Electronic Journal of Differential Equations, Vol. 2007(2007), No. 44, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

ALMOST SURE EXPONENTIAL STABILITY OF DELAYED CELLULAR NEURAL NETWORKS

CHUANGXIA HUANG, YIGANG HE, LIHONG HUANG

Abstract. The stability of stochastic delayed Cellular Neural Networks (DCNN) is investigated in this paper. Using suitable Lyapunov functional and the semimartingale convergence theorem, we obtain some sufficient conditions for checking the almost sure exponential stability of the DCNN.

1. Introduction

Since the seminal work for Cellular Neural Networks in[4, 5], the past nearly two decades have witnessed the successful applications of Cellular Neural Networks in many areas such as combinatorial optimization, signal processing and pattern recognition, see e.g. [3, 9, 11, 12]. Recently, it has been realized that the axonal signal transmission delays often occur in various neural networks, and may cause undesirable dynamic network behaviors such as oscillation and instability. Consequently, the stability analysis problems for delayed Cellular neural networks(DCNN) have gained considerable research attention. Up to now, a great deal of results have been reported in the literature, see e.g.[2, 7, 13] and references therein, where the DCNN has been largely restricted to deterministic differential equations. These models do not take into account the inherent randomness that is associated with signal transmission.

Just as pointed out by Haykin [6], in real nervous systems and in the implementation of artificial neural networks, noise is unavoidable and should be taken into consideration in modelling. In this paper, we propose a system of stochastic differential equations for modelling DCNN as follows

$$\begin{aligned} \mathrm{d}x_i(t) &= -c_i(t)x_i(t)\mathrm{d}t + \sum_{j=1}^{n} a_{ij}(t)f_{ij}(x_j(t))\mathrm{d}t \\ &\quad + \sum_{j=1}^{n} b_{ij}(t)f_{ij}(x_j(t-\tau_{ij}))\mathrm{d}t + \sum_{j=1}^{n} \sigma_{ij}(x_j(t))\mathrm{d}w_j(t), t \geq 0, \end{aligned} \tag{1.1}$$

where $i = 1, \dots, n$; n corresponds to the number of units in a neural network; $x_i(t)$ denotes the potential (or voltage) of the cell i at time t; $f_{ij}(\cdot)$ denotes a non-linear

---

2000 *Mathematics Subject Classification.* 34K50, 60H10.
Key words and phrases. Stochastic; cellular neural networks; almost sure exponential stability; Lyapunov functional.

Submitted February 27, 2007. Published March 15, 2007.

output function between cell i and j; $\inf\{c_i(t)\} > 0$ denotes the rate with which cell i resets its potential to the resting state when isolated from other cells and inputs at time t; $a_{ij}(t)$ and $b_{ij}(t)$ denotes the strengths of connectivity between cell i and j at time t respectively; τ_{ij} is time delay and satisfies $0 \le \tau_{ij} \le \tau$. $\sigma(t) = (\sigma_{ij}(t))_{n\times n}$ is the diffusion coefficient matrix and $\omega(t) = (\omega_1(t), \dots, \omega_n(t))^T$ is an m-dimensional Brownian motion defined on a complete probability space$(\mathbf{\Omega}, \mathcal{F}, \mathbf{P})$ with a natural filtration $\{\mathcal{F}_t\}_{t\ge 0}$ (i.e. $\mathcal{F}_t = \sigma\{w(s) : 0 \le s \le t\}$).

There are various kinds convergence concepts to describe limiting behaviors of stochastic differential equations. The almost sure convergence is the most useful because it is closer to the real situation during computation than other forms of convergence (see [10]). Therefore it is very important to study almost sure convergence for stochastic DCNN.

To the best our knowledge, few authors discuss almost sure exponential stability for stochastic DCNN. Motivated by the above discussions, under the help of suitable Lyapunov functional and the semimartingale convergence theorem, we obtain some sufficient criteria ensuring the almost sure exponential stability for the model.

2. Preliminaries

Let $C := C([-\tau, 0], \mathbb{R}^n)$ be the Banach space of continuous functions which maps $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. For $(x_1(t), \dots, x_n(t))^T \in \mathbb{R}^n$, we define $\|x(t)\| = \sum_{i=1}^n |x_i(t)|$. For any $\varphi(t) \in C$, we define $\|\varphi\| = \sum_{i=1}^n |\varphi_i|$, where $|\varphi_i| = \sup_{-\tau\le s\le 0} |\varphi_i(s)|$.

The initial conditions for system (1.1) are $x(t) = \varphi(t)$, $-\tau \le t \le 0$, $\varphi \in L^2_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$, here $L^2_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$ is regarded as a $\mathbb{R}^n-$ valued stochastic process $\varphi(t), -\tau \le t \le 0$, moreover, $\varphi(t)$ is $\mathcal{F}_0$ measurable, $\int_{-\tau}^0 E|\xi(t)|^2 dt < \infty$. Throughout this paper, we always assume that $f_{ij}(0) = \sigma_{ij}(0) = 0$ and f_{ij}, σ_{ij} are globally Lipschitz, and $c_i(\cdot), a_{ij}(\cdot), b_{ij}(\cdot)$ are bounded functions. We also assume there exist positive constants $p_{ij}, i, j = 1, \dots, n$, such that $|f_{ij}(u) - f_{ij}(v)| \le p_{ij}|u-v|, \quad \forall u, v \in R$. This implies that (1.1) has a unique global solution on $t \ge 0$ for the initial conditions[10]. Clearly, (1.1) admit an equilibrium solution $x(t) \equiv 0$.

Definition 2.1 ([10]). The trivial solution of (1.1) is said to be almost surely exponentially stable if for almost all sample paths of the solution $x(t)$, we have

$$\limsup_{t\to\infty} \frac{1}{t} \log \|x(t)\| < 0.$$

Lemma 2.2 (Semimartingale convergence theorem [10]). *Let $A(t)$ and $U(t)$ be two continuous adapted increasing processes on $t \ge 0$ with $A(0) = U(0) = 0$ a.s. Let $M(t)$ be a real-valued continuous local martingale with $M(0) = 0$ a.s. Let ξ be a nonnegative $\mathcal{F}_0$-measurable random variable. Define*

$$X(t) = \xi + A(t) - U(t) + M(t), \quad \textit{for} \quad t \ge 0$$

If $X(t)$ is nonnegative, then

$$\{\lim_{t\to\infty} A(t) < \infty\} \subset \{\lim_{t\to\infty} X(t) < \infty\} \cap \{\lim_{t\to\infty} U(t) < \infty\}, \quad \text{a.s.},$$

where $B \subset D$ a.s. means $P(B \cap D^c) = 0$. In particular, If $\lim_{t\to\infty} A(t) < \infty$ a.s., then for almost all $\omega \in \Omega$

$$\lim_{t\to\infty} X(t) < \infty \quad \textit{and} \quad \lim_{t\to\infty} U(t) < \infty,$$

that is both $X(t)$ and $U(t)$ converge to finite random variables.

Lemma 2.3 ([8]). *If $\rho(K) < 1$ for matrix $K = (k_{ij})_{n\times n} \geq 0$, then $(E-K)^{-1} \geq 0$, where E denotes the identity matrix of size n.*

3. Main results

Theorem 3.1. *Let $k_{ij} = c_i^{-1}(a_{ji}+b_{ji})p_{ji}$, $K = (k_{ij})_{n\times n}$, where, $c_i = \inf\{c_i(t)\}$, $a_{ji} = \sup\{|a_{ji}(t)|\}$, $b_{ji} = \sup\{|b_{ji}(t)|\}$. If $\rho(K) < 1$, then the equilibrium point O of system* (1.1) *is almost surely exponentially stable.*

Proof. From $\rho(K) < 1$, it follows that $(E-K)$ is an M-matrix[1], where E denotes an identity matrix of size n. Therefore, using Lemma 2.3, there exists a diagonal matrix $M =\mathrm{diag}(m_1,\dots,m_n)$ with positive diagonal elements such that the product $(E-K)M$ is strictly diagonally dominant with positive diagonal entries. Namely,

$$m_i > \sum_{j=1}^{n} m_j c_i^{-1}(a_{ji}+b_{ji})p_{ji}, \quad i = 1,2,\dots,n. \tag{3.1}$$

Then, there exists a constant $\mu > 0$ such that

$$-m_i c_i + \sum_{j=1}^{n} m_j(a_{ji}+b_{ji})p_{ji} < -\mu, \quad i = 1,2,\dots,n. \tag{3.2}$$

Thus, we can choose a constant $0 < \lambda \ll 1$ such that

$$m_i(\lambda - c_i) + \sum_{j=1}^{n} m_j(a_{ji}+b_{ji})p_{ji}e^{\lambda\tau} < 0, \quad i = 1,2,\dots,n. \tag{3.3}$$

We define a positive definite Lyapunov function $V(x(t),t) = e^{\lambda t}\sum_{i=1}^{n} m_i|x_i(t)|$. By Itô formula, we can calculate the upper right differential D^+V of V along (1.1) as follows

$$\begin{aligned} D^+V(x(t),t) &= \lambda e^{\lambda t}\sum_{i=1}^{n} m_i|x_i(t)|\mathrm{d}t + e^{\lambda t}\sum_{i=1}^{n} m_i \mathrm{sign}(x_i(t))\mathrm{d}x_i(t)\\ &\leq e^{\lambda t}\{\sum_{i=1}^{n} m_i[(\lambda-c_i)|x_i(t)| + \sum_{j=1}^{n} a_{ij}p_{ij}|x_j(t)|\mathrm{d}t\\ &\quad + \sum_{j=1}^{n} b_{ij}p_{ij}|x_j(t-\tau_{ij})|]\mathrm{d}t\} + e^{\lambda t}\sum_{i=1}^{n} m_i\sum_{j=1}^{n}|\sigma_{ij}(x_j(t))|\mathrm{d}w_j(t), \end{aligned} \tag{3.4}$$

On the other hand, for $T > 0$, it is easy to see that

$$\begin{aligned} \int_0^T e^{\lambda t}|x_j(t-\tau_{ij})|\mathrm{d}t &= \int_{-\tau_{ij}}^{T-\tau_{ij}} e^{\lambda+\tau_{ij}}|x_j(t)|\mathrm{d}t\\ &\leq e^{\lambda\tau}\int_{-\tau}^{0} e^{\lambda t}|x_j(t)|\mathrm{d}t + e^{\lambda\tau}\int_0^T e^{\lambda t}|x_j(t)|\mathrm{d}t. \end{aligned} \tag{3.5}$$

Calculating the integral of inequality (3.4) from 0 to T and noticing inequality (3.3) and inequality (3.5), we have

$$\begin{aligned}
V(x(T),T) &\le V(x(0),0) + \int_0^T e^{\lambda t}\sum_{i=1}^n\sum_{j=1}^n m_i|\sigma_{ij}(x_j(t))|\mathrm{d}w_j(t)\\
&\quad + \int_0^T e^{\lambda t}\{\sum_{i=1}^n m_i[(\lambda - c_i)|x_i(t)| + \sum_{j=1}^n a_{ij}p_{ij}|x_j(t)|\mathrm{d}t\\
&\quad + \sum_{j=1}^n b_{ij}p_{ij}|x_j(t-\tau_{ij})|]\mathrm{d}t\}\mathrm{d}t\\
&\le \sum_{i=1}^n m_i|\varphi_i(0)| + \int_0^T e^{\lambda t}\sum_{i=1}^n\sum_{j=1}^n m_i|\sigma_{ij}(x_j(t))|\mathrm{d}w_j(t)\\
&\quad + \int_0^T e^{\lambda t}\{\sum_{i=1}^n m_i[(\lambda - c_i)|x_i(t)| + \sum_{j=1}^n e^{\lambda\tau}(a_{ij}+b_{ij})p_{ij}|x_j(t))|]\mathrm{d}t\\
&\quad + \int_{-\tau}^0 \sum_{i=1}^n\sum_{j=1}^n e^{\lambda\tau}m_i b_{ij}p_{ij}|x_j(t))|\}\mathrm{d}t\\
&\le \sum_{i=1}^n m_i|\varphi_i(0)| + \int_0^T e^{\lambda t}\sum_{i=1}^n\sum_{j=1}^n m_i|\sigma_{ij}(x_j(t))|\mathrm{d}w_j(t)\\
&\quad + \int_0^T e^{\lambda t}\{\sum_{i=1}^n [m_i(\lambda - c_i) + \sum_{j=1}^n e^{\lambda\tau}m_j(a_{ji}+b_{ji})p_{ji}]|x_i(t))|\}\mathrm{d}t\\
&\quad + \int_{-\tau}^0 \sum_{i=1}^n\sum_{j=1}^n e^{\lambda\tau}m_i b_{ij}p_{ij}|x_j(t))|\mathrm{d}t\\
&\le \sum_{i=1}^n m_i|\varphi_i(0)| + \int_0^T e^{\lambda t}\sum_{i=1}^n\sum_{j=1}^n m_i|\sigma_{ij}(x_j(t))|\mathrm{d}w_j(t)\\
&\quad + \int_{-\tau}^0 \sum_{i=1}^n\sum_{j=1}^n e^{\lambda\tau}m_i b_{ij}p_{ij}|x_j(t))|\mathrm{d}t.
\end{aligned}$$

The right hand of the above expression is a nonnegative martingale, and Lemma 2.2 shows

$$\lim_{T\to\infty} X(T) < \infty \quad \text{a.s.},$$

where

$$\begin{aligned}
X(T) &= \sum_{i=1}^n m_i|\varphi_i(0)| + \int_0^T e^{\lambda t}\sum_{i=1}^n\sum_{j=1}^n m_i|\sigma_{ij}(x_j(t))|\mathrm{d}w_j(t)\\
&\quad + \int_{-\tau}^0 \sum_{i=1}^n\sum_{j=1}^n e^{\lambda\tau}m_i b_{ij}p_{ij}|x_j(t))|\mathrm{d}t.
\end{aligned}$$

It follows that $\lim_{T\to\infty}(e^{\lambda t}\sum_{i=1}^n m_i|x_i(t)|) < \infty$, which implies

$$\lim_{T\to\infty}(e^{\lambda t}\sum_{i=1}^n |x_i(t)|) < \infty \qquad \text{a.s.}$$

That is, $\limsup_{T\to\infty} \frac{1}{t}\log\|x(T)\| < -\lambda$. This completes the proof. □

Remark 3.2. Note that for a given matrix M, its spectral radius $\rho(M)$ is equal to the minimum of its all matrix norms of M, i.e., for any norm $\|\cdot\|$, $\rho(M) \le \|M\|$. Therefore, we have the following corollary.

Corollary 3.3. *Suppose that there exist positive real numbers m_i $(i = 1, 2, \dots, n)$ such that one of the following inequalities is satisfied:*

(1) $m_i c_i > \sum_{j=1}^n m_j(a_{ji} + b_{ji})p_{ji}$, $i = 1, 2, \dots, n$.
(2) $m_i c_i > \sum_{j=1}^n m_j(a_{ij} + b_{ij})p_{ij}$, $i = 1, 2, \dots, n$.
(3) $\sum_{i=1}^n m_j(a_{ij} + b_{ij})p_{ij}/(c_i m_i)$, $i = 1, 2, \dots, n$.
(4) $\sum_{i=1} \sum_{j=1}^n ((a_{ij} + b_{ij})p_{ij}m_j/(c_i m_i))^2$,

then the equilibrium point O of system (1.1) *is almost surely exponentially stable.*

Remark 3.4. By Theorem 3.1 and Corollary 3.3, we conclude if the delay neural network satisfy the conditions, the stability of system (1.1) are independent of the magnitude of noise, and therefore, the noise fluctuations is harmless.

Acknowledgments. This research was supported by grant NCET-04-0767 from Program for New Century Excellent Talents in University of China, by grant 50677014 from Nature Science Foundation Council of China, by grant 20020532016 from Doctoral Special Found of Ministry of Education and Hunan Postdoctoral Scientific Program.

References

[1] A. Berman, R. Plemmons; *Nonnegative Matrices in the Mathematical Science*, Academic Press, New York, 1979.
[2] J. Cao; *New results concerning expontential stability and periodic solutions of delayed cellular neural networks*, Phys. Lett. A, 307 (2003), 136-147.
[3] H. Chen, et al.; *Image-processing algorithms realized by discrete-time cellular neural networks and their circuit implementations*, Chaos, Solitons, Fractals, 29(2006), 1100-1108.
[4] L. Chua, L. Yang; *Cellular neural networks: Theory*, IEEE Trans. Circuits Syst., 35(1988), 1257-1272.
[5] L. Chua, L. Yang, ; *Cellular neural networks: Applications*, IEEE Trans. Circuits Syst., 35(1988), 1273-1290.
[6] S. Haykin; *Neural Networks*, Prentice-Hall, NJ, 1994.
[7] L. Huang, C. Huang, B. Liu; *Dynamics of a class of cellular neural networks with time-varying delays*, Phys. Lett. A, 345(2005), 330-344.
[8] J. LaSalle; *The Stability of Dynamical System*, SIAM, Philadelphia, 1976.
[9] D. Liu, A. Michel; *Celular neural networks for associative memories*, IEEE Trans. Circuits. Syst., 40 (1993), 119-121.
[10] X. Mao; *Stochastic Differential Equation and Application*, Horwood Publishing, Chichester, 1997.
[11] T. Roska, L. Chua; *Cellular neural networks with nonlinear and delay-type template*, Int. J. Circuit Theor. Appl., 20(1992), 469-481.
[12] P. Venetianer, T. Roska; *Image compression by delayed CNNs*, IEEE Trans. Circuits. Syst. I, 45 (1998), 205-215.
[13] H. Zhao, J. Cao; *New conditions for global exponential stability of cellular neural networks with delays*, Neural Networks, 18(2005)1332-1340.

Chuangxia Huang
College of Mathematics and Computing Science, Changsha University of Science and Technology, Changsha, Hunan 410076, China; and College of Electrical and Information Engineering, Hunan University, Changsha, Hunan 410082, China
E-mail address: huangchuangxia@sina.com.cn

Yigang He
College of Electrical and Information Engineering, Hunan University, Changsha, Hunan 410082, China
E-mail address: yghe@hnu.cn

Lihong Huang
College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, China
E-mail address: lhhuang@hnu.cn