta a^2 + \frac{b^2}{\delta}$ for all $\delta > 0$ we have

$$\begin{aligned} \lambda\int \Psi^2 f''(u)|\nabla u|^2 dx &\le \int \sum_{i=1}^n a_{ij}(x)\frac{\partial u}{\partial x_j}\Psi^2\frac{\partial u}{\partial x_i}f''(u)dx \\ &= -2\int \sum_{i=1}^n a_{ij}(x)\Psi f'(u)\frac{\partial u}{\partial x_j}\frac{\partial\Psi}{\partial x_i}dx \\ &\le 2\int \sum_{i=1}^n \Big|a_{ij}(x)\Psi f'(u)\frac{\partial u}{\partial x_j}\frac{\partial\Psi}{\partial x_i}\Big|dx \\ &\le \Lambda\Big(\int \delta^2\Psi^2|f'(u)|^2|\nabla u|^2 dx + \int \frac{|\nabla\Psi|^2}{\delta^2}dx\Big). \end{aligned}$$

Then we obtain

$$\int \big[\lambda\Psi^2 f''(u) - \Lambda\delta^2\Psi^2|f'(u)|^2\big]|\nabla u|^2 dx \le \int \frac{|\nabla\Psi|^2}{\delta^2}dx.$$

If $\nabla u \neq 0$, then $f''(u) = |f'(u)|^2$. Taking $\delta^2 = \frac{\lambda}{2\Lambda}$ we have

$$\int \Psi^2|f'(u)|^2|\nabla u|^2 dx \le C(\lambda,\Lambda)\int |\nabla\Psi|^2 dx. \tag{4.11}$$

By (4.11) and the fact that $\Psi \equiv 1$ in Q_{Mr,x_0}, it results that

$$\begin{aligned}\int_{Q_{Mr,x_0}} |\nabla v|^2 dx &= \int_{Q_{Mr,x_0}} |\nabla (f(u))|^2 dx \\ &= \int_{Q_{Mr,x_0}} |f'(u)|^2 |\nabla u|^2 dx \\ &\leq \int_{Q_{2Mr,x_0}} \Psi^2 |f'(u)|^2 |\nabla u|^2 dx \\ &\leq C(\lambda,\Lambda) \int_{Q_{2Mr,x_0}} |\nabla \Psi|^2 dx \\ &\leq C(\lambda,\Lambda)(Mr)^{n-2}.\end{aligned}$$

□

We remark that Lemma 4.4 is a necessary tool for the proof of Theorem 3.2.

5. Proof of Theorem 3.2

Property 1.1: It is sufficient to prove that $v = f(u) = \max\{-\log(u+\epsilon), 0\}$ is bounded for all $x \in Q_{\frac{r}{2},x_0}$. In fact, if this is true we have that $-\log(u+\epsilon) \leq v(x) < C$ for all $x \in Q_{\frac{r}{2},x_0}$, then $\log(u+\epsilon)^{-1} < C$ and so $u > \frac{1}{10^C} = C_1$ in $Q_{\frac{r}{2},x_0}$.

Let $A = \{x \in Q_{r,x_0} : u(x) > 1\}$. If $x \in A$, we have $v(x) = 0$, then

$$|Q_{r,x_0} - A| = |\{x \in Q_{r,x_0} : u(x) \leq 1\}| < (1-\gamma_0) r^n. \tag{5.1}$$

By Lemma 4.4 we have that v is a positive weak subsolution of $Lu = 0$ in Q_{2r,x_0}, then by (4.3) we obtain that,

$$\sup_{Q_{\frac{r}{2},x_0}} v^2 \leq \frac{c}{r^n} \int_{Q_{r,x_0}} v^2 dx. \tag{5.2}$$

Furthermore, if we prove that there exists a constant γ_0 such that

$$\frac{c}{r^n} \int_{Q_{r,x_0}} v^2 dx \leq r^{2-n} \int_{Q_{r,x_0}} |\nabla v|^2 dx, \tag{5.3}$$

and we use the estimation (4.9) with $M = 1$ we have

$$\int_{Q_{r,x_0}} |\nabla v|^2 dx \leq C r^{n-2}; \tag{5.4}$$

then by (5.2), (5.3) and (5.4) we have that $\sup_{Q_{\frac{r}{2},x_0}} v$ is bounded.

Finally we have only to show (5.3). The Hölder's inequality, estimate (5.1) and the Sobolevs inequality allow us to obtain that

$$\begin{aligned}
\frac{1}{r^n}\int_{Q_{r,x_0}} v^2 dx &= \frac{1}{r^n}\int_{Q_{r,x_0}-A} v^2 dx \\
&\le \frac{1}{r^n}\Big(\int_{Q_{r,x_0}-A} v^{\frac{2n}{n-2}}dx\Big)^{(n-2)/n}|Q_{r,x_0}-A|^{2/n} \\
&\le \frac{c\left((1-\gamma_0)r^n\right)^{2/n}}{r^n}\Big(\int_{Q_{r,x_0}} v^{\frac{2n}{n-2}}dx\Big)^{(n-2)/n} \\
&\le \frac{c\beta^2\left((1-\gamma_0)r^n\right)^{2/n}}{r^n}\Big(\frac{1}{r^2}\int_{Q_{r,x_0}} v^2 dx + \int_{Q_{r,x_0}} |\nabla v|^2 dx\Big).
\end{aligned}$$

Then we have

$$\Big(\frac{1}{r^n} - \frac{c\beta^2(1-\gamma_0)^{\frac{n}{2}}}{r^n}\Big)\int_{Q_{r,x_0}} v^2 dx \le \frac{c\beta^2 r^2(1-\gamma_0)^{2/n}}{r^n}\int_{Q_{r,x_0}}|\nabla v|^2 dx.$$

If we choose γ_0 such that $1 - c\beta^2(1-\gamma_0)^{\frac{n}{2}} \ge \frac{1}{2}$ we obtain (5.3).

Property 1.2: The main estimate that we need is the following

$$\frac{1}{(Mr)^n}\int_{Q_{Mr,x_0}} v^2 dx \le \tilde{C}(1-M^{-n+1})(Mr)^{2-n}\int_{Q_{Mr,x_0}} |\nabla v|^2\,dx, \tag{5.5}$$

where $v = \max\{-\log(u+\epsilon), 0\}$. The above estimate, (4.3) and (4.9) allow us to obtain the result in the following inequality

$$\begin{aligned}
\|v\|^2_{L^\infty(Q_{\frac{Mr}{2},x_0})} &\le \frac{c}{(Mr)^n}\|v\|^2_{L^2(Q_{Mr,x_0})} \\
&= \frac{c}{(Mr)^n}\int_{Q_{Mr,x_0}} v^2 dx \\
&\le \tilde{C}(1-M^{-n+1})(Mr)^{2-n}\int_{Q_{Mr,x_0}}|\nabla v|^2\,dx \\
&\le \tilde{C}(1-M^{-n+1})(Mr)^{2-n}(Mr)^{n-2} \\
&= \tilde{C}(1-M^{-n+1}).
\end{aligned}$$

Then $\sup_{Q_{\frac{Mr}{2},x_0}} v \le C$. As in the proof of Property 1.1 we have $\inf_{Q_{\frac{Mr}{2},x_0}} u > C_2$.

Now we need only to prove (5.5). Since $u \ge 1$ in Q_{r,x_0} then $v = \max\{-\log(u+\epsilon), 0\} = 0$ in Q_{r,x_0}. In particular $v(x_0) = 0$ then for all $x \in Q_{Mr,x_0} - Q_{r,x_0}$ and m such that $m > 1$ we can write,

$$v(x) = \int_{\frac{1}{m}}^1 \frac{\partial v}{\partial t}(tx + (1-t)x_0)dt = \int_{\frac{1}{m}}^1 \nabla v(tx - (1-t)x_0).(x - x_0)dt.$$

By the last identity, the chain rule and the Fubini's Theorem, we obtain (5.5) as follows

$$
\begin{aligned}
\int_{Q_{Mr,x_0}} |v(x)|^2 dx &= \int_{Q_{Mr,x_0}-Q_{r,x_0}} |v(x)|^2 dx \\
&\le \int_{Q_{Mr,x_0}-Q_{r,x_0}} \int_{\frac{1}{m}}^{1} \left|\nabla v(tx+(1-t)x_0)\right|^2 |x-x_0|^2 \, dt dx \\
&\le Cn\Big(\frac{Mr}{2}\Big)^2 \int_{\frac{1}{m}}^{1} \int_{Q_{Mr,x_0}-Q_{r,x_0}} \left|\nabla v(tx+(1-t)x_0)\right|^2 dx dt \\
&= C(Mr)^2 \int_{\frac{1}{m}}^{1} \int_{Q_{Mrt,x_0}-Q_{rt,x_0}} |\nabla v(y)|^2 \frac{dy}{t^n} dt \\
&= C(Mr)^2 \int_{Q_{Mr,x_0}-Q_{\frac{r}{m},x_0}} |\nabla v(y)|^2 \Big[\int_{\frac{2\|y-x_0\|_\infty}{Mr}}^{\frac{2\|y-x_0\|_\infty}{r}} \frac{dt}{t^n}\Big] dy \\
&\le C(Mr)^2(1-M^{-n+1}) \int_{Q_{Mr,x_0}} |\nabla v(y)|^2 \, dy.
\end{aligned}
$$

References

[1] H. Aimar , L. Forzani, R. Toledano; *Hölder regularity of Pde's: A geometrical view*, Communications in Partial Differential Equations, Marcel Dekker, 2001.

[2] L. Caffarelli; *Métodos de Continuaçao em equaçoes elpticas nao-lineares*, VIII Escola Latino - Americana de matemática - Ro de Janeiro, Brasil, 1986.

[3] L. Caffarelli, X. Cabre; *Fully nonlinear Elliptic Equations*, American Mathematic Society, Colloquium Publications, 1995.

[4] E. DeGiorgi; *Sulla Differenziabilitá e Lanaliticit delle Estremali degli Integrali Multipli Regolaritá*, Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. (3), 3, 1957, 25-43.

[5] L. Evans; *Partial Differential Equations*, American Mathematical Society, Rhode Island, 1998.

[6] O. D. Kellogg; *Foundations of Potential Theory*, Dover, New York, 1954.

[7] N. Krylov, M. Safonov; *Certain Properties of Solutions of Parabolic Equations with Measurable Coefficients*, Izvestia Akad. Nauk., 40, 1980, 161-175.

[8] D. Gilbarg; N. Trudinger; *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.

[9] J. Moser; *On Harnacks Theorem for Elliptic Differential Equations*, Communications on Pure and Applied Mathematics, 1961, VOL XIV, 577-591.

[10] L. Nirenberg; *On elliptic partial Differential equations*, Annalli Scuola Normali Superiori Pisa. Serie(3), 1959, XIII, 931-953.

[11] S. Salsa; *Propietá locali delle soluzioni di equazioni ellittiche del secondo ordine*, 1995, Dottorato di Ricerca in Matematica, Milano, Pavia.

[12] M. Taylor; *Partial Differential Equations III*, Springer Verlag, 1996.

[13] R. Toledano; *Desigualdad de Harnack elptica y parabólica: enfoque abstracto*, Tesis doctoral - Universidad Nacional del Litoral, Santa Fe - Argentina, 1999.

Raquel Crescimbeni

Departmento de Matemática, Universidad Nacional del Comahue, Neuquén(8300), Argentina

E-mail address: `rcrescim@uncoma.edu.ar`

Liliana Forzani

Departmento de Matemática, Universidad Nacional del Litoral, Santa Fe (3000), Argentina

E-mail address: `liliana.forzani@gmail.com`

Alejandra Perini
Departmento de Matemática, Universidad Nacional del Comahue, Neuquén (8300), Argentina
E-mail address: alejandraperini@gmail.com aperini@uncoma.edu.ar