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EXISTENCE OF ψ-BOUNDED SOLUTIONS FOR NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this article we present a necessary and sufficient condition for the existence of ψ-bounded solution on $\mathbb{R}$ of the nonhomogeneous linear differential equation $x' = A(t)x + f(t)$. We associate that with the condition of the concept ψ-dichotomy on $\mathbb{R}$ of the homogeneous linear differential equation $x' = A(t)x$.

1. Introduction

The existence of ψ-bounded and ψ-stable solutions on $\mathbb{R}_+$ for systems of ordinary differential equations has been studied by many authors; see for example Akinyele [1], Avramescu [2], Constantin [4], Diamandescu [5, 6, 7]. Denote by $\mathbb{R}^d$ the d-dimensional Euclidean space. Elements in this space are denoted by $x = (x_1, x_2, \dots, x_d)^T$ and their norm by $\|x\| = \max\{|x_1|, |x_2|, \dots, |x_d|\}$. For real $d \times d$ matrices, we define norm $|A| = \sup_{\|x\| \leqslant 1} \|Ax\|$. Let $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_- = (-\infty, 0]$, $J = \mathbb{R}_-, \mathbb{R}_+$ or $\mathbb{R}$ and $\psi_i : J \to (0, \infty)$, $i = 1, 2, \dots, d$ be continuous functions. Set

$$\psi = \operatorname{diag}[\psi_1, \psi_2, \dots, \psi_d].$$

Definition 1.1. A function $f : J \to \mathbb{R}^d$ is said to be

- ψ-bounded on J if $\psi(t)f(t)$ is bounded on J.
- ψ-integrable on J if $f(t)$ is measurable and $\psi(t)f(t)$ is Lebesgue integrable on J.
- ψ-integrally bounded on J if $f(t)$ is measurable and the Lebesgue integrals $\int_t^{t+1} \|\psi(u)f(u)\|du$ are uniformly bounded for any $t, t+1 \in J$.

In $\mathbb{R}^d$, consider the following equations

$$x' = A(t)x + f(t), \tag{1.1}$$

$$x' = A(t)x. \tag{1.2}$$

where $A(t)$ is continuous matrix on J, $f(t)$ is a continuous function on J. Let $Y(t)$ be fundamental matrix of (1.2) with $Y(0) = I_d$, the identity $d \times d$ matrix. The

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$d \times d$ matrices P_1, P_2 is said to be the pair of the supplementary projections if $P_1^2 = P_1, P_2^2 = P_2, P_1 + P_2 = I_d$.

Definition 1.2. The equation (1.2) is said to have a ψ-exponential dichotomy on J if there exist positive constants K, L, α, β and a pair of the supplementary projections P_1, P_2 such that

$$|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)| \leqslant Ke^{-\alpha(t-s)} \quad \text{for } s \leqslant t, s, t \in J, \tag{1.3}$$

$$|\psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)| \leqslant Le^{\beta(t-s)} \quad \text{for } t \leqslant s, s, t \in J. \tag{1.4}$$

The equation (1.2) is said to have a ψ-ordinary dichotomy on J if (1.3), (1.4) hold with $\alpha = \beta = 0$.

We say that (1.2) has a ψ-bounded grow if for some fixed $h > 0$ there exists a constant $C \geqslant 1$ such that every solution $x(t)$ of (1.2) is satisfied

$$\|\psi(t)x(t)\| \leqslant C\|\psi(s)x(s)\| \text{ for } s \leqslant t \leqslant s+h, s, t \in J. \tag{1.5}$$

Remark 1.3. It is easy to see that if (1.2) has a ψ-exponential dichotomy on $\mathbb{R}_+$ and on $\mathbb{R}_-$ with a pair of the supplementary projections P_1, P_2 then (1.2) has a ψ-exponential dichotomy on $\mathbb{R}$ with the pair of the supplementary projections P_1, P_2.

Theorem 1.4 ([3, 5, 7])**.** *(a) The equation* (1.1) *has at least one ψ-bounded solution on $\mathbb{R}_+$ for every ψ-integrable function f on $\mathbb{R}_+$ if and only if* (1.2) *has a ψ-ordinary dichotomy on $\mathbb{R}_+$.*
(b) The equation (1.1) *has at least one ψ-bounded solution on $\mathbb{R}_+$ for every ψ- integrally bounded function f on $\mathbb{R}_+$ if and only if* (1.2) *has a ψ-exponential dichotomy on $\mathbb{R}_+$.*
(c) Suppose that (1.2) *has a ψ-bounded grow on $\mathbb{R}_+$. Then,* (1.1) *has at least one ψ-bounded solution on $\mathbb{R}_+$ for every ψ-bounded function f on $\mathbb{R}_+$ if and only if* (1.2) *has a ψ-exponential dichotomy on $\mathbb{R}_+$.*

Theorem 1.5 ([7])**.** *Suppose that* (1.1) *has a ψ-exponential dichotomy on $\mathbb{R}_+$ and, $P_1 \neq 0, P_2 \neq 0$. If $lim_{t\to\infty}\|\psi(t)f(t)\| = 0$ then every ψ-bounded solution $x(t)$ of* (1.1) *is such that $lim_{t\to\infty}\|\psi(t)x(t)\| = 0$.*

2. Preliminaries

Lemma 2.1. *(a) Let* (1.2) *has a ψ-exponential dichotomy on $\mathbb{R}_+$ with a pair of the supplementary projections P_1, P_2. If Q_1, Q_2 is a pair of the supplementary projections such that $ImP_1 = ImQ_1$, then* (1.2) *also has a ψ-exponential dichotomy on $\mathbb{R}_+$ with the pair of the supplementary projections Q_1, Q_2.*
(b) Let (1.2) *have a ψ-exponential dichotomy on $\mathbb{R}_-$ with a pair of the supplementary projections P_1, P_2. If Q_1, Q_2 is a pair of supplementary projections such that $ImP_2 = ImQ_2$, then* (1.2) *also has a ψ-exponential dichotomy on $\mathbb{R}_-$ with the pair of the supplementary projections Q_1, Q_2.*

Proof. First, we prove in the case of $J = \mathbb{R}_+$. Note that (1.2) has a ψ-exponential dichotomy on $\mathbb{R}_+$ with the pair of the supplementary projections P_1, P_2 if only if following statements are satisfied:

$$\|\psi(t)Y(t)P_1\xi\| \leqslant K'e^{-\alpha(t-s)}\|\psi(s)Y(s)\xi\| \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } t \geqslant s \geqslant 0, \tag{2.1}$$

$$\|\psi(t)Y(t)P_2\xi\| \leqslant L'e^{\beta(t-s)}\|\psi(s)Y(s)\xi\| \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } s \geqslant t \geqslant 0. \tag{2.2}$$

In fact, if (1.3) and (1.4) are true, we have for any vector $y \in \mathbb{R}^d$

$$\|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)y\| \leqslant Ke^{-\alpha(t-s)}\|y\| \quad \text{for } t \geqslant s \geqslant 0,$$
$$\|\psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)y\| \leqslant Le^{\beta(t-s)}\|y\| \quad \text{for } s \geqslant t \geqslant 0.$$

Choose $y = \psi(s)Y(s)\xi$, we obtain (2.1), (2.2). Conversely, suppose that inequalities (2.1), (2.2) are true. For any vector $y \in \mathbb{R}^d$, putting $\xi = Y^{-1}(s)\psi^{-1}(s)y$ we get (1.3), (1.4).

Now prove the lemma. It follows from $KerP_2 = ImP_1 = ImQ_1 = KerQ_2$ that $P_2Q_1 = 0$. Hence $P_1Q_1 = P_1Q_1 + P_2Q_1 = Q_1$. Similarly $Q_1P_1 = P_1$. Then

$$P_1 - Q_1 = P_1^2 - P_1Q_1 = P_1(P_2 - Q_2), \tag{2.3}$$
$$P_1 - Q_1 = -Q_1P_2 = P_1P_2 - Q_1P_2 = (P_1 - Q_1)P_2. \tag{2.4}$$

For each $u \in \mathbb{R}^d$, put $\xi = (P_1 - Q_1)u$. The relation (2.3) implies that $\xi \in ImP_1$, then $P_1\xi = \xi$. Result from (2.1), for $s = 0$ that

$$\|\psi(t)Y(t)[P_1 - Q_1]u\| \leqslant K'e^{-\alpha t}\|\psi(0)[P_1 - Q_1]u\|, t \geqslant 0. \tag{2.5}$$

By (2.4) we conclude

$$\begin{aligned} K'e^{-\alpha t}\|\psi(0)[P_1 - Q_1]u\| &= K'e^{-\alpha t}\|\psi(0)[P_1 - Q_1]P_2u\| \\ &\leqslant K'|\psi(0)||P_1 - Q_1|e^{-\alpha t}\|P_2u\|, \quad t \geqslant 0. \end{aligned} \tag{2.6}$$

Applying (2.2), for $t = 0$, we get

$$\begin{aligned} \|P_2u\| &= \|\psi^{-1}(0)\psi(0)P_2u\| \\ &\leqslant |\psi^{-1}(0)|\|\psi(0)P_2u\| \\ &\leqslant L'e^{-\beta s}|\psi^{-1}(0)|\|\psi(s)Y(s)u\|, \quad \text{for} s \geqslant 0. \end{aligned} \tag{2.7}$$

The relations (2.5)–(2.7) imply

$$\begin{aligned} \|\psi(t)Y(t)[P_1 - Q_1]u\| &\leqslant K'L'|\psi(0)||\psi^{-1}(0)||P_1 - Q_1|e^{-\alpha t}e^{-\beta t}\|\psi(s)Y(s)u\| \\ &\leqslant K_1e^{\beta(t-s)}\|\psi(s)Y(s)u\|, \quad \text{for } t, s \geqslant 0. \end{aligned} \tag{2.8}$$

On the other hand, by (2.2) we get

$$\|\psi(t)Y(t)P_2u\| \leqslant L'e^{\beta(t-s)}\|\psi(sY(s))u\|, \quad \text{for } 0 \leqslant t \leqslant s. \tag{2.9}$$

It follows from $Q_2 = P_2 + P_1 - Q_1$, (2.8) and (2.9) that

$$\begin{aligned} \|\psi(t)Y(t)Q_2u\| &\leqslant \|\psi(t)Y(t)P_2u\| + \|\psi(t)Y(t)[P_1 - Q_1]u\| \\ &\leqslant (L' + K_1)e^{\beta(t-s)}\|\psi(s)Y(s)u\| \\ &\leqslant L_2e^{\beta(t-s)}\|\psi(s)Y(s)u\|, \quad \text{for } 0 \leqslant t \leqslant s. \end{aligned} \tag{2.10}$$

Similarly, for $u \in \mathbb{R}^d$, we have

$$\|\psi(t)Y(t)Q_1u\| \leqslant K_2e^{-\alpha(t-s)}\|\psi(s)Y(s)u\|, \quad \text{for } 0 \leqslant s \leqslant t. \tag{2.11}$$

Then from this inequality, (2.10) and the preceding note it follows that (1.2) has a ψ-exponential dichotomy on $\mathbb{R}_+$ with the pair of the supplementary projections Q_1, Q_2. In the case of $J = \mathbb{R}_-$, the proof is similar. $\square$

Remark 2.2. (a) Suppose that (1.2) has a ψ-exponential dichotomy on $\mathbb{R}_+$ with a pair of supplementary projections P_1, P_2. The set $P_1\mathbb{R}^d$ is the subspace of $\mathbb{R}^d$ consisting of the values $x(0)$ of all ψ-bounded solutions $x(t)$ on $\mathbb{R}_+$ of (1.2). In fact, denote by X_1 this subspace, if $v \in P_1\mathbb{R}^d$ then $v \in X_1$ by virtue of (2.1). Conversely if $u \in X_1$, we have to show that $P_2u = 0$. Suppose otherwise that $P_2u \neq 0$, by (2.1), (2.2) we have $\|\psi(t)Y(t)P_1u\|$ is bounded and the limit of $\|\psi(t)Y(t)P_2u\|$ is ∞, as t tend to ∞. Denote y the solution of (1.2), $y(0) = u$. The relation $\psi(t)y(t) - \psi(t)Y(t)P_1u = \psi(t)Y(t)P_2u$ follows that y is nonψ-bounded on $\mathbb{R}_+$, which is a contradiction.
(b) Similarly if (1.2) has a ψ-exponential dichotomy on $\mathbb{R}_-$ with a pair of supplementary projections P_1, P_2 then the set $P_2\mathbb{R}^d$ is the subspace of $\mathbb{R}^d$ consisting of the values $x(0)$ of all ψ-bounded solutions $x(t)$ on $\mathbb{R}_-$ of (1.2).

(c) Suppose that (1.2) has a ψ-exponential dichotomy on $\mathbb{R}$, then (1.2) has no nontrivial ψ-bounded solution on $\mathbb{R}$. In fact if $x(t)$ is the ψ-bounded solution of (1.2) on $\mathbb{R}$ then it is ψ-bounded on $\mathbb{R}_+$ and on $\mathbb{R}_-$. Because equation (1.2) has a ψ-exponential dichotomy on $\mathbb{R}_+$, and on $\mathbb{R}_-$ with a pair of supplementary projections P_1, P_2, by preceding notice we have $P_2x(0) = 0$ and $P_1x(0) = 0$. Hence $x(0) = 0$, then $x(t)$ is the trivial solution of (1.2).

Lemma 2.3 ([8]). *Let $h(t)$ be a non-negative, locally integrable such that*
$$\int_t^{t+1} h(s)ds \leqslant c, \quad \text{for all } t \in \mathbb{R}$$
If $\theta > 0$ then, for all $t \in \mathbb{R}$,
$$\int_t^{\infty} e^{-\theta(s-t)}h(s)ds \leqslant c[1 - e^{-\theta}]^{-1}, \tag{2.12}$$
$$\int_{-\infty}^{t} e^{-\theta(t-s)}h(s)ds \leqslant c[1 - e^{-\theta}]^{-1}. \tag{2.13}$$

Proof. We prove (2.12), the proof of (2.13) is similar.
$$\begin{aligned}\int_{t+m}^{t+m+1} e^{-\theta(s-t)}h(s)ds &\leqslant \int_{t+m}^{t+m+1} e^{-\theta(t+m)}e^{\theta t}h(s)ds\\ &= \int_{t+m}^{t+m+1} e^{-\theta m}h(s)ds \leqslant ce^{-\theta m}\end{aligned}$$
implies that
$$\int_t^{\infty} e^{-\theta(s-t)}h(s)ds = \sum_{m=0}^{\infty}\int_{t+m}^{t+m+1} e^{-\theta(s-t)}h(s)ds \leqslant c\sum_{m=0}^{\infty} e^{-\theta m} = c[1 - e^{-\theta}]^{-1}$$
□

Lemma 2.4. *Equation* (1.1) *has at least one ψ-bounded solution on $\mathbb{R}$ for every ψ-integrally bounded function f on $\mathbb{R}$ if and only if the following three conditions are satisfied:*

1. *Equation* (1.1) *has at least one solution on $\mathbb{R}$, ψ-bounded on $\mathbb{R}_+$ for every ψ- integrally bounded function f on $\mathbb{R}_+$*
2. *Equation* (1.1) *has at least one solution on $\mathbb{R}$, ψ-bounded on $\mathbb{R}_-$ for every ψ-integrally bounded function f on $\mathbb{R}_-$.*

(3) *Every solution of* (1.2) *is the sum of two solution of* (1.2)*, one of that is ψ-bounded on $\mathbb{R}_+$, another is $\psi-$ bounded on $\mathbb{R}_-$.*

Proof. Suppose the three conditions are satisfied we have to prove that (1.1) has at least one ψ-bounded solution on $\mathbb{R}$ for every ψ-integrally bounded function f on $\mathbb{R}$. Every ψ-integrally bounded function f on $\mathbb{R}$ is ψ-integrally bounded function f on $\mathbb{R}_+$ and on $\mathbb{R}_-$. Then for each ψ-integrally bounded function f on $\mathbb{R}$ exists the solution y_1 and y_2 of (1.1), which is defined on $\mathbb{R}$ and corresponding ψ-bounded on $\mathbb{R}_+$ and on $\mathbb{R}_-$. Denote by $x(t)$ the solution of (1.2) such that $x(0) = y_2(0) - y_1(0)$. By 3, we get $x(t) = x_1(t) + x_2(t)$, here x_1, x_2 are two solutions of (1.2), that are corresponding ψ-bounded solution on $\mathbb{R}_+$ and $\mathbb{R}_-$. Set $z_1 = y_1 + x_1, z_2 = y_2 - x_2$.

Hence z_1 and z_2 are the solutions of (1.1) corresponding ψ-bounded solution on $\mathbb{R}_+$ and on $\mathbb{R}_-$. Further, $z_2(0) = y_2(0) - x_2(0) = y_1(0) + x_1(0) = z_1(0)$, then $z_1 = z_2$. Consequently z_1 is a ψ-bounded solution on $\mathbb{R}$ of (1.1).

Conversely, now if (1.1) has at least one ψ-bounded solution on $\mathbb{R}$ for every ψ-integrally bounded function f on $\mathbb{R}$ we have to prove three condition are satisfied. The conditions 1, 2 are satisfied since every ψ-integrally bounded function f on $\mathbb{R}_+$, or $\mathbb{R}_-$ is the restriction of a ψ- integrally bounded function f on $\mathbb{R}$. We prove that the condition 3 is satisfied. Set

$$h(t) = \begin{cases} 0 & \text{for } |t| \geqslant 1 \\ 1 & \text{for } t = 0 \\ \text{linear} & \text{for } t \in [-1, 0], t \in [0, 1] \end{cases}$$

Fix a solution $x(t)$ of (1.2) . Then $h(t)x(t)$ is a ψ-integrally bounded function on $\mathbb{R}$. Set $y(t) = x(t) \int_0^t h(s)ds$, we have

$$y'(t) = A(t)x(t) \int_0^t h(s)ds + h(t)x(t) = A(t)y(t) + h(t)x(t).$$

By hypothesis, the equation

$$y'(t) = A(t)y(t) + h(t)x(t)$$

has a solution $\widetilde{y}(t)$, which is ψ-bounded on $\mathbb{R}$. Set $x_1(t) = \widetilde{y}(t) - y(t) + \frac{1}{2}x(t)$ and $x_2(t) = \widetilde{y}(t) + y(t) + \frac{1}{2}x(t)$. It follows from $\int_{-1}^0 h(t)dt = \int_0^1 h(t)dt = \frac{1}{2}$ that $x_1(t) = \widetilde{y}(t)$ for $t \geqslant 1$; $x_2(t) = \widetilde{y}(t)$ for $t \leqslant -1$. Then x_1, x_2 are the corresponding ψ-bounded solutions on $\mathbb{R}_+$, $\mathbb{R}_-$ of (1.2). Consequently the solution $x(t)$ of (1.2) is the sum of two solutions $x_1(t)$ and $x_2(t)$ of (1.2), those solutions satisfy the condition 3. The lemma is proved. □

3. Main results

Theorem 3.1. *Equation* (1.1) *has at least one ψ-bounded solution on $\mathbb{R}_-$ for every ψ-integrally bounded function f on $\mathbb{R}_-$ if and only if* (1.2) *has a ψ-exponential dichotomy on $\mathbb{R}_-$.*

Proof. This Theorem can be shown as in [3, Theorem 3.3]. We give the main steps of the proof as follows. In the proof of "if part": Suppose that $\int_{t-1}^t \|\psi(s)f(s)\|ds \leqslant c$

for $t \leqslant 0$. By using Lemma 2.3 we get

$$\begin{aligned}\|\int_{-\infty}^{t}\psi(t)Y(t)P_1Y^{-1}(s)ds\| &\leqslant \int_{-\infty}^{t}|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)|\|\psi(s)f(s)\|ds\\ &\leqslant \int_{-\infty}^{t}e^{-\alpha(t-s)}\|\psi(s)f(s)\|ds \leqslant c(1-e^{-\alpha})^{-1}\end{aligned}$$

and

$$\begin{aligned}\|\int_{t}^{0}\psi(t)Y(t)P_2Y^{-1}(s)f(s)ds\| &\leqslant \int_{t}^{0}e^{-\beta(s-t)}\|\psi(s)f(s)\|ds\\ &\leqslant \int_{t}^{\infty}e^{-\beta(s-t)}\|\psi(s)f(s)\|ds \leqslant c(1-e^{-\beta})^{-1}.\end{aligned}$$

It follows that the function

$$\widetilde{x}(t)=\int_{-\infty}^{t}\psi(t)Y(t)P_1Y^{-1}(s)f(s)ds-\int_{t}^{0}\psi(t)Y(t)P_2Y^{-1}(s)f(s)ds$$

is bounded on $\mathbb{R}_-$. Hence the function

$$\begin{aligned}x(t)&=\psi^{-1}(t)\widetilde{x}(t)\\ &=\int_{-\infty}^{t}\psi(t)Y(t)P_1Y^{-1}(s)f(s)ds-\int_{t}^{0}\psi(t)Y(t)P_2Y^{-1}(s)f(s)ds\end{aligned}$$

is ψ-bounded on $\mathbb{R}_-$. On the other hand

$$\begin{aligned}x'(t)&=A(t)(\int_{-\infty}^{t}Y(t)P_1Y^{-1}(s)f(s)ds-\int_{t}^{0}Y(t)P_2Y^{-1}(s)f(s)ds)\\ &\quad+Y(t)P_1Y^{-1}(t)f(t)+Y(t)P_2Y^{-1}(t)f(t)\\ &=A(t)x(t)+f(t),\end{aligned}$$

it implies that $x(t)$ is a solution of (1.1).

In the proof of "only if part": The set

$$\widetilde{C}_\psi=\{x:\mathbb{R}_-\to\mathbb{R}^d : x$$

is ψ-bounded and continuous on $\mathbb{R}_-\}$. It is a Banach space with the norm $\|x\|_{\widetilde{C}_\psi}=sup_{t\leqslant 0}\|\psi(t)x(t)\|$. The first step: we show that (1.1) has a unique ψ-bounded solution $x(t)$ with $x(0)\in\widetilde{X}_1=P_1\mathbb{R}^d$ for each $f\in\widetilde{C}_\psi$ and $\|x\|_{\widetilde{C}_\psi}\leqslant r\|f\|_{\widetilde{C}_\psi}$, here r is a positive constant independent of f.

The next steps of the proof are similar to the proof of [3, Theorem 3.3], with the corresponding replacement (for example replace $t\geqslant t_0\geqslant 0$ by $0\geqslant t_0\geqslant t$, P_1 by $-P_2, P_2$ by $-P_1$, ∞ by $-\infty$, $-\infty$ by ∞, ...). □

Theorem 3.2. *The equation* (1.1) *has a unique ψ-bounded solution on $\mathbb{R}$ for every ψ-integrally bounded function f on $\mathbb{R}$ if and only if* (1.2) *has a ψ-exponential dichotomy on $\mathbb{R}$.*

Proof. First, we prove the "if" part. By Lemma 2.3 and in the same way as in the proof of Theorem 3.1, the function

$$x(t)=\int_{-\infty}^{t}Y(t)P_1Y^{-1}(s)f(s)ds-\int_{t}^{\infty}Y(t)P_2Y^{-1}(s)f(s)ds$$

is ψ-bounded and continuous on $\mathbb{R}$. Moreover,

$$\begin{aligned} x'(t) &= A(t)\Big(\int_{-\infty}^{t} Y(t)P_1Y^{-1}(s)f(s)ds - \int_{t}^{\infty} Y(t)P_2Y^{-1}(s)f(s)ds\Big) \\ &\quad + Y(t)P_1Y^{-1}(t)f(t) - Y(t)P_2Y^{-1}(t)f(t) \\ &= A(t)x(t) + f(t), \end{aligned}$$

it follows that $x(t)$ is a solution of (1.1).

The uniqueness of the solution $x(t)$ result from (1.2) having no nontrivial ψ-bounded solution on $\mathbb{R}$ (Remark 2.2). Suppose that y is a ψ-bounded solution of (1.1) then $x - y$ is a ψ-bounded solution of (1.2) on $\mathbb{R}$. We conclude $x = y$ since $x - y$ is the trivial solution of (1.2).

We prove the "only if "part. Suppose that (1.1) has unique ψ-bounded solution on $\mathbb{R}$ for every ψ-integrally bounded function f on $\mathbb{R}$, we have to prove that (1.1) has a ψ-exponential dichotomy on $\mathbb{R}$. By Lemma 2.4, Theorem 1.4 and Theorem 3.1 we get (1.2) has a ψ-exponential dichotomy on $\mathbb{R}_+$ with a pair of the supplementary projections P_1, P_2 and has a ψ-exponential dichotomy on $\mathbb{R}_-$. with a pair of the supplementary projections Q_1, Q_2. Remark 2.2 follows that $P_1\mathbb{R}^d$ is the subspace of $\mathbb{R}^d$ consisting of the values $x(0)$ of all ψ-bounded solutions $x(t)$ on $\mathbb{R}_+$ of (1.2) and $Q_2\mathbb{R}^d$ is the subspace of $\mathbb{R}^d$ consisting of the values $x(0)$ of all ψ-bounded solutions $x(t)$ on $\mathbb{R}_-$ of (1.2). We are going to prove that

$$\mathbb{R}^d = P_1\mathbb{R}^d \oplus Q_2\mathbb{R}^d. \tag{3.1}$$

For each $u \in \mathbb{R}^d$, denote by $x = x(t)$ the solution of (1.2), $x(0) = u$. By Lemma 2.4 we get $x = x_1 + x_2$, where x_1, x_2 are the solutions of (1.2) corresponding ψ-bounded on $\mathbb{R}_+$,$\mathbb{R}_-$. It follows from Remark 2.2 that $x_1(0) \in P_1\mathbb{R}^d$ and $x_2(0) \in Q_2\mathbb{R}^d$. It follows from $u = x_1(0) + x_2(0)$, that

$$\mathbb{R}^d = P_1\mathbb{R}^d + Q_2\mathbb{R}^d. \tag{3.2}$$

By hypothesis (1.1) with $f = 0$ has unique ψ-bounded solution on $\mathbb{R}$ i.e. (1.2) have no nontrivial ψ-bounded solution on $\mathbb{R}$. For any $v \in P_1\mathbb{R}^d \cap Q_2\mathbb{R}^d$, denote by $x(t)$ the solution of (1.2) such that $x(0) = v$. Then $x(t)$ is the ψ-bounded solution of (1.2), it implies that $x(t)$ is the trivial solution. Hence $v = 0$. Consequently

$$P_1\mathbb{R}^d \cap Q_2\mathbb{R}^d = 0. \tag{3.3}$$

The relations (3.2) and (3.3) imply (3.1). Now, we prove the existence of a pair supplementary projections, for which (1.1) has a ψ-exponential dichotomy on $\mathbb{R}$. Choose the projection P of $\mathbb{R}^d$ such that $ImP = P_1\mathbb{R}^d$, $\ker P = Q_2\mathbb{R}^d$. By Lemma 2.1, (1.2) has a ψ-exponential dichotomy on $\mathbb{R}_+$, and have a ψ-exponential dichotomy on $\mathbb{R}_-$ with the pair of the supplementary projections $P, I_d - P$. From Remark 1.3 it follows that (1.2) has a ψ-exponential dichotomy on $\mathbb{R}$ with the pair of the supplementary projections $P, I_d - P$. The proof is complete. □

Theorem 3.3. *Suppose that* (1.2) *has a ψ-exponential dichotomy on $\mathbb{R}$. If*

$$\lim_{t\to\pm\infty} \int_{t}^{t+1} \|\psi(s)f(s)\|ds = 0 \tag{3.4}$$

then the ψ-bounded solution of (1.1) *is such that*

$$\lim_{t\to\pm\infty} \|\psi(t)x(t)\| = 0. \tag{3.5}$$

Proof. By Theorem 3.2, the unique solution of (1.1) is

$$x(t)=\int_{-\infty}^{t} Y(t)P_1Y^{-1}(s)f(s)ds-\int_{t}^{\infty} Y(t)P_2Y^{-1}(s)f(s)ds.$$

$$\begin{aligned}\|\psi(t)x(t)\| &\leqslant \int_{-\infty}^{t}\|\psi(t)Y(t)P_1Y^{-1}(s)f(s)\|ds+\int_{t}^{\infty}\|\psi(t)Y(t)P_2Y^{-1}(s)f(s)\|ds\\ &\leqslant K\int_{-\infty}^{t} e^{-\alpha(t-s)}\|\psi(s)f(s)\|ds+L\int_{t}^{\infty} e^{-\beta(s-t)}\|\psi(s)f(s)\|ds\\ &\leqslant K_1\{\int_{-\infty}^{t} e^{-\alpha(t-s)}\|\psi(s)f(s)\|ds+\int_{t}^{\infty} e^{-\beta(s-t)}\|\psi(s)f(s)\|ds\},\end{aligned}\tag{3.6}$$

where $K1=\max\{K,L\}$. Denote by $\gamma=\min\{\alpha,\beta\}$. Under the hypothesis (3.4), for a given $\varepsilon>0$, there exists $T>0$ such that

$$\int_{t}^{t+1}\|\psi(s)f(s)\|ds<\frac{\varepsilon}{2K_1}(1-e^{-\gamma})\quad \text{for } |t|>T.$$

Then from Lemma 2.3 and inequality (3.6) it follow that

$$\begin{aligned}\|\psi(t)x(t)\| &\leqslant K_1\frac{\varepsilon}{2K_1}(1-e^{-\gamma})[(1-e^{-\alpha})^{-1}+(1-e^{-\beta})^{-1}]\\ &\leqslant K_1\frac{\varepsilon}{2K_1}(1-e^{-\gamma})2(1-e^{-\gamma})^{-1}=\varepsilon\quad \text{for all } |t|>T,\end{aligned}$$

this implies (3.5). The proof is complete. □

Corollary 3.4. *Suppose that* (1.2) *has a ψ-exponential dichotomy on $\mathbb{R}$. If*

$$\lim_{t\to\pm\infty}\|\psi(t)f(t)\|=0 \tag{3.7}$$

then the ψ-bounded solution of (1.1) *is such that*

$$\lim_{t\to\pm\infty}\|\psi(t)x(t)\|=0. \tag{3.8}$$

Proof. It is easy to see that (3.7) implies (3.4) □

Now, we consider the perturbed equation

$$x'(t)=[A(t)+B(t)]x(t) \tag{3.9}$$

where $B(t)$ is a $d\times d$ continuous matrix function on $\mathbb{R}$. We have the following result.

Theorem 3.5. *Suppose that* (1.2) *has a ψ-exponential dichotomy on $\mathbb{R}$. If $\delta=\sup_{t\in\mathbb{R}}\int_t^{t+1}|\psi(s)B(s)\psi^{-1}(s)|ds$ is sufficiently small, then* (3.9) *has a ψ-exponential dichotomy on $\mathbb{R}$.*

Proof. By Theorem 3.2 it suffices to show that the equation

$$x'(t)=[A(t)+B(t)]x(t)+f(t) \tag{3.10}$$

has a unique ψ-bounded solution on $\mathbb{R}$ for every ψ-integrally bounded f function on $\mathbb{R}$. Denote by G_ψ the set

$$G_\psi=\{x:\mathbb{R}\to\mathbb{R}^d: x \text{ is } \psi\text{-bounded and continuous on } \mathbb{R}\}.$$

It is well-known that G_ψ is a real Banach space with the norm

$$\|x\|_{G_\psi} = \sup_{t\in R} \|\psi(t)x(t)\|.$$

Consider the mapping $T : G_\psi \to G_\psi$ which is defined by

$$\begin{aligned} Tz(t) = &\int_{-\infty}^{t} Y(t)P_1Y^{-1}(s)[B(s)z(s) + f(s)]ds \\ &- \int_{t}^{\infty} Y(t)P_2Y^{-1}(s)[B(s)z(s) + f(s)]ds. \end{aligned}$$

It is easy verified that $Tz \in G_\psi$. More ever if $z_1, z_2 \in G_\psi$ then

$$\begin{aligned} &\|Tz_1 - Tz_2\|_{G_\psi} \\ &\leqslant \int_{-\infty}^{t} |\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)||\psi(s)B(s)\psi^{-1}(s)|\|\psi(s)z_1(s) - \psi(s)z_2(s)\|ds \\ &\quad + \int_{t}^{\infty} |\psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)||\psi(s)B(s)\psi^{-1}(s)|\|\psi(s)z_1(s) - \psi(s)z_2(s)\|ds \end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned} \|Tz_1 - Tz_2\|_{G_\psi} &\leqslant K\|z_1 - z_2\|_{G_\psi} \int_{-\infty}^{t} e^{-\alpha(t-s)}|\psi(s)B(s)\psi^{-1}(s)|ds \\ &\quad + L\|z_1 - z_2\|_{G_\psi} \int_{t}^{\infty} e^{\beta(t-s)}|\psi(s)B(s)\psi^{-1}(s)|ds \\ &\leqslant \delta[K(1 - e^{-\alpha})^{-1} + L(1 - e^{-\beta})^{-1}]\|z_1 - z_2\|_{G_\psi} \end{aligned}$$

Hence, by the contraction principle, if $\delta[K(1 - e^{-\alpha})^{-1} + L(1 - e^{-\beta})^{-1}] < 1$, then the mapping T has a unique fixed point. Denoting this fixed point by z, we have

$$\begin{aligned} z(t) = &\int_{-\infty}^{t} Y(t)P_1Y^{-1}(s)[B(s)z(s) + f(s)]ds \\ &- \int_{t}^{\infty} Y(t)P_2Y^{-1}(s)[B(s)z(s) + f(s)]ds. \end{aligned}$$

It follows that $z(t)$ is a solution on $\mathbb{R}$ of (3.10).

Now, we prove the uniqueness of this solution. Suppose that $x(t)$ is a arbitrary ψ-bounded solution on $\mathbb{R}$ of (3.10). Consider the function

$$\begin{aligned} y(t) = x(t) - &\int_{-\infty}^{t} Y(t)P_1Y^{-1}(s)[B(s)x(s) + f(s)]ds \\ &+ \int_{t}^{\infty} Y(t)P_2Y^{-1}(s)[B(s)x(s) + f(s)]ds. \end{aligned}$$

It is easy to see that $y(t)$ is a ψ-bounded solution on $\mathbb{R}$ of (1.2). Then from Theorem 3.2 follows that $y(t)$ is the trivial solution. Then

$$\begin{aligned} x(t) = &\int_{-\infty}^{t} Y(t)P_1Y^{-1}(s)[B(s)x(s) + f(s)]ds \\ &- \int_{t}^{\infty} Y(t)P_2Y^{-1}(s)[B(s)x(s) + f(s)]ds. \end{aligned}$$

Hence $x(t)$ is the fixed point of mapping T. From the uniqueness of this point, it follows that $x = z$. The proof is complete. □

Corollary 3.6. *Suppose that* (1.2) *has a ψ-exponential dichotomy on $\mathbb{R}$. If* $\delta = \sup_{t\in\mathbb{R}} |\psi(t)B(t)\psi^{-1}(t)|$ *is sufficiently small, then* (3.9) *has a ψ-exponential dichotomy on $\mathbb{R}$.*

References

[1] O. Akinyele; *On partial stability and boundedness of degree k*, Atti. Acad. Naz. Lincei Rend. Cl. Sei. Fis. Mat. Natur., (8), 65 (1978), 259-264.

[2] C. Avramescu; *Asupra comportãrii asimptotice a solutiilor unor ecuatii funcionable*, Analele Universitãtii din Timisoara, Seria Stiinte Matamatice-Fizice, Vol. VI, 1968, 41-55.

[3] P. N. Boi; *On the ψ- dichotomy for homogeneous linear differential equations.* Electron. J. of Differential Equations, vol. 2006 (2006), No. 40, 1-12.

[4] A. Constantin; *Asymptotic properties of solution of differential equation*, Analele Universitãtii din Timisoara, Seria Stiinte Matamatice, Vol. XXX, fasc. 2-3,1992, 183-225.

[5] A. Diamandescu; *Existence of ψ-bounded solutions for a system of differential equations*; Electron. J. of Differetial Equaitons, vol. 2004 (2004), No. 63, 1-6.

[6] A. Diamandescu; *On the ψ-stability of a nonlinear Volterra integro-differential system*, Electron. J. of Differetial Equaitons, Vol. 2005 (2005), No. 56, 1-14.

[7] A. Dimandescu; *Note on the ψ-boundedness of the solutions of a system of differential equations.* Acta Math. Univ. Comenianea. vol. LXXIII, 2 (2004), 223-233.

[8] J. L. Massera and J. J. Schaffer; *Linear differential equations and functional analysis*, Ann. Math. 67 (1958), 517-573.

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