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POSITIVITY OF LYAPUNOV EXPONENTS FOR ANDERSON-TYPE MODELS ON TWO COUPLED STRINGS

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Abstract. We study two models of Anderson-type random operators on two deterministically coupled continuous strings. Each model is associated with independent, identically distributed four-by-four symplectic transfer matrices, which describe the asymptotics of solutions. In each case we use a criterion by Gol'dsheid and Margulis (i.e. Zariski denseness of the group generated by the transfer matrices in the group of symplectic matrices) to prove positivity of both leading Lyapunov exponents for most energies. In each case this implies almost sure absence of absolutely continuous spectrum (at all energies in the first model and for sufficiently large energies in the second model). The methods used allow for singularly distributed random parameters, including Bernoulli distributions.

1. Introduction

Localization for one-dimensional Anderson models is well understood, while important physical conjectures remain open in dimension $d \geq 2$. In particular, there is no proof yet of the physical conjecture that, as for $d = 1$, localization (in spectral or dynamical sense) holds at all energies and arbitrary disorder for $d = 2$. It is physically even more convincing that localization should hold for Anderson models on strips, which should behave like one-dimensional models.

In fact, Anderson localization has been established rigorously for discrete strips of arbitrary width in [18], for related work see also [10]. However, an interesting open problem is to understand the localization properties of Anderson-models on continuum strips. Consider, for example, the operator

$$-\Delta + \sum_{n \in \mathbb{Z}} \omega_n f(x - n, y) \tag{1.1}$$

on $L^2(\mathbb{R} \times [0,1])$ with, say, Dirichlet boundary conditions on $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$, i.i.d. random couplings ω_n and a single site potential f supported in $[0,1] \times [0,1]$. Under weak additional assumptions on f and the distribution of the ω_n, this operator, describing a physically one-dimensional disordered system, should be localized at *all* energies. But, with the exception of the easily separable case of y-independent f, this question is open. Technically, the main problem arising is that, while physically one-dimensional, the model is mathematically multi-dimensional

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in the sense that the underlying PDE can not be easily reduced to ODEs. Thus the rich array of tools for ODEs (coming mostly from dynamical systems) isn't available. PDE methods like multiscale analysis will show localization at the bottom of the spectrum (e.g. by adapting the general approach described in [21]), but can't fully grasp the consequences of physical one-dimensionality.

Thus, one reason for writing this note is to promote the further study of Anderson models on continuum strips.

Concretely, we take a rather modest step in this direction by studying two particular models of Anderson-type random operators on a semi-discretized strip, i.e. models of the form

$$-\frac{d^2}{dx^2} + \sum_{n\in\mathbb{Z}} V(x-n,\omega_n) \tag{1.2}$$

acting on vector-valued functions in $L^2(\mathbb{R},\mathbb{C}^N)$ for some positive integer N. Here $V(\cdot,\omega)$ is a compactly supported, $N\times N$-symmetric-matrix-valued random potential, and $\omega=(\omega_n)_{n\in\mathbb{Z}}$ is a (generally vector-valued) sequence of i.i.d. random variables, to be specified more explicitly in the models below.

Compared to (1.1), model (1.2) is discretized in the y-direction, remaining continuous in the x-direction. This allows to use ODE methods, in particular transfer matrices and the theory of Lyapunov exponents of products of independent random matrices. Here the transfer matrices are $2N$-dimensional and symplectic, leading to N pairs of Lyapunov exponents $\gamma_1(E)\geq\cdots\geq\gamma_N(E)\geq 0\geq-\gamma_N(E)\geq\cdots\geq-\gamma_1(E)$. Kotani-theory for such operators was developed in [19]. For a non-deterministic random potential, in particular for model (1.2), the general theory of [19] implies that $\gamma_1(E)>0$ for almost every $E\in\mathbb{R}$. However, for Anderson-type models one expects that all of the first N Lyapunov exponents are positive for most energies. This incompleteness of Kotani-theory on the strip is also pointed out as Problem 3 in the recent review of Kotani theory in [6].

An abstract criterion for the latter in terms of the groups generated by the random transfer matrices has been provided by Gol'dsheid and Margulis [11]. It is exactly this criterion which allowed to prove Anderson localization for discrete strips [18].

To the best of our knowledge, our results below are the first applications of the Gol'dsheid-Margulis criterion to continuum models. We depend on very explicit calculations and can so far only handle the case of two coupled strings, i.e. $N=2$. Our first model involves random point interactions, where we can show $\gamma_2(E)>\gamma_1(E)>0$ for all but an explicitly characterized discrete set of exceptional energies (Section 3). For our second model, two deterministically coupled single string Anderson models, we get in Section 4 that $\gamma_2(E)>\gamma_1(E)>0$ for all but a countable set of energies $E>2$. As explained at the end of Section 4.1, the latter is a technical restriction and we expect the same to hold for energies less than 2.

For both models we conclude the absence of absolutely continuous spectrum as a consequence of Kotani theory. Discreteness of the set of exceptional energies, established here for the point interaction model, should imply that the spectrum is almost surely pure point. This should follow by extending existing methods, e.g. [4, 18, 9], but we leave this to a future work.

We start in Section 2 with a discussion of the necessary background on products of i.i.d. symplectic matrices and, in particular, with a statement of the Gol'dsheid-Margulis criterion.

We mention that quite different methods, going back to the works [20] and [16], have been used to prove localization properties for random operators on strips in [17]. While [17] only considers discrete strips, the methods used have potential to be applicable to the continuum strip model (1.1). One difference between these methods and the ones used here is that we have aimed at handling singular distributions of the random parameters, in particular Bernoulli distributions. This excludes the use of spectral averaging techniques, which are quite central to the approach in [17].

The examples studied here are of a very special nature and we hope to get further reaching results in the future. Still, our simple examples should be of some interest from the point of view of exceptional energies where one or several Lyapunov exponents vanish.

It seems that larger N will lead to richer sets of exceptional energies, as might be expected physically due to the added (at least partial) transversal degree of freedom of a particle in a strip. Our examples show that the discrete strip is somewhat untypical in having no exceptional energies. It has been shown (for models with $N = 1$) that the existence of exceptional energies leads to weaker localization properties or, more precisely, stronger transport [14, 7]. A further study of the models proposed here for larger N, with the possibility of observing how this weakens localization effects, would be quite interesting.

2. Separability of Lyapunov exponents

We will first review the main results which allow to prove simplicity of the Lyapunov spectrum of a sequence of *i.i.d.* symplectic matrices, and thus, in particular, positivity of the first N Lyapunov exponents.

2.1. **Lyapunov exponents.** Let N be a positive integer. Let $\mathrm{Sp}_N(\mathbb{R})$ denote the group of $2N \times 2N$ real symplectic matrices. It is the subgroup of $\mathrm{GL}_{2N}(\mathbb{R})$ of matrices M satisfying
$$^{t}MJM = J,$$
where J is the matrix of order $2N$ defined by $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Here, I is the identity matrix of order N.

Recall that for $p \in \{1, \dots, N\}$, $\wedge^p \mathbb{R}^N$ is the vector space of alternating p-linear forms on $(\mathbb{R}^N)^*$. For $u_1, \dots, u_p$ in $\mathbb{R}^N$ and $f_1, \dots, f_p$ in $(\mathbb{R}^N)^*$, set
$$(u_1 \wedge \cdots \wedge u_p)(f_1, \dots, f_p) = \det((f_i(u_j))_{i,j})$$
We call $u_1 \wedge \cdots \wedge u_p$ a decomposable p-vector. We define a basis of $\wedge^p \mathbb{R}^N$ with those decomposable p-vectors in the following way : if $(u_1, \dots, u_N)$ is a basis of $\mathbb{R}^N$, $\{u_{i_1} \wedge \cdots \wedge u_{i_p} : 1 \le i_1 < \dots i_p \le N\}$ is a basis of $\wedge^p \mathbb{R}^N$. This allows to define all linear operations on $\wedge^p \mathbb{R}^N$ on the set of decomposable p-vectors.

First, we define a scalar product on $\wedge^p \mathbb{R}^N$ by the formula
$$(u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_p) = \det((\langle u_i, v_j \rangle)_{i,j})$$
where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on $\mathbb{R}^N$. The norm associated with $(\cdot, \cdot)$ will be denoted by $\| \cdot \|$.

Now we define how an element of the linear group $\mathrm{GL}_N(\mathbb{R})$ acts on $\wedge^p \mathbb{R}^N$. If $M \in \mathrm{GL}_N(\mathbb{R})$, an automorphism $\wedge^p M$ of $\wedge^p \mathbb{R}^N$ is given by
$$(\wedge^p M)(u_1 \wedge \cdots \wedge u_p) = M u_1 \wedge \cdots \wedge M u_p.$$

We have $\wedge^p(MN) = (\wedge^p M)(\wedge^p N)$. We also introduce the p-Lagrangian manifold. Let $(e_1, \dots, e_{2N})$ be the canonical basis of $\mathbb{R}^{2N}$. For any p in $\{1, \dots, N\}$ let L_p be the subspace of $\wedge^p \mathbb{R}^{2N}$ spanned by $\{Me_1 \wedge \dots \wedge Me_p : M \in \mathrm{Sp}_N(\mathbb{R})\}$. It is called the p-Lagrangian submanifold of $\mathbb{R}^{2N}$. The projective space $\mathbb{P}(L_p)$ is the set of isotropic spaces of dimension p in $\mathbb{R}^{2N}$.

We can now define the Lyapunov exponents.

Definition 2.1. Let $(A_n^\omega)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random matrices in $\mathrm{Sp}_N(\mathbb{R})$ with

$$\mathbb{E}(\log^+ \|A_0^\omega\|) < \infty.$$

The Lyapunov exponents $\gamma_1, \dots, \gamma_{2N}$ associated with $(A_n^\omega)_{n\in\mathbb{N}}$ are defined inductively by

$$\sum_{i=1}^{p} \gamma_i = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}(\log \| \wedge^p (A_{n-1}^\omega \dots A_0^\omega)\|).$$

One has $\gamma_1 \geq \dots \geq \gamma_{2N}$ and, due to symplecticity of the random matrices $(A_n)_{n\in\mathbb{N}}$, the symmetry property $\gamma_{2N-i+1} = -\gamma_i$, $\forall i \in \{1, \dots, N\}$ (see [3] p.89, Prop. 3.2).

2.2. **A criterion for separability of Lyapunov exponents.** In this section we will follow Bougerol and Lacroix [3]. For the definitions of L_p-strong irreducibility and p-contractivity we refer to [3], definitions $A.IV.3.3$ and $A.IV.1.1$, respectively.

Let μ be a probability measure on $\mathrm{Sp}_N(\mathbb{R})$. We denote by G_μ the smallest closed subgroup of $\mathrm{Sp}_N(\mathbb{R})$ which contains the topological support of μ, $\operatorname{supp}\mu$.

Now we can set forth the main result on separability of Lyapunov exponents, which is a generalization of Furstenberg's theorem to the case $N > 1$.

Proposition 2.2. *Let $(A_n^\omega)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random symplectic matrices of order $2N$ and p be an integer in $\{1, \dots, N\}$. We denote by μ the common distribution of the A_n^ω. Suppose that G_μ is p-contracting and L_p-strongly irreducible and that $\mathbb{E}(\log \|A_0^\omega\|) < \infty$. Then the following holds :*

(i) $\gamma_p > \gamma_{p+1}$
(ii) *For any non zero x in L_p :*

$$\lim_{n\to\infty} \frac{1}{n} \log \| \wedge^p (A_{n-1}^\omega \dots A_0^\omega)x\| = \sum_{i=1}^{p} \gamma_i\,.$$

This is [3, Proposition 3.4], where a proof can be found. As a corollary we have that if G_μ is p-contracting and L_p-strongly irreducible for all $p \in \{1, \dots, N\}$ and if $\mathbb{E}(\log \|A_0^\omega\|) < \infty$, then $\gamma_1 > \gamma_2 > \dots > \gamma_N > 0$ (using the symmetry property of Lyapunov exponents).

For explicit models (that is, explicit μ) it will typically be quite difficult to check p-contractivity and L_p-strong irreducibility for all p. That is why we will use the Gol'dsheid-Margulis theory presented in [11] which gives us an algebraic argument to verify these assumptions. The idea is that if the group G_μ is large enough in an algebraic sense then it is p-contractive and L_p-strongly irreducible for all p.

We recall that the algebraic closure or Zariski closure of a subset G of an algebraic manifold is the smallest algebraic submanifold that contains G. We denote it by $\mathrm{Cl_Z}(G)$. In other words, if G is a subset of an algebraic manifold, its Zariski closure $\mathrm{Cl_Z}(G)$ is the set of the zeros of polynomials vanishing on G. A subset $G' \subset G$ is said to be Zariski-dense in G if $\mathrm{Cl_Z}(G') = \mathrm{Cl_Z}(G)$, *i.e.* each polynomial vanishing on G' vanishes on G. More precisely, from the results of Gol'dsheid and Margulis one easily gets

Proposition 2.3 (Gol'dsheid-Margulis criterion)**.** *If G_μ is Zariski dense in the group $\mathrm{Sp}_N(\mathbb{R})$, then for all p, G_μ is p-contractive and L_p-strong irreducible.*

Proof. According to Lemma 6.2 and Theorem 6.3 on page 57 of [11], it suffices to prove that the connected component of the identity of $\mathrm{Sp}_N(\mathbb{R})$ is irreducible in L_p and that $\mathrm{Sp}_N(\mathbb{R})$ has the p-contracting property, for all p. For the p-contractivity it suffices to say that $\mathrm{Sp}_N(\mathbb{R})$ contains an element whose eigenvalues have all distincts moduli (as an example, $\mathrm{diag}(2,3,\dots,N+1,\frac{1}{2},\frac{1}{3},\dots,\frac{1}{N+1}) \in \mathrm{Sp}_N(\mathbb{R})$) and to use Corollary 2.2 in [3], p. 82. Next we recall that $\mathrm{Sp}_N(\mathbb{R})$ is connected and so its connected component of the identity is itself. And so we have to prove that $\mathrm{Sp}_N(\mathbb{R})$ is irreducible in L_p for all p. This is exactly what is proven in [3, Proposition 3.5, p. 91.]. $\square$

Now we will adopt this algebraic point of view to study two explicit models.

3. A model with random point interactions

3.1. **The model.** First, we will study a model of two deterministically coupled strings with i.i.d. point interactions at all integers on both strings. Formally, this model is given by the random Schrödinger operator

$$H_\omega^P = -\frac{d^2}{dx^2} + V_0 + \sum_{n\in\mathbb{Z}} \begin{pmatrix} \omega_1^{(n)}\delta_0(x-n) & 0 \\ 0 & \omega_2^{(n)}\delta_0(x-n) \end{pmatrix} \tag{3.1}$$

acting on $L^2(\mathbb{R},\mathbb{C}^2)$. Here V_0 is the constant-coefficient multiplication operator by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and δ_0 is the Dirac distribution at 0. Also, $\omega^{(n)} = (\omega_1^{(n)}, \omega_2^{(n)})$, $n \in \mathbb{Z}$, is a sequence of i.i.d. $\mathbb{R}^2$-valued random variables with common distribution ν on $\mathbb{R}^2$ such that $\operatorname{supp}\nu \subset \mathbb{R}^2$ is bounded and not co-linear, i.e.

$$\{x-y : x,y \in \operatorname{supp}\nu\} \tag{3.2}$$

spans $\mathbb{R}^2$. For example, this holds if the components $\omega_1^{(n)}$ and $\omega_2^{(n)}$ are independent non-trivial real random variables (i.e. each supported on more than one point).

More rigorously,

$$H_\omega^P = H_{\omega_1} \oplus H_{\omega_2} + V_0 \tag{3.3}$$

acting on $L^2(\mathbb{R},\mathbb{C}^2) = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, where H_{ω_i}, $i=1,2$, are operators in $L^2(\mathbb{R})$ with domain

$$\begin{aligned} D(H_{\omega_i}) = \{ & f \in L^2(\mathbb{R}) : f, f' \text{ are absolutely continuous on } \mathbb{R}\setminus\mathbb{Z},\ f'' \in L^2(\mathbb{R}), \\ & f \text{ is continuous on } \mathbb{R},\ f'(n^+) = f'(n^-) + \omega_i^{(n)} f(n) \text{ for all } n \in \mathbb{Z}\}, \end{aligned} \tag{3.4}$$

where existence of the left and right limits $f'(n^-)$ and $f'(n^+)$ at all integers is assumed. On this domain the operator acts by $H_{\omega_i} f = -f''$. These operators are self-adjoint and bounded from below, see e.g. [1], where boundedness of the distribution ν is used. The matrix operator V_0 is bounded and self-adjoint. Thus H_ω^P in (3.3) is self-adjoint for all ω.

Note here that this model, containing point interactions, is not covered by the assumptions made in [19], but that the proofs easily extend to our setting. Also, [19] considers $\mathbb{R}$-ergodic systems, while our model is $\mathbb{Z}$-ergodic. However, the suspension method provided in [15] to extend Kotani-theory to $\mathbb{Z}$-ergodic operators, also applies to the systems in [19] and our model. In particular, non-vanishing of all Lyapunov exponents allows to conclude absence of absolutely continuous spectrum via an extended version of Theorem 7.2 of [19].

In order to study the Lyapunov exponents associated with this operator we need to introduce the sequence of transfer matrices associated to the equation

$$H_\omega^P u = Eu, \ E \in \mathbb{R}. \tag{3.5}$$

Here, we incorporate the point interactions into the concept of solution. Thus a function $u = (u_1, u_2) : \mathbb{R} \to \mathbb{C}^2$ (not necessarily square-integrable) is called a solution of (3.5) if

$$-\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}'' + V_0 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = E \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{3.6}$$

on $\mathbb{R} \setminus \mathbb{Z}$ and u satisfies the same "interface conditions" as the elements of $D(H_\omega^P)$, i.e. it is continuous on $\mathbb{R}$ and

$$u_i'(n^+) = u_i'(n^-) + \omega_i^{(n)} u_i(n) \tag{3.7}$$

for $i = 1, 2$ and all $n \in \mathbb{Z}$.

If $u = (u_1, u_2)$ is a solution of (3.5), we define the transfer matrix $A^\omega_{(n,n+1]}(E)$ from n to $n+1$ by the relation

$$\begin{pmatrix} u_1((n+1)^+) \\ u_2((n+1)^+) \\ u_1'((n+1)^+) \\ u_2'((n+1)^+) \end{pmatrix} = A^\omega_{(n,n+1]}(E) \begin{pmatrix} u_1(n^+) \\ u_2(n^+) \\ u_1'(n^+) \\ u_2'(n^+) \end{pmatrix}.$$

Thus we include the effect of the point interaction at $n+1$, but not at n, insuring the usual multiplicative property of transfer matrices over multiple intervals. The sequence of *i.i.d.* random matrices $A^\omega_{(n,n+1]}(E)$ will determine the Lyapunov exponents at energy E.

By first solving the system (3.6) over $(0,1)$ and then accounting for the interface condition (3.7) one can see that the matrix $A^\omega_{(n,n+1]}(E)$ splits into a product of two matrices:

$$A^\omega_{(n,n+1]}(E) = M(\operatorname{diag}(\omega_1^{(n)}, \omega_2^{(n)})) A_{(0,1)}(E)\,. \tag{3.8}$$

Here, for any 2×2-matrix Q, we define the 4×4-matrix $M(Q) := \begin{pmatrix} I & 0 \\ Q & I \end{pmatrix}$, where I is the 2×2-unit matrix. Thus the first factor in (3.8) depends only on the random parameters and

$$A_{(0,1)}(E) = \exp\left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -E & 1 & 0 & 0 \\ 1 & -E & 0 & 0 \end{pmatrix} \right) \tag{3.9}$$

depends only on the energy E.

The matrices $A^\omega_{(n,n+1]}(E)$ are symplectic, which in our case can be seen directly from their explicit form (but also is part of the general theory built up in [19]). The distribution μ_E of $A^\omega_{(0,1]}(E)$ in $\mathrm{Sp}_2(\mathbb{R})$ is given by

$$\mu_E(\Gamma) = \nu(\{\omega^{(0)} \in \mathbb{R}^2 : \ M(\operatorname{diag}(\omega_1^{(0)}, \omega_2^{(0)})) A_{(0,1)}(E) \in \Gamma\})\,. \tag{3.10}$$

The closed group generated by the support of μ_E is

$$G_{\mu_E} = \overline{\langle M(\mathrm{diag}(\omega_1^{(0)}, \omega_2^{(0)}))A_{(0,1)}(E) |\, \omega^{(0)} \in \operatorname{supp} \nu \rangle} \,. \tag{3.11}$$

The following is our main result for model (3.1):

Theorem 3.1. *There exists a discrete set $\mathcal{S} \subset \mathbb{R}$ such that for all $E \in \mathbb{R} \setminus \mathcal{S}$, G_{μ_E} is Zariski-dense in $\mathrm{Sp}_2(\mathbb{R})$. Therefore we have $\gamma_1(E) > \gamma_2(E) > 0$ for all $E \in \mathbb{R} \setminus \mathcal{S}$ and the operator H^P_ω almost surely has no absolutely continuous spectrum.*

All we have to prove below is the first statement of Theorem 3.1 about Zariski-denseness. Positivity of Lyapunov exponents then follows from the results reviewed in Section 2. As discussed above, Theorem 7.2 of [19] applies to our model. Thus the essential support of the a.c. spectrum of H^P_ω is contained in the discrete set $\mathcal{S}$, implying that the a.c. spectrum is almost surely empty.

The exponential (3.9) will have different forms for $E > 1$, $E \in (-1, 1)$ and $E < -1$. Below we will consider the case $E > 1$ in detail and then briefly discuss the necessary changes for the other cases. We don't discuss the energies $E = \pm 1$, as we can include them in the discrete set $\mathcal{S}$.

3.2. **Proof of Theorem 3.1 for $E > 1$.** To study the group G_{μ_E} we begin by giving an explicit expression for the transfer matrices. To do this we have to compute the exponential defining $A_{(0,1)}(E)$. We assume now that $E > 1$. We begin by diagonalizing the real symmetric matrix V_0 in an orthonormal basis:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = U \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U \,.$$

Here $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is orthogonal as well as symmetric. By computing the successive powers of

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -E & 1 & 0 & 0 \\ 1 & -E & 0 & 0 \end{pmatrix}$$

with each block expressed in the orthonormal basis defined by U one gets

$$A_{(0,1)}(E) = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} R_{\alpha,\beta} \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \,, \tag{3.12}$$

where $\alpha = \sqrt{E-1}$, $\beta = \sqrt{E+1}$, and

$$R_{\alpha,\beta} = \begin{pmatrix} \cos(\alpha) & 0 & \frac{1}{\alpha}\sin(\alpha) & 0 \\ 0 & \cos(\beta) & 0 & \frac{1}{\beta}\sin(\beta) \\ -\alpha\sin(\alpha) & 0 & \cos(\alpha) & 0 \\ 0 & -\beta\sin(\beta) & 0 & \cos(\beta) \end{pmatrix} \,.$$

Now that we have an explicit form for our transfer matrices let us explain the strategy for proving the Zariski-denseness of G_{μ_E} in $\mathrm{Sp}_2(\mathbb{R})$. As $\mathrm{Sp}_2(\mathbb{R})$ is a connected Lie group, to show Zariski-denseness it is enough to prove that the Lie algebra of $\mathrm{Cl_Z}(G_{\mu_E})$ is equal to the Lie algebra of $\mathrm{Sp}_2(\mathbb{R})$. The latter is explicitly given by

$$\mathfrak{sp}_2(\mathbb{R}) = \{ \begin{pmatrix} a & b_1 \\ b_2 & -{}^t a \end{pmatrix} ,\ a \in \mathrm{M}_2(\mathbb{R}),\ b_1 \text{ and } b_2 \text{ symmetric} \} \,,$$

which is of dimension 10. So our strategy will be to prove that the Lie algebra of $\mathrm{Cl}_{\mathrm{Z}}(G_{\mu_E})$, which will be denoted by $\mathfrak{S}_2(E)$, is of dimension 10 and to do that we will explicitly construct 10 linearly independent elements in this Lie algebra. First we prove

Lemma 3.2. *For a two-by-two matrix Q one has $M(Q) \in Cl_Z(G_{\mu_E})$ if and only if* $\begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix} \in \mathfrak{S}_2(E)$.

Proof. If $\begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix} \in \mathfrak{S}_2(E)$, then $M(Q) = \exp\left(M(Q) - I\right) \in Cl_Z(G_{\mu_E})$. Conversely, if $M(Q) \in Cl_Z(G_{\mu_E})$, consider the subgroup $G_Q := \{M(nQ) = M(Q)^n : n \in \mathbb{Z}\}$ of $Cl_Z(G_{\mu_E})$. It follows that $M(xQ) \in Cl_Z(G_Q)$ for all $x \in \mathbb{R}$. To see this, let p be a polynomial in 4×4 variables such that $p(A) = 0$ for all $A \in G_Q$. Then the polynomial in one variable $\tilde{p}(x) := p(M(xQ))$ has roots in all integers and must therefore vanish identically. Thus $p(M(xQ)) = 0$ for all $x \in \mathbb{R}$. $M(xQ) \in Cl_Z(G_Q) \subset Gl_Z(G_{\mu_E})$ now follows from the definition of Zariski closure. Then, by differentiating at the identity element of $Cl_Z(G_{\mu_E})$, we find $\begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix} \in \mathfrak{S}_2(E)$. $\square$

Proof of Theorem 3.1 for $E > 1$. **Step 1.** By (3.8),

$$A_{(0,1]}^{\tilde{\omega}^{(0)}}(E) A_{(0,1]}^{\omega^{(0)}}(E)^{-1} = M(\mathrm{diag}(\tilde{\omega}_1^{(0)} - \omega_1^{(0)}, \tilde{\omega}_2^{(0)} - \omega_2^{(0)})) \in G_{\mu_E} \tag{3.13}$$

for all $\omega^{(0)}, \tilde{\omega}^{(0)} \in \operatorname{supp} \nu$. As $\mathfrak{S}_2(E)$ is an algebra, Lemma 3.2 and assumption (3.2) imply that $\begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix} \in \mathfrak{S}_2(E)$ for arbitrary diagonal matrices Q.

Step 2. Using Step 1 and Lemma 3.2 shows that $M(Q) \in Cl_Z(G_{\mu_E})$ for arbitrary diagonal Q. In particular, we conclude

$$A_{(0,1)}(E) = M(\mathrm{diag}(\omega_1^{(0)}, \omega_2^{(0)}))^{-1} A_{(0,1]}(E) \in Cl_Z(G_{\mu_E}).$$

Step 3. By a general property of matrix Lie groups we know that

$$XMX^{-1} \in \mathfrak{S}_2(E) \tag{3.14}$$

whenever $M \in \mathfrak{S}_2(E)$ and $X \in G_{\mu_E}$. Thus, by Steps 1 and 2, for $l \in \mathbb{Z}$,

$$\begin{aligned} &\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} R_{\alpha,\beta}^{l} \begin{pmatrix} 0 & 0 \\ UQU & 0 \end{pmatrix} R_{\alpha,\beta}^{-l} \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \\ &= A_{(0,1)}(E)^{l} \begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix} A_{(0,1)}(E)^{-l} \in \mathfrak{S}_2(E), \end{aligned} \tag{3.15}$$

where $Q = \mathrm{diag}(\omega_1^{(0)}, \omega_2^{(0)})$. But we also have, as U is orthogonal and symmetric,

$$\mathfrak{S}_2(E) = \mathfrak{sp}_2(\mathbb{R}) \Leftrightarrow \tilde{\mathfrak{S}}_2(E) := \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \mathfrak{S}_2(E) \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} = \mathfrak{sp}_2(\mathbb{R})$$

Thus we are left with having to show the latter. To this end, we know from (3.15) that

$$S(l, Q) := R_{\alpha,\beta}^{l} \begin{pmatrix} 0 & 0 \\ UQU & 0 \end{pmatrix} R_{\alpha,\beta}^{-l} \in \tilde{\mathfrak{S}}_2(E) \tag{3.16}$$

for all $l \in \mathbb{Z}$ and all four matrices $Q = \mathrm{diag}(\omega_1^{(0)}, \omega_2^{(0)})$.

Step 4. By Step 3, for $(\omega_1^{(0)},\omega_2^{(0)})=(1,1)$ and all $l\in\mathbb{Z}$,

$$\begin{aligned}
&A_1(l):=S(l,I)\\
&=\begin{pmatrix}\frac{1}{\alpha}\sin(l\alpha)\cos(l\alpha) & 0 & -\frac{1}{\alpha^2}\sin^2(l\alpha) & 0\\ 0 & \frac{1}{\beta}\sin(l\beta)\cos(l\beta) & 0 & -\frac{1}{\beta^2}\sin^2(l\beta)\\ \cos^2(l\alpha) & 0 & -\frac{1}{\alpha}\sin(l\alpha)\cos(l\alpha) & 0\\ 0 & \cos^2(l\beta) & 0 & -\frac{1}{\beta}\sin(l\beta)\cos(l\beta)\end{pmatrix}\\
&\in\tilde{\mathfrak{S}}_2(E).
\end{aligned}$$

Also, as $\tilde{\mathfrak{S}}_2(E)$ is an algebra, $A_2(l):=2S(l,\begin{pmatrix}1&0\\0&0\end{pmatrix})-S(l,I)\in\tilde{\mathfrak{S}}_2(E)$ for all $l\in\mathbb{Z}$. These matrices take the form

$$\begin{aligned}
&A_2(l)=\\
&\begin{pmatrix}0 & \frac{1}{\alpha}\sin(l\alpha)\cos(l\beta) & 0 & -\frac{1}{\alpha\beta}\sin(l\alpha)\sin(l\beta)\\ \frac{1}{\beta}\cos(l\alpha)\sin(l\beta) & 0 & -\frac{1}{\alpha\beta}\sin(l\alpha)\sin(l\beta) & 0\\ 0 & \cos(l\alpha)\cos(l\beta) & 0 & -\frac{1}{\beta}\cos(l\alpha)\sin(l\beta)\\ \cos(l\alpha)\cos(l\beta) & 0 & -\frac{1}{\alpha}\sin(l\alpha)\cos(l\beta) & 0\end{pmatrix}\\
&\in\tilde{\mathfrak{S}}_2(E).
\end{aligned}$$

Step 5. We remark that the space generated by the family $(A_1(l))_{l\in\mathbb{Z}}$ is orthogonal to the one generated by $(A_2(l))_{l\in\mathbb{Z}}$ in $\mathfrak{sp}_2(\mathbb{R})$. We can work independently with each of these two families to find enough linearly independent matrices in $\tilde{\mathfrak{S}}_2(E)$ to generate a subspace of dimension 10. We begin with the family $(A_2(l))_{l\in\mathbb{Z}}$. We want to prove that for all but a discrete set of energies $E\in\mathbb{R}$, $A_2(0),A_2(1),A_2(2),A_2(3)$ are linearly independent. Because of the symmetries in the coefficients of these matrices, their linear independence is equivalent to the linear independence of the vectors

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}\cos(\alpha)\cos(\beta)\\ -\frac{1}{\alpha\beta}\sin(\alpha)\sin(\beta)\\ \frac{1}{\alpha}\sin(\alpha)\cos(\beta)\\ \frac{1}{\beta}\sin(\beta)\cos(\alpha)\end{pmatrix},\ \begin{pmatrix}\cos(2\alpha)\cos(2\beta)\\ -\frac{1}{\alpha\beta}\sin(2\alpha)\sin(2\beta)\\ \frac{1}{\alpha}\sin(2\alpha)\cos(2\beta)\\ \frac{1}{\beta}\sin(2\beta)\cos(2\alpha)\end{pmatrix},\ \begin{pmatrix}\cos(3\alpha)\cos(3\beta)\\ -\frac{1}{\alpha\beta}\sin(3\alpha)\sin(3\beta)\\ \frac{1}{\alpha}\sin(3\alpha)\cos(3\beta)\\ \frac{1}{\beta}\sin(3\beta)\cos(3\alpha)\end{pmatrix}.$$

The determinant of the matrix generated by those four vectors, best found with the help of a computer algebra system, is

$$\frac{4}{\alpha^2\beta^2}\sin^2(\alpha)\sin^2(\beta)(\cos^2(\alpha)-\cos^2(\beta))\,. \tag{3.17}$$

This function is real-analytic in $E>1$ with roots not accumulating at 1, thus it vanishes only for a discrete set $\mathcal{S}_1$ of energies $E>1$.

Step 6. By Step 5 we know that $(A_2(0),A_2(1),A_2(2),A_2(3))$ generate a subspace of dimension four of $\tilde{\mathfrak{S}}_2(E)$ for $E\in(1,\infty)\setminus\mathcal{S}_1$, i.e. they generate all matrices of the form

$$\begin{pmatrix}0&a&0&d\\ b&0&d&0\\ 0&c&0&-b\\ c&0&-a&0\end{pmatrix},$$

with $a,b,c,d \in \mathbb{R}$. In particular, for $a=1$, $b=c=d=0$,

$$B_0 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \in \tilde{\mathfrak{S}}_2(E)\,.$$

It follows that $B := \frac{1}{2}[B_0, A_2(0)] \in \tilde{\mathfrak{S}}_2(E)$, with $[\cdot,\cdot]$ denoting the matrix commutator bracket. We calculate

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\,.$$

Step 7. In this step we will prove that $A_1(0), A_1(1), A_1(2), A_1(3), A_1(4), B$ are linearly independent for all but a discrete set of energies. Due to the symmetries and zeros in matrices of the form $A_1(l)$, it suffices to show linear independence of the vectors formed by the entries $(3,1),(4,2),(1,1),(2,2),(1,3),(2,4)$ of these six matrices. The determinant of the matrix spanned by these six columns is found to be

$$\frac{64}{\alpha^3\beta^3}\sin^3(\alpha)\sin^3(\beta)\cos(\alpha)\cos(\beta)(\cos^2(\alpha)-\cos^2(\beta))^2\,. \tag{3.18}$$

Similar to Step 5 one argues that this vanishes only for E in a discrete subset $\mathcal{S}_2 \subset (1,\infty)$. Thus, for $E \in (1,+\infty)\setminus \mathcal{S}_2$, $A_1(0), A_1(1), A_1(2), A_1(3), A_1(4), B$ are linearly independent.

Step 8. First we can see that $\mathcal{S}_1 \subset \mathcal{S}_2$. Having complementary sets of non-zero entries, the subspaces generated by $A_1(0)$, $A_1(1)$, $A_1(2)$, $A_1(3)$, $A_1(4), B$ as well as $A_2(0)$, $A_2(1)$, $A_2(2)$, $A_2(3)$ are orthogonal. If $E \in (1,\infty)\setminus\mathcal{S}_2$, then both sets of matrices are linearly independent and contained in $\tilde{\mathfrak{S}}_2(E)$. Thus the latter has at least dimension 10 and therefore is equal to $\mathfrak{sp}_2(\mathbb{R})$. This concludes the proof of Theorem 3.1 for the case $E > 1$. $\square$

3.3. **The cases** $E \in (-1,1)$ **and** $E < -1$. We now turn to the proof of the Theorem for $E \in (-1,1)$. Here the expression for the matrix $A_{(0,1)}(E)$ changes slightly. We now set $\alpha = \sqrt{1-E}$ and, as before, $\beta = \sqrt{E+1}$. Also, U remains unchanged. But we replace $R_{\alpha,\beta}$ by

$$\tilde{R}_{\alpha,\beta} = \begin{pmatrix} \cosh(\alpha) & 0 & \frac{1}{\alpha}\sinh(\alpha) & 0 \\ 0 & \cos(\beta) & 0 & \frac{1}{\beta}\sin(\beta) \\ \alpha\sinh(\alpha) & 0 & \cosh(\alpha) & 0 \\ 0 & -\beta\sin(\beta) & 0 & \cos(\beta) \end{pmatrix}\,.$$

Proof of Theorem 3.1 for $E \in (-1,1)$. In fact, we can follow the proof for the first case very closely. We will briefly comment on the changes. Steps 1 and 2 remain unchanged. In the Step 3 we replace $R_{\alpha,\beta}$ by $\tilde{R}_{\alpha,\beta}$ and so in Step 4 we get that for all $l \in \mathbb{Z}$,

$$\tilde{A}_1(l) := \begin{pmatrix} \frac{1}{\alpha}\sinh(l\alpha)\cosh(l\alpha) & 0 & -\frac{1}{\alpha^2}\sinh^2(l\alpha) & 0 \\ 0 & \frac{1}{\beta}\sin(l\beta)\cos(l\beta) & 0 & -\frac{1}{\beta^2}\sin^2(l\beta) \\ \cosh^2(l\alpha) & 0 & -\frac{1}{\alpha}\cosh(l\alpha)\sinh(l\alpha) & 0 \\ 0 & \cos^2(l\beta) & 0 & -\frac{1}{\beta}\sin(l\beta)\cos(l\beta) \end{pmatrix}$$

is in $\tilde{\mathfrak{S}}_2(E)$, and

$\tilde{A}_2(l) :=$

$$\begin{pmatrix} 0 & \frac{1}{\alpha}\sinh(l\alpha)\cos(l\beta) & 0 & -\frac{1}{\alpha\beta}\sinh(l\alpha)\sin(l\beta) \\ \frac{1}{\beta}\cosh(l\alpha)\sin(l\beta) & 0 & -\frac{1}{\alpha\beta}\sinh(l\alpha)\sin(l\beta) & 0 \\ 0 & \cosh(l\alpha)\cos(l\beta) & 0 & -\frac{1}{\beta}\cosh(l\alpha)\sin(l\beta) \\ \cosh(l\alpha)\cos(l\beta) & 0 & -\frac{1}{\alpha}\sinh(l\alpha)\cos(l\beta) & 0 \end{pmatrix}$$

is in $\tilde{\mathfrak{S}}_2(E)$. In Step 5 we again get that for all but a discrete set of energies, $\tilde{A}_2(0)$, $\tilde{A}_2(1)$, $\tilde{A}_2(2)$, $\tilde{A}_2(3)$ are linearly independent. The determinant set up from the entries in exactly the same way as in Step 5 above is now

$$\frac{4}{\alpha^2\beta^2}\sinh^2(\alpha)\sin^2(\beta)(\cosh^2(\alpha)-\cos^2(\beta)) \,, \tag{3.19}$$

which vanishes only on a finite set $\mathcal{S}_3$ of values $E \in (-1,1)$.
Step 6 remains unchanged except we now get $B \in \tilde{\mathfrak{S}}_2(E)$ for all $E \in (-1,1)\setminus\mathcal{S}_3$. In Step 7 we set up a 6×6-matrix from the entries of $\tilde{A}_1(0),\dots,\tilde{A}_1(4), B$ in exactly the same way as in Step 7 above and find for its determinant

$$\frac{64}{\alpha^3\beta^3}\sinh^3(\alpha)\sin^3(\beta)\cosh(\alpha)\cos(\beta)(\cosh^2(\alpha)-\cos^2(\beta))^2 \,. \tag{3.20}$$

The roots of this function are a discrete subset $\mathcal{S}_4$ of $(-1,1)$, which contains $\mathcal{S}_3$. As in Step 8 we conclude that for all energies $E \in (-1,1)\setminus\mathcal{S}_4$, $\mathrm{Cl_Z}(G_{\mu_E}) = \mathrm{Sp}_2(\mathbb{R})$. $\square$

Finally, without providing further details, we note that very similar changes can be used to cover the remaining case $E < -1$.

4. Matrix-valued continuum Anderson model

While in our first model the randomness acted through point interactions on a discrete set, we now turn to a model with more extensive randomness. We consider two independent continuum Anderson models on single strings, with the single site potentials given by characteristic functions of unit intervals, and couple the two strings with the deterministic off-diagonal matrix V_0 already used above.

4.1. **The model.** Let

$$H_\omega^A = -\frac{d^2}{dx^2} + V_0 + \sum_{n\in\mathbb{Z}} \begin{pmatrix} \omega_1^{(n)}\chi_{[0,1]}(x-n) & 0 \\ 0 & \omega_2^{(n)}\chi_{[0,1]}(x-n) \end{pmatrix} \tag{4.1}$$

be a random Schrödinger operator acting in $L^2(\mathbb{R},\mathbb{C}^2)$, where $\chi_{[0,1]}$ is the characteristic function of the interval $[0,1]$, V_0 is as in the previous model, and $(\omega_1^{(n)})_{n\in\mathbb{Z}}$ and $(\omega_2^{(n)})_{n\in\mathbb{Z}}$ are two sequences of i.i.d. random variables (also independent from each other) with common distribution $\tilde{\nu}$ such that $\{0,1\}\subset \operatorname{supp}\tilde{\nu}$.

This operator is a bounded perturbation of $(-\frac{d^2}{dx^2})\oplus(-\frac{d^2}{dx^2})$ and thus self-adjoint on the $\mathbb{C}^2$-valued second order L^2-Sobolev space.

For this model we have the following result:

Theorem 4.1. *There exists a countable set $\mathcal{C}$ such that for all $E \in (2,\infty)\setminus\mathcal{C}$, $\gamma_1(E) > \gamma_2(E) > 0$. Therefore, H_ω^A has no absolutely continuous spectrum in the interval $(2,\infty)$.*

The transfer matrices $A^{\omega}_{n,2}(E)$, mapping $(u_1(n), u_2(n), u_1'(n), u_2'(n))$ to $(u_1(n+1), u_2(n+1), u_1'(n+1), u_2'(n+1))$ for solutions $u = (u_1, u_2)$ of the equation $H^A_\omega u = Eu$, are i.i.d. and symplectic [19]. Denote the distribution of $A^{\omega}_{0,2}(E)$ in $\mathrm{Sp}_2(\mathbb{R})$ by $\tilde{\mu}_E$. As before, by $G_{\tilde{\mu}_E}$ we denote the closed subgroup of $\mathrm{Sp}_2(\mathbb{R})$ generated by $\operatorname{supp} \tilde{\mu}_E$. As $\{0,1\} \subset \operatorname{supp} \tilde{\nu}$ we have that

$$\{A^{(0,0)}_{0,2}(E), A^{(1,0)}_{0,2}(E), A^{(0,1)}_{0,2}(E), A^{(1,1)}_{0,2}(E)\} \subset G_{\tilde{\mu}_E}\,.$$

Here we also write $A^{\omega^{(0)}}_{0,2}(E)$ for the transfer matrices from 0 to 1, where $\omega^{(0)} = (\omega_1^{(0)}, \omega_2^{(0)})$. We will denote the Lie algebra of the Zariski closure $\mathrm{Cl_Z}(G_{\tilde{\mu}_E})$ of $G_{\tilde{\mu}_E}$ by $\mathfrak{A}_2(E)$.

To give an explicit description of the matrices $A^{\omega^{(0)}}_{0,2}(E)$ we define,

$$M_{\omega^{(0)}} = \begin{pmatrix} \omega_1^{(0)} & 1 \\ 1 & \omega_2^{(0)} \end{pmatrix} = S_{\omega^{(0)}} \begin{pmatrix} \lambda_1^{\omega^{(0)}} & 0 \\ 0 & \lambda_2^{\omega^{(0)}} \end{pmatrix} S^{-1}_{\omega^{(0)}},$$

with orthogonal matrices $S_{\omega^{(0)}}$ and the real eigenvalues $\lambda_2^{\omega^{(0)}} \le \lambda_1^{\omega^{(0)}}$ of $M_{\omega^{(0)}}$. Explicitly, we get

$$\begin{gathered}
S_{(0,0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ \lambda_1^{(0,0)} = 1,\ \lambda_2^{(0,0)} = -1, \\
S_{(1,1)} = S_{(0,0)} \qquad , \ \lambda_1^{(1,1)} = 2,\ \lambda_2^{(1,1)} = 0, \\
S_{(1,0)} = \begin{pmatrix} \frac{2}{\sqrt{10-2\sqrt{5}}} & \frac{2}{\sqrt{10+2\sqrt{5}}} \\ \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} \end{pmatrix}, \ \lambda_1^{(1,0)} = \frac{1+\sqrt{5}}{2},\ \lambda_2^{(1,0)} = \frac{1-\sqrt{5}}{2}\,.
\end{gathered}$$

We do not compute $S_{(0,1)}, \lambda_1^{(0,1)}, \lambda_2^{(0,1)}$ because we will not use them in the following.

We also introduce the block matrices

$$R_{\omega^{(0)}} = \begin{pmatrix} S_{\omega^{(0)}} & 0 \\ 0 & S_{\omega^{(0)}} \end{pmatrix}.$$

Let $E > 2$ (and thus larger than all eigenvalues of all $M_{\omega^{(0)}}$). With the abbreviation $r_i = r_i(E, \omega^{(0)}) := \sqrt{E - \lambda_i^{\omega^{(0)}}}$, $i = 1, 2$, the transfer matrices become

$$A^{\omega^{(0)}}_{0,2}(E) = R_{\omega^{(0)}} \begin{pmatrix} \cos(r_1) & 0 & \frac{1}{r_1}\sin(r_1) & 0 \\ 0 & \cos(r_2) & 0 & \frac{1}{r_2}\sin(r_2) \\ -r_1\sin(r_1) & 0 & \cos(r_1) & 0 \\ 0 & -r_2\sin(r_2) & 0 & \cos(r_2) \end{pmatrix} R^{-1}_{\omega^{(0)}}\,. \tag{4.2}$$

For $E < 2$ one can still write explicit expressions for the transfer matrices, where some of the sines and cosines are replaced by the respective hyperbolic functions, depending on the relative location of E to the various $\lambda_i^{\omega^{(0)}}$. This would lead to various cases, for each of which the arguments of the following subsection are not quite as easily adjustable as in the cases of Section 3. We therefore left the nature of Lyapunov exponents for $E \in (-1, 2)$ open, even if we fully expect similar results. As in Section 3 it is seen that the minimum of the almost sure spectrum of H^A_ω is again -1.

4.2. **Proof of Theorem 4.1.** Using the Gol'dsheid-Margulis criterion and the results from [19] and [15] as in Section 3 above, it will suffice to prove the following:

Proposition 4.2. *There exists a countable set $\mathcal{C}$ such that for all $E \in (2,\infty) \setminus \mathcal{C}$, $G_{\tilde{\mu}_E}$ is Zariski-dense in* $\mathrm{Sp}_2(\mathbb{R})$.

Proof. **Step 1.** We fix $E \in (2,\infty)$. For $\omega^{(0)} = (0,0)$ we have

$$A_{0,2}^{(0,0)}(E) = R_{(0,0)} \begin{pmatrix} \cos(\alpha_1) & 0 & \frac{1}{\alpha_1}\sin(\alpha_1) & 0 \\ 0 & \cos(\alpha_2) & 0 & \frac{1}{\alpha_2}\sin(\alpha_2) \\ -\alpha_1\sin(\alpha_1) & 0 & \cos(\alpha_1) & 0 \\ 0 & -\alpha_2\sin(\alpha_2) & 0 & \cos(\alpha_2) \end{pmatrix} R_{(0,0)}^{-1} \tag{4.3}$$

where $\alpha_1 = \sqrt{E - \lambda_1^{(0,0)}} = \sqrt{E-1}$ and $\alpha_2 = \sqrt{E - \lambda_2^{(0,0)}} = \sqrt{E+1}$.

Let $\mathcal{C}_1$ be the set of energies such that $(2\pi, \alpha_1, \alpha_2)$ is a rationally dependent set. It is easily checked that $\mathcal{C}_1$ is countable.

We now assume that $E \in (2,\infty) \setminus \mathcal{C}_1$. Rational independence of $(2\pi, \alpha_1, \alpha_2)$ implies that there exists a sequence $(n_k) \in \mathbb{N}^{\mathbb{N}}$, such that

$$(n_k\alpha_1, n_k\alpha_2) \xrightarrow[k\to\infty]{} (\frac{\pi}{2}, 0)$$

with convergence in $\mathbb{R}^2/(2\pi\mathbb{Z})^2$. There also exists $(m_k) \in \mathbb{N}^{\mathbb{N}}$, such that

$$(m_k\alpha_1, m_k\alpha_2) \xrightarrow[k\to\infty]{} (0, \frac{\pi}{2}) \,.$$

Then, as $G_{\tilde{\mu}_E}$ is closed, we conclude that

$$\Big(A_{0,2}^{(0,0)}(E)\Big)^{n_k} \xrightarrow[k\to\infty]{} R_{(0,0)} \begin{pmatrix} 0 & 0 & \frac{1}{\alpha_1} & 0 \\ 0 & 1 & 0 & 0 \\ -\alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} R_{(0,0)}^{-1} \in G_{\tilde{\mu}_E} \tag{4.4}$$

and

$$\Big(A_{0,2}^{(0,0)}(E)\Big)^{m_k} \xrightarrow[k\to\infty]{} R_{(0,0)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha_2} \\ 0 & 0 & 1 & 0 \\ 0 & -\alpha_2 & 0 & 0 \end{pmatrix} R_{(0,0)}^{-1} \in G_{\tilde{\mu}_E} \,. \tag{4.5}$$

For $\omega^{(0)} = (1,1)$ we have

$$A_{0,2}^{(1,1)}(E) = R_{(0,0)} \begin{pmatrix} \cos(\beta_1) & 0 & \frac{1}{\beta_1}\sin(\beta_1) & 0 \\ 0 & \cos(\beta_2) & 0 & \frac{1}{\beta_2}\sin(\beta_2) \\ -\beta_1\sin(\beta_1) & 0 & \cos(\beta_1) & 0 \\ 0 & -\beta_2\sin(\beta_2) & 0 & \cos(\beta_2) \end{pmatrix} R_{(0,0)}^{-1} \,, \tag{4.6}$$

where $\beta_1 = \sqrt{E - \lambda_1^{(1,1)}} = \sqrt{E-2}$ and $\beta_2 = \sqrt{E - \lambda_2^{(1,1)}} = \sqrt{E}$. Similarly, working with powers of $A_{0,2}^{(1,1)}(E)$, we see that for E such that $(2\pi, \beta_1, \beta_2)$ is rationally independent (which occurs away from a countable set $\mathcal{C}_2$)

$$R_{(0,0)} \begin{pmatrix} 0 & 0 & \frac{1}{\beta_1} & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} R_{(0,0)}^{-1} \in G_{\tilde{\mu}_E} \tag{4.7}$$

and

$$R_{(0,0)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta_2} \\ 0 & 0 & 1 & 0 \\ 0 & -\beta_2 & 0 & 0 \end{pmatrix} R_{(0,0)}^{-1} \in G_{\tilde{\mu}_E} \,. \tag{4.8}$$

Step 2. Multiplying (4.4) by the inverse of (4.7) we get

$$R_{(0,0)} \begin{pmatrix} \frac{\beta_1}{\alpha_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\alpha_1}{\beta_1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} R_{(0,0)}^{-1} \in G_{\tilde{\mu}_E} \,.$$

As $\alpha_1 > \beta_1 > 0$, by an argument similar to the one used in the proof of Lemma 3.2, this implies that for all $x > 0$,

$$C_1(x) = R_{(0,0)} \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{x} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} R_{(0,0)}^{-1} \in \mathrm{Cl}_{\mathrm{Z}}(G_{\tilde{\mu}_E}) \,.$$

We remark that $C_1(1) = I$. Thus, by differentiating at I,

$$C_1 := R_{(0,0)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} R_{(0,0)}^{-1} \in \mathfrak{A}_2(E) \,.$$

In the same way, using (4.5) and (4.8),

$$C_2 := R_{(0,0)} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} R_{(0,0)}^{-1} \in \mathfrak{A}_2(E) \,.$$

Step 3. Now we conjugate C_1 by $A_{0,2}^{(0,0)}(E)$ to find

$$\begin{aligned} &A_{0,2}^{(0,0)}(E)\, C_1 \,(A_{0,2}^{(0,0)}(E))^{-1} \\ &= R_{(0,0)} \begin{pmatrix} \cos^2(\alpha_1) - \sin^2(\alpha_1) & 0 & -\frac{2}{\alpha_1}\sin(\alpha_1)\cos(\alpha_1) & 0 \\ 0 & 0 & 0 & 0 \\ -2\alpha_1 \sin(\alpha_1)\cos(\alpha_1) & 0 & \sin^2(\alpha_1) - \cos^2(\alpha_1) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} R_{(0,0)}^{-1} \in \mathfrak{A}_2(E) \end{aligned}$$

by (3.14). We can subtract from this a multiple of C_1, $(\cos^2(\alpha_1) - \sin^2(\alpha_1))C_1$, and divide the result by $2\alpha_1 \sin(\alpha_1)\cos(\alpha_1) \neq 0$ to find

$$C_3 := R_{(0,0)} \begin{pmatrix} 0 & 0 & \frac{1}{\alpha_1} & 0 \\ 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} R_{(0,0)}^{-1} \in \mathfrak{A}_2(E) \,.$$

Conjugating C_1 by $A_{0,2}^{(1,1)}(E)$ and repeating the same arguments we find

$$C_4 := R_{(0,0)} \begin{pmatrix} 0 & 0 & \frac{1}{\beta_1} & 0 \\ 0 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} R_{(0,0)}^{-1} \in \mathfrak{A}_2(E)\,.$$

Conjugating C_2 in the same way shows that

$$C_5 := R_{(0,0)} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha_2} \\ 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \end{pmatrix} R_{(0,0)}^{-1} \in \mathfrak{A}_2(E)$$

and

$$C_6 := R_{(0,0)} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta_2} \\ 0 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 \end{pmatrix} R_{(0,0)}^{-1} \in \mathfrak{A}_2(E)\,.$$

Step 4. As $|\alpha_1| \neq |\beta_1|$ and $|\alpha_2| \neq |\beta_2|$ it is clear that the matrices $C_1, \dots, C_6$ are linearly independent. It follows that

$$R_{(0,0)} \begin{pmatrix} a & 0 & b & 0 \\ 0 & \tilde{a} & 0 & \tilde{b} \\ c & 0 & -a & 0 \\ 0 & \tilde{c} & 0 & -\tilde{a} \end{pmatrix} R_{(0,0)}^{-1} \in \mathfrak{A}_2(E) \tag{4.9}$$

for all $(a, \tilde{a}, b, \tilde{b}, c, \tilde{c}) \in \mathbb{R}^6$.
Step 5. Let $\mathcal{C}_3$ be the countable set of energies E such that $(2\pi, \sqrt{E - \frac{1+\sqrt{5}}{2}}, \sqrt{E - \frac{1-\sqrt{5}}{2}})$ is rationally dependent. Then for $E \in (2, +\infty) \setminus \mathcal{C}_3$, using the same argument as in (4.4) for the powers of $A_{0,2}^{(1,0)}(E)$, we have

$$M_1 := R_{(1,0)} \begin{pmatrix} 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 1 & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} R_{(1,0)}^{-1} \in G_{\tilde{\mu}_E}$$

where $\alpha := \sqrt{E - \frac{1+\sqrt{5}}{2}}$.

In addition to $C_1, \dots, C_6$, we will find four more linearly independent elements of $\mathfrak{A}_2(E)$ by conjugating particular matrices of the form (4.9) with M_1. Let X be an arbitrary matrix of the form (4.9). First we remark that

$$R_{(1,0)} = R_{(0,0)} \begin{pmatrix} S_{(0,0)}^{-1} S_{(1,0)} & 0 \\ 0 & S_{(0,0)}^{-1} S_{(1,0)} \end{pmatrix}.$$

Then a calculation shows that

$$M_1 X M_1^{-1} = R_{(0,0)} \begin{pmatrix} B & \frac{1}{\alpha} A \\ -\alpha A & B \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & \tilde{a} & 0 & \tilde{b} \\ c & 0 & -a & 0 \\ 0 & \tilde{c} & 0 & -\tilde{a} \end{pmatrix} \begin{pmatrix} B & -\frac{1}{\alpha} A \\ \alpha A & B \end{pmatrix} R_{(0,0)}^{-1}\,, \tag{4.10}$$

where $A = T^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T$, $B = T^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} T$ and $T = S_{(1,0)} S_{(0,0)}^{-1}$.

To construct our last four elements we will take particular values for $a, \tilde{a}, b, \tilde{b}, c, \tilde{c}$. Letting X_1 be the special case of X where $c = 1, a = 0, \tilde{a} = 0, b = 0, \tilde{b} = 0, \tilde{c} = 0$, we get, after tedious calculations,

$$\begin{aligned} C_7 &:= M_1 X_1 M_1^{-1} \\ &= \frac{1}{4(5-\sqrt{5})^2} R_{(0,0)} \begin{pmatrix} * & -\frac{2+2\sqrt{5}}{\alpha} & * & * \\ \frac{-22+10\sqrt{5}}{\alpha} & * & -\frac{2+2\sqrt{5}}{\alpha^2} & * \\ * & 22-10\sqrt{5} & * & * \\ * & * & * & * \end{pmatrix} R_{(0,0)}^{-1} \in \mathfrak{A}_2(E). \end{aligned}$$

Here we only keep track of the four matrix elements which are crucial for establishing linear independence from $C_1, \dots, C_6$ as the corresponding matrix-elements of these matrices all vanish. Similarly to X_1 we choose X_2 such that $\tilde{c} = 1$ and all other parameters are 0. This gives

$$\begin{aligned} C_8 &:= M_1 X_2 M_1^{-1} \\ &= \frac{1}{4(5-\sqrt{5})^2} R_{(0,0)} \begin{pmatrix} * & \frac{2+2\sqrt{5}}{\alpha} & * & * \\ \frac{22-10\sqrt{5}}{\alpha} & * & \frac{22-10\sqrt{5}}{\alpha^2} & * \\ * & -2-2\sqrt{5} & * & * \\ * & * & * & * \end{pmatrix} R_{(0,0)}^{-1} \in \mathfrak{A}_2(E). \end{aligned}$$

Under the assumption that $(2\pi, \sqrt{E - \frac{1+\sqrt{5}}{2}}, \sqrt{E - \frac{1-\sqrt{5}}{2}})$ is rationally independent, as in (4.5), for the powers of $A_{0,2}^{(1,0)}(E)$ we can prove that

$$M_2 := R_{(1,0)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta} \\ 0 & 0 & 1 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix} R_{(1,0)}^{-1} \in G_{\tilde{\mu}_E} ,$$

where $\beta := \sqrt{E - \frac{1-\sqrt{5}}{2}})$. Then, as before, we have

$$M_2 X M_2^{-1} = R_{(0,0)} \begin{pmatrix} A & \frac{1}{\beta} B \\ -\beta B & A \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & \tilde{a} & 0 & \tilde{b} \\ c & 0 & -a & 0 \\ 0 & \tilde{c} & 0 & -\tilde{a} \end{pmatrix} \begin{pmatrix} A & -\frac{1}{\beta} B \\ \beta B & A \end{pmatrix} R_{(0,0)}^{-1}$$

with the same A and B as in (4.10). Let X_3 be the special case of X with $b = 1$ and all the other parameters equal to 0 and, similarly, X_4 with $\tilde{b} = 1$ instead of $b = 1$. This gives

$$\begin{aligned} C_9 &:= M_2 X_3 M_2^{-1} \\ &= \frac{1}{4(5-\sqrt{5})^2} R_{(0,0)} \begin{pmatrix} * & -\beta \frac{1+\sqrt{5}}{8} & * & * \\ \beta \frac{125-41\sqrt{5}}{8} & * & \frac{1+\sqrt{5}}{8} & * \\ * & \beta^2 \frac{125-41\sqrt{5}}{8} & * & * \\ * & * & * & * \end{pmatrix} R_{(0,0)}^{-1} \in \mathfrak{A}_2(E) \end{aligned}$$

and

$$C_{10} := M_2 X_4 M_2^{-1} = \frac{1}{4(5-\sqrt{5})^2} R_{(0,0)} \begin{pmatrix} * & \beta\frac{95-29\sqrt{5}}{8} & * & * \\ \beta\frac{11-5\sqrt{5}}{8} & * & \frac{-11+5\sqrt{5}}{8} & * \\ * & \beta^2\frac{95-29\sqrt{5}}{8} & * & * \\ * & * & * & * \end{pmatrix} R_{(0,0)}^{-1} \in \mathfrak{A}_2(E).$$

Step 6. It can be verified that for most E the four $\mathbb{R}^4$-vectors composed of the four tracked matrix-elements of C_7, C_8, C_9 and C_{10} are linearly independent. In fact we have

$$\begin{aligned} &\begin{vmatrix} -\frac{1}{\alpha}(2+2\sqrt{5}) & \frac{1}{\alpha}(2+2\sqrt{5}) & -\beta\frac{1+\sqrt{5}}{8} & \beta\frac{95-29\sqrt{5}}{8} \\ \frac{1}{\alpha}(-22+10\sqrt{5}) & \frac{1}{\alpha}(22-10\sqrt{5}) & \beta\frac{125-41\sqrt{5}}{8} & \beta\frac{11-5\sqrt{5}}{8} \\ 22-10\sqrt{5} & -2-2\sqrt{5} & \beta^2\frac{125-41\sqrt{5}}{8} & \beta^2\frac{95-29\sqrt{5}}{8} \\ -\frac{1}{\alpha^2}(2+2\sqrt{5}) & \frac{1}{\alpha^2}(22-10\sqrt{5}) & \frac{1+\sqrt{5}}{8} & \frac{-11+5\sqrt{5}}{8} \end{vmatrix} \\ &= \frac{2\beta(780-349\sqrt{5}(\alpha+\beta)(121\alpha-13664\sqrt{5}\beta-71805\beta))}{121(4(5-\sqrt{5})^2)^4\alpha^3}. \end{aligned}$$

The right hand side is algebraic as a function of E and therefore has a discrete set $\mathcal{C}_4$ of zeros.

Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$. We fix $E \in (2, +\infty) \setminus \mathcal{C}$. Then $C_1, \dots, C_6$ are linearly independent and so are $C_7, \dots, C_{10}$. As the corresponding matrix-elements of $C_1, \dots, C_6$ all vanish, it follows that $(C_1, \dots, C_{10})$ is linearly independent. Thus

$$10 \le \dim \mathfrak{A}_2(E) \le \dim \mathrm{Sp}_2(\mathbb{R}) = 10$$

and therefore $\mathfrak{A}_2(E) = \mathfrak{sp}_2(\mathbb{R})$. Then by connectedness of $\mathrm{Sp}_2(\mathbb{R})$ we have that $\mathrm{Cl_Z}(G_{\tilde{\mu}_E}) = \mathrm{Sp}_2(\mathbb{R})$. We have proved the proposition. □

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References

[1] S.Albeverio, F.Gesztesy, R.Hoegh-Krohn and H.Holden, *Solvable Models in Quantum Mechanics*, Texts and Monographs in Physics) Springer, Berlin, 1988.
[2] Y. Benoist, *Sous-groupes discrets des groupes de Lie*, European Summer School in Group Theory, 1997.
[3] P.Bougerol and J.Lacroix, *Products of Random Matrices with Applications to Schrödinger Operators*, Progr. Probab. Statist. **8**, Birkhäuser, Boston, 1985.
[4] R.Carmona, A.Klein and F.Martinelli, *Anderson localization for Bernoulli and other singular potentials*, Commun. Math. Phys. **108** (1987), 41-66
[5] R.Carmona and J.Lacroix, *Spectral Theory of Random Schrödinger Operators*, Probability and Its Applications, Birkhäuser, Boston, 1990.
[6] D.Damanik, *Lyapunov exponents and spectral analysis of ergodic Schrödinger operators: A survey of Kotani theory and its applications*, Preprint 2006, arXiv:math-ph/0605054
[7] D.Damanik, D.Lenz and G.Stolz, *Lower Transport Bounds for One-dimensional Continuum Schrödinger Operators*, Math. Ann. **336** (2006), 361-389.

[8] D.Damanik, R.Sims and G.Stolz, *Lyapunov exponents in continuum Bernoulli-Anderson models*, Operator Theory : Advances and Applications **132**, 121-130, Birkhaüser 2002.
[9] D.Damanik, R. Sims and G. Stolz, *Localization for one-dimensional, continuum, Bernoulli-Anderson models*, Duke Mathematical Journal **114** (2002), 59-99.
[10] C.Glaffig, *Smoothness of the integrated density of states on strips*, J. Funct. Anal. **92** (1990), 509-534.
[11] I. Ya. Gol'dsheid and G.A. Margulis, *Lyapunov indices of a product of random matrices*, Russian Math. Survey **44:5** (1989), 11-71.
[12] Y.Guivarc'h and A.Raugi, *Frontière de Furstenberg, propriétés de contraction et théorèmes de convergence*, Z. Wahrsch. Verw. Gebiete **69** (1985), no. 2, p.187-242.
[13] Y.Guivarc'h and A.Raugi, *Propriétés de contraction d'un semi-groupe de matrices inversibles. Coefficients de Liapunoff d'un produit de matrices aléatoires indépendantes*, Israel J. Math. **65** (1989), no. 2, 165-196.
[14] S.Jitomirskaya, H.Schulz-Baldes and G.Stolz, *Delocalization in random polymer models*, Commun. Math. Phys. **233** (2003), 27-48
[15] W.Kirsch, *On a class of Random Schrödinger Operators*, Advances in Applied Mathematics **6** (1985), p.177-187
[16] W.Kirsch, S.Molchanov and L.Pastur, *One dimensional Schrödinger operator with unbounded potential: pure point spectrum, I.*, Funct. Anal. Appl., **3** (1990), 24-28
[17] W.Kirsch, S.Molchanov, L.Pastur and B.Vainberg, *Quasi 1D localization: Deterministic and random potentials*, Markov Processes Relat. Fields, **9** (2003), 687-708
[18] A.Klein, J.Lacroix and A.Speis, *Localization for the Anderson model on a Strip with Singular Potentials*, J. Func. Anal. **94** (1990), 135-155.
[19] S.Kotani and B.Simon, *Stochastic Schrödinger operators and Jacobi Matrices on the Strip*, Commun. Math. Phys. **119** (1988), no. 3, 403-429.
[20] B.Simon and T.Spencer, *Trace class perturbation and the absence of absolutely continuous spectrum*, Commun. Math. Phys. **125** (1989), 113-125
[21] P.Stollmann, *Caught by Disorder — Bound States in Random Media*, Progress in Mathematical Physics, Vol.**20**, Birkhäuser, 2001
[22] G. Stolz, *Strategies in localization proofs for one-dimensional random Schrödinger operators*, Proc. Indian Acad. Sci, **112** (2002), 229-243.

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