Electronic Journal of Differential Equations, Vol. 2007(2007), No. 57, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

SUFFICIENT CONDITIONS FOR NONEXISTENCE OF GRADIENT BLOW-UP FOR NONLINEAR PARABOLIC EQUATIONS

ARIS S. TERSENOV

Abstract. In this paper we study the initial-boundary value problems for nonlinear parabolic equations without Bernstein-Nagumo condition. Sufficient conditions guaranteeing the nonexistence of gradient blow-up are formulated. In particular, we show that for a wide class of nonlinearities the Lipschitz continuity in the space variable together with the strict monotonicity with respect to the solution guarantee that gradient blow-up cannot occur at the boundary or in the interior of the domain.

1. Introduction

In the present paper we consider the nonlinear equation

$$u_t = F(t, x, u, u_x, u_{xx}) \quad \text{in } Q_T = (-l, l) \times (0, T), \tag{1.1}$$

coupled with one of the boundary conditions

$$u_x(t, -l) = u_x(t, l) = 0, \tag{1.2}$$

$$u_x + \sigma_1(t, x, u)\big|_{x=-l} = u_x + \sigma_2(t, x, u)\big|_{x=l} = 0, \tag{1.3}$$

$$u(t, -l) = u(t, l) = 0 \tag{1.4}$$

and the initial condition

$$u(0, x) = u_0(x). \tag{1.5}$$

We assume that $F(t, x, u, p, r)$ is continuously differentiable with respect to r and satisfies the parabolicity condition, i.e.

$$F_r(t, x, u, p, r) > 0 \quad \text{for} \quad (t, x, u, p, r) \in \overline{Q}_T \times [-M, M] \times \mathbb{R}^2. \tag{1.6}$$

Write equation (1.1) in the form

$$u_t = F_r(t, x, u, u_x, \lambda u_{xx})u_{xx} + F(t, x, u, u_x, 0), \quad \lambda \in [0, 1], \tag{1.7}$$

using the mean value theorem. The well known Bernstein-Nagumo condition [4, 5, 19] (see also [6, 13, 15, 17, 18, 20]) in the case of equation (1.7) appears as

$$\frac{|F(t, x, u, p, 0)|}{F_r(t, x, u, p, r)} \le \phi(|p|) \quad \text{for} \quad (t, x, u, p, r) \in \overline{Q}_T \times [-M, M] \times \mathbb{R}^2, \tag{1.8}$$

2000 *Mathematics Subject Classification.* 35K55, 35K15, 35A05.
Key words and phrases. Bernstein-Nagumo condition; gradient blow-up; a priori estimates nonlinear parabolic equation.

Submitted February 8, 2006. Published April 17, 2007.

where $\phi(\rho)$ is a nondecreasing positive function such that

$$\int^{+\infty} \frac{\rho d\rho}{\phi(\rho)} = +\infty.$$

Condition (1.8) guarantees global a priori estimate for the gradient of bounded solutions. There are examples showing that a violation of the Bernstein-Nagumo condition can imply the gradient blow-up on the boundary as well as at interior points of the domain (see [1, 2, 9, 12, 16, 21, 23]), while the solution itself remains bounded. Recently, in [22], condition (1.8) was substituted by a less restrictive one that allows an arbitrary growth of $F(t,x,u,p,0)$ with respect to p (see also [3]).

Let us recall some of the main results that were established in [22]. Suppose that the right hand side of equation (1.7) can be represented as follows

$$F(t,x,u,p,0) = f_1(t,x,u,p) + f_2(t,x,u,p), \tag{1.9}$$

where f_2 satisfies the restrictions

$$f_2(t,y,u_1,p) - f_2(t,x,u_2,p) \geq 0, \tag{1.10}$$

$$f_2(t,x,u_1,-p) - f_2(t,y,u_2,-p) \geq 0 \tag{1.11}$$

for $t \in [0,T]$, $-l \leq y < x \leq l$, $-M \leq u_1 < u_2 \leq M$, $p \in [q_0, q_1]$. For the Dirichlet boundary value problem we additionally suppose that

$$uf_2(t,x,u,p) \leq 0, \quad \text{for} \quad x \in [-l, -l + min\{\tau_0, 2l\}] \bigcup [l - min\{\tau_0, 2l\}, l], \tag{1.12}$$

for $t \in [0,T]$, $|u| \leq M$ and $p \in [-q_1, q_0] \cup [q_0, q_1]$, where τ_0, q_0, q_1 are specified below.

Concerning the function f_1 we assume that

$$|f_1(t,x,u,p)| \leq F_r(t,x,u,p,r)\psi(|p|) \tag{1.13}$$

for $(t,x) \in \overline{Q}_T$, $|u| \leq M$ and arbitrary (p,r), where $\psi(\rho) \in \mathbf{C}^1(0,+\infty)$ is a nondecreasing nonnegative function that satisfies the following condition: there exist q_0 and q_1 such that $0 < K \leq q_0 < q_1 < +\infty$ and

$$\int_{q_0}^{q_1} \frac{\rho d\rho}{\psi(\rho)} \geq \operatorname{osc}(u) \equiv \max u - \min u. \tag{1.14}$$

Here K is a Lipschitz constant of the initial function which satisfies the assumption

$$|u_0(x) - u_0(y)| \leq K|x-y|. \tag{1.15}$$

Introduce $h(\tau)$ as a solution of the following problem

$$h'' + \psi(|h'|) = 0, \quad h(0) = 0, \quad h(\tau_0) = \operatorname{osc}(u).$$

Represent the solution of $h'' + \psi(|h'|) = 0$ in parametrical form (using the standard substitution $h'(\tau) = q(h)$, $\frac{dq}{d\tau} = q\frac{dq}{dh}$):

$$h(q) = \int_q^{q_1} \frac{\rho d\rho}{\psi(\rho)}, \quad \tau(q) = \int_q^{q_1} \frac{d\rho}{\psi(\rho)}.$$

The parameter q varies in the interval $[q_0, q_1]$ and we select q_0, q_1 such that $0 < K \leq q_0 < q_1 < +\infty$, $h(q_0) = \operatorname{osc}(u)$ (this is possible due to (1.14)). Put $\tau_0 \equiv \tau(q_0)$.

If conditions (1.13)-(1.15) as well as conditions (1.10), (1.11) are fulfilled, then the gradient of a bounded solution of problem (1.1), (1.2), (1.5) is bounded by a constant depending only on K, ψ, $\operatorname{osc}(u)$. In the case of problem (1.1), (1.3), (1.5) we need additional assumption on q_0 in terms of functions σ_i (see [22, Lemma 2]).

For problem (1.1), (1.4), (1.5) assumptions (1.10)-(1.15) guarantee the gradient estimate of a bounded solution depending only on K, ψ and $\operatorname{osc}(u)$.

Note that if f_1 satisfies (1.8), then for an arbitrary Lipschitz continuous function $u_0(x)$ condition (1.14) is automatically fulfilled for any K (taking q_1 large enough and using the divergence of the integral in (1.14) in this case).

Consider conditions (1.10), (1.11). These conditions guarantee that the gradient of a bounded solution of equation (1.1) cannot blow-up in the interior of Q_T for any $T > 0$ in the case of problems (1.1), (1.2), (1.5) and (1.1), (1.3), (1.5) (see also Remark 2.3). When f_2 is independent of x, one can easily see that (1.10), (1.11) mean that $f_2(t, u, p)$ is a nonincreasing function with respect to u. Unfortunately if f_2 depends also on x and satisfies (1.10), (1.11), its behavior becomes rather complicated.

The goal of this paper is to show that under some additional assumptions the strict monotonicity of $f_2(t, x, u, p)$ in u is sufficient for nonexistence of the gradient blow-up of a bounded solution. In order to motivate these additional assumptions we will recall some facts from the theory of viscosity solutions. One can easily see that in the case where f_2 is independent of x, conditions (1.10), (1.11) reminds us one of the main assumptions (the properness, see [7]) under which the notion of viscosity solution is introduced. For example, if we assume in (1.7) that F_r is independent of u and $F(t, x, u, p, 0) = f_2(t, u, p)$ satisfies (1.10), (1.11), then

$$u_t - F_r(t, x, u_x, \lambda u_{xx})u_{xx} - f_2(t, u, u_x) = 0, \quad \lambda \in [0, 1], \tag{1.16}$$

is proper. Moreover in [8], in particular, it was shown that if $F_r = F_r(p, r)$ is locally strictly elliptic and $F(t, x, u, p, 0) = f_2(u, p)$ satisfies (1.10), (1.11), then there exists a unique continuous viscosity solution to the Dirichlet problem

$$\begin{gathered} u_t - F_r(u_x, \lambda u_{xx})u_{xx} - f_2(u, u_x) = 0, \quad \lambda \in [0, 1], \\ u(t, -l) = u(t, l) = 0, \quad u(0, x) = u_0(x), \end{gathered} \tag{1.17}$$

provided (1.17) has a sub- and supersolution satisfying initial-boundary data. Comparing this result with the results of [22], we conclude that a viscosity solution of the mentioned above problem becomes classical, if additionally f_2 satisfies (1.12) and the coefficients have sufficient smoothness. The situation becomes more complicated, when F_r and f_2 depends also on t and x. First of all we have to assume that the elliptic operator is uniformly proper. It means that f_2 is strictly decreasing in u

$$f_2(t, x, u_1, p) - f_2(t, x, u_2, p) \geq \gamma_0(u_2 - u_1) \tag{1.18}$$

for $u_2 \geq u_1$, $x \in [-l, l]$, $p \in \mathbf{R}$, for fixed $t \in [0, T]$, where γ_0 is a positive constant. The second assumption is a structure condition on the continuity of the elliptic operator in x (see [7]). Assumptions that we use in order to improve (1.10), (1.11) were inspired by these two assumptions under which the existence of a viscosity solution can be proved.

We proceed now to the statement of main results of the paper. Assume that $F(t, x, u, p, r)$ is defined for $(t, x) \in \overline{Q}_T$, $u \in [-M, M]$ and arbitrary (p, r) and is bounded on every compact set in $\overline{Q}_T \times [-M, M] \times \mathbf{R}^2$. Suppose that

$$|f_2(t, x, u, p) - f_2(t, y, u, p)| \leq K_1(t, x, y, u, p)|x - y| \tag{1.19}$$

for $t \in [0, T]$, $x, y \in [-l, l]$, $0 < x - y < \tau_0$, $|u| \leq M$, $p \in [-q_1, -q_0] \cup [q_0, q_1]$, where $K_1 \geq 0$,

$$f_2(t, x, u_1, p) - f_2(t, x, u_2, p) \geq \gamma(t, x, u_1, u_2, p)(u_2 - u_1) \tag{1.20}$$

for $t \in [0,T]$, $x \in [-l,l]$, $|u_1|,|u_2| \le M$, $u_2 \ge u_1$, $p \in [-q_1,-q_0] \cup [q_0,q_1]$, where $\gamma(t,x,u_1,u_2,p) \ge \gamma_0 > 0$. Denote by $\mathbf{V}$ the following set

$$\mathbf{V} = \{(t,x,y) \in \overline{Q}_T, 0 < x-y < \tau_0, |u_1|,|u_2| \le M, u_2 \ge u_1, p \in [-q_1,-q_0] \cup [q_0,q_1]\}.$$

Assume that

$$\max_{\mathbf{V}} \frac{K_1(t,x,y,u_1,p)}{\gamma(t,x,u_1,u_2,p)} \le C|p|^{\alpha}, \tag{1.21}$$

where $\alpha < 1$ and C is a positive constant.

Consider problem (1.1), (1.2), (1.5).

Theorem 1.1. *Let $u(t,x)$ be a classical solution of problem* (1.1), (1.2), (1.5). *Suppose that conditions* (1.6), (1.9), (1.13)-(1.15), (1.19)-(1.21) *are fulfilled. Then in $\overline{Q}_T$ the inequality*

$$|u_x(t,x)| \le C_1$$

holds, where the constant C_1 depends on $\operatorname{osc}(u)$, *ψ, C and α.*

From this theorem it follows that the gradient of a bounded solution of (1.7) cannot blow-up in the interior of Q_T for any $T > 0$ for problem (1.1), (1.2), (1.5). Analogous results we obtain for problem (1.1), (1.3), (1.5) (see Corollary 2.1). In the case of problem (1.1), (1.4), (1.5) we also need assumption (1.12) to obtain the nonexistence of the gradient blow-up on the boundary as well as in the interior of Q_T (see Corollary 2.2). Note (see Remark 2.3) that if $f_2(t,x,0,p) = 0$, then condition (1.12) is a simple consequence of the strict monotonicity of f_2 in u (see condition (1.20)).

Consider now the case when $f_2(t,x,u,p) = X(t,x)U(u)H(p)$ (special case). Suppose that

$$|f_2(t,x,u,p) - f_2(t,y,u,p)| \le K_2|U(u)||H(p)||x-y| \tag{1.22}$$

for $t \in [0,T]$, $x,y \in [-l,l]$, $0 < x-y < \tau_0$, $|u| \le M$, $p \in [-q_1,-q_0] \cup [q_0,q_1]$, where K_2 is a Lipschitz constant of $X(t,x)$ with respect to x,

$$\begin{gathered} f_2(t,x,u_1,p) - f_2(t,x,u_2,p) = X(t,x)H(p)(U(u_1) - U(u_2)) \ge \\ \gamma_1 X(t,x)H(p)(u_2 - u_1) \ge \gamma_0(u_2 - u_1) \end{gathered} \tag{1.23}$$

for $t \in [0,T]$, $x \in [-l,l]$, $|u_1|,|u_2| \le M$, $u_2 > u_1$, $p \in [-q_1,-q_0] \cup [q_0,q_1]$, where $\gamma_1, \gamma_0 > 0$ and without loss of generality we assume that $U(u(t,x))$ is a strictly decreasing function. Put $\gamma_2 = \min_{t,x \in [0,T]\times[-l,l]} |X(t,x)| > 0$.

Consider problem (1.1), (1.2), (1.5).

Theorem 1.2. . *Let $u(t,x)$ be a classical solution of problem* (1.1), (1.2), (1.5). *Suppose that conditions* (1.6), (1.9), (1.13) - (1.15), (1.22), (1.23) *are fulfilled. Then in $\overline{Q}_T$ the inequality*

$$|u_x(t,x)| \le C_4$$

holds, where the constant C_4 depends on $\operatorname{osc}(u)$, *M, ψ, K_2, γ_1 and γ_2.*

From Theorem 1.2 it follows that the gradient of a bounded solution of (1.7) cannot blow-up in the interior of Q_T for any $T > 0$ if $f_2 = X(t,x)U(u)H(p)$ is Lipschitz continuous in x and strictly decreasing in u. Concerning the special case see also Remark 3.1.

Comparing Theorem 1.1 with Theorem 1.2 one can easily see that in the case where f_2 is an arbitrary function of variables t, x, u, p, besides the Lipschitz continuity in x and strict monotonicity in u, we need to impose some additional structure conditions regarding the behavior of f_2 in p (see also Remark 2.5) .

In Section 1 we obtain the a priori estimate of the gradient of a bounded solution in the general case $f_2 = f_2(t, x, u, p)$. In Section 2 we obtain the a priori estimate of the gradient of a bounded solution in the case where $f_2 = X(t, x)U(u)H(p)$.

We remark that based on these a priori estimates one can prove the existence theorems for the initial-boundary value problems for (1.1), using the well-known fixed point theorem (see [18]). The proofs are exactly the same as in [22].

2. Gradient estimates in the general case

In this section we obtain global a priori estimates of the gradient of classical solutions for boundary value problems for (1.1), in the case where $f_2(t, x, u, p)$ is an arbitrary bounded function of variables t, x, u, p. Recall that a classical solution is a function belonging to $C^{1,2}_{t,x}(Q_T) \cap C^{0,1}_{t,x}(\bar{Q}_T)$ in the case of problem (1.1), (1.2), (1.5) or (1.1), (1.3), (1.5) and to $C^{1,2}_{t,x}(Q_T) \cap C^0(\bar{Q}_T)$ for problem (1.1), (1.4), (1.5). We use here Kruzhkov's idea of introducing a new spatial variable [13, 14] and the technique developed in [22].

Proof of Theorem 1.1. Consider equation (1.1) in the form (1.7) at two different points (t, x) and (t, y):

$$u_t = F_r(t, x, u, u_x, \lambda u_{xx})u_{xx} + F(t, x, u, u_x, 0), \quad \lambda \in [0, 1], \quad u = u(t, x), \tag{2.1}$$

$$u_t = F_r(t, y, u, u_y, \mu u_{yy})u_{yy} + F(t, y, u, u_y, 0), \quad \mu \in [0, 1], \quad u = u(t, y). \tag{2.2}$$

Introduce the function $v(t, x, y) = u(t, x) - u(t, y)$. In $\Omega = \{(t, x, y) : 0 < t < T, 0 < x - y, |x| < l, |y| < l\}$ the function $v(t, x, y)$ satisfies the equation

$$\begin{aligned} &- v_t + F_r(t, x, u(t, x), u_x(t, x), \lambda u_{xx}(t, x))v_{xx} \\ &+ F_r(t, y, u(t, y), u_y(t, y), \mu u_{yy}(t, y))v_{yy} \\ &= F(t, y, u(t, y), u_y(t, y), 0) - F(t, x, u(t, x), u_x(t, x), 0). \end{aligned} \tag{2.3}$$

Put

$F_r^{(x)} = F_r(t, x, u(t, x), u_x(t, x), \lambda u_{xx}(t, x))$, $F_r^{(y)} = F_r(t, y, u(t, y), u_y(t, y), \mu u_{yy}(t, y))$.

Define the operator

$$L(v) \equiv -v_t + F_r^{(x)}[v_{xx} + \psi(|v_x|)] + F_r^{(y)}[v_{yy} + \psi(|v_y|)].$$

From (1.9), (1.13) it follows that

$$L(v) \geq f_2(t, y, u(t, y), u_y(t, y)) - f_2(t, x, u(t, x), u_x(t, x)). \tag{2.4}$$

Let the function $h(\tau)$ be a solution of the ordinary differential equation

$$h''(\tau) + \psi(|h'(\tau)|) = 0 \tag{2.5}$$

on the interval $[0, \tau_0]$ and satisfies conditions:

$$h(0) = 0, \quad h(\tau_0) = \operatorname{osc}(u), \quad h' > 0 \quad \text{for} \quad \tau \in [0, \tau_0]. \tag{2.6}$$

Represent the solution of (2.5), (2.6) in parametrical form

$$h(q) = \int_q^{q_1} \frac{\rho d\rho}{\psi(\rho)}, \quad \tau(q) = \int_q^{q_1} \frac{d\rho}{\psi(\rho)}.$$

The parameter q varies in the interval $[q_0, q_1]$, where $K^* < q_0 < q_1 < +\infty$, $K^* = \max\{K, C^{\frac{1}{1-\alpha}}\}$ and

$$h(q_0) = \int_{q_0}^{q_1} \frac{\rho d\rho}{\psi(\rho)} = osc(u). \tag{2.7}$$

Put

$$\tau_0 \equiv \tau(q_0) = \int_{q_0}^{q_1} \frac{d\rho}{\psi(\rho)}.$$

Consider the function $w(t,x,y) = v(t,x,y) - h(x-y)$ in $P = \{(t,x,y) : 0 < t < T, 0 < x-y < \tau_0, |x| < l, |y| < l\}$. Due to the fact that $h(\tau)$ satisfies (2.5) we have $L(h(x-y)) = 0$. Hence, using (2.4) we obtain

$$\begin{aligned}\widetilde{L}(w) &\equiv L(v) - L(h) \equiv -w_t + F_r^{(x)}[w_{xx} + \alpha_1 w_x] + F_r^{(y)}[w_{yy} + \alpha_2 w_y] \\ &\geq f_2(t,y,u(t,y),u_y(t,y)) - f_2(t,x,u(t,x),u_x(t,x)).\end{aligned}$$

Where $|\alpha_i| < +\infty$, $i = 1, 2$, by virtue of the mean value theorem and of the fact that ψ is a smooth function and u is a classical solution of (1.1), (1.2), (1.5). Let $\tilde{w} = we^{-t}$, then

$$\begin{aligned}\widetilde{L}_1(\tilde{w}) &\equiv -\tilde{w}_t + F_r^{(x)}[\tilde{w}_{xx} + \alpha_1 \tilde{w}_x] + F_r^{(y)}[\tilde{w}_{yy} + \alpha_2 \tilde{w}_y] - \tilde{w} \\ &\geq e^{-t}[f_2(t,y,u(t,y),u_y(t,y)) - f_2(t,x,u(t,x),u_x(t,x))].\end{aligned} \tag{2.8}$$

Denote by Γ the parabolic boundary of P (i.e. $\Gamma = \partial P \setminus \{(t,x,y) : t = T, 0 < x-y < \tau_0, |x| < l, |y| < l\}$). Suppose that the function $\tilde{w}$ attains its positive maximum at some point $(t_1,x_1,y_1) \in \overline{P} \setminus \Gamma$. Obviously it should be $\widetilde{L}_1(\tilde{w})\big|_{(t_1,x_1,y_1)} < 0$. On the other hand, at this point we have

$$-\tilde{w} < 0, \quad \tilde{w}_x = \tilde{w}_y = 0, \quad \tilde{w}_{xx} \leq 0, \quad \tilde{w}_{yy} \leq 0, \quad -\tilde{w}_t \leq 0;$$

i.e.,

$$\begin{gathered}\tilde{w}(t_1,x_1,y_1) = e^{-t}[u(t_1,x_1) - u(t_1,y_1) - h(x_1-y_1)] > 0, \\ \tilde{w}_x(t_1,x_1,y_1) = e^{-t}[u_x(t_1,x_1) - h'(x_1-y_1)] = 0, \\ \tilde{w}_y(t_1,x_1,y_1) = e^{-t}[-u_y(t_1,y_1) + h'(x_1-y_1)] = 0\end{gathered}$$

and as a consequence

$$u(t_1,x_1) > u(t_1,y_1), \quad u_x(t_1,x_1) = u_y(t_1,y_1) = h'(x_1-y_1) > 0. \tag{2.9}$$

Represent the right-hand side of inequality (2.8) in the following way

$$\begin{aligned}&e^{-t}[f_2(t,y,u(t,y),u_y(t,y)) - f_2(t,x,u(t,x),u_x(t,x))] \\ &= e^{-t}[f_2(t,y,u(t,y),u_y(t,y)) - f_2(t,x,u(t,y),u_y(t,y)) \\ &\quad + f_2(t,x,u(t,y),u_y(t,y)) - f_2(t,x,u(t,x),u_x(t,x))],\end{aligned} \tag{2.10}$$

where we subtract and add the term $f_2(t,x,u(t,y),u_y(t,y))$. So at the maximum point (t_1,x_1,y_1), using (1.19), (1.20), (2.9), we obtain

$$\begin{aligned}&\widetilde{L}_1(\tilde{w}) \\ &\geq e^{-t_1}[f_2(t_1,y_1,u(t_1,y_1),u_y(t_1,y_1)) - f_2(t_1,x_1,u(t_1,y_1),u_y(t_1,y_1)) \\ &\quad + f_2(t_1,x_1,u(t_1,y_1),u_y(t_1,y_1)) - f_2(t_1,x_1,u(t_1,x_1),u_x(t_1,x_1))] \\ &\geq e^{-t_1}\Big[- K_1\Big(t_1,x_1,y_1,u(t_1,y_1),h'(x_1-y_1)\Big)(x_1-y_1) \\ &\quad + \gamma\Big(t_1,x_1,y_1,u(t_1,x_1),u(t_1,y_1),h'(x_1-y_1)\Big)(u(t_1,x_1) - u(t_1,y_1))\Big].\end{aligned} \tag{2.11}$$

Consider now the difference $u(t_1,x_1) - u(t_1,y_1)$. Due to the fact that

$$\tilde{w}(t_1,x_1,y_1) = e^{-t}[u(t_1,x_1) - u(t_1,y_1) - h(x_1-y_1)] > 0,$$

we have

$$u(t_1,x_1)-u(t_1,y_1)>h(x_1-y_1)=h(x_1-y_1)-h(0)=h'(\xi)(x_1-y_1)$$

for some $\xi\in[0,\tau_0]$. Thus one can rewrite the inequality (2.11) in the following way

$$\begin{aligned}\widetilde{L}_1(\tilde w)\ge e^{-t_1}\Big[&-K_1\Big(t_1,x_1,y_1,u(t_1,y_1),h'(x_1-y_1)\Big)\\&+\gamma\Big(t_1,x_1,y_1,u(t_1,x_1),u(t_1,y_1),h'(x_1-y_1)\Big)h'(\xi)\Big](x_1-y_1).\end{aligned}\tag{2.12}$$

Using now (1.21) we conclude that

$$\widetilde{L}_1(\tilde w)\ge e^{-t_1}\Big[-Ch'^{\alpha}(x_1-y_1)+h'(\xi)\Big]\gamma_0(x_1-y_1).\tag{2.13}$$

To obtain the contradiction with $\widetilde{L}_1(\tilde w)\big|_{(t_1,x_1,y_1)}<0$ we have to show that (recall that $x_1>y_1$)

$$-Ch'^{\alpha}(x_1-y_1)+h'(\xi)\ge 0.$$

Using the fact that $q_0\le h'\le q_1$, we arrive to the inequality

$$-Cq_1^{\alpha}+q_0\ge 0.$$

Thus if

$$q_0\ge Cq_1^{\alpha},\tag{2.14}$$

then $\widetilde{L}_1(\tilde w)\big|_{(t_1,x_1,y_1)}\ge 0$. Obviously, when $\alpha<1$, there exist q_0 and $q_1>q_0$ such that inequality (2.14) takes place. Solving the system

$$q_0\ge Cq_1^{\alpha},\quad q_1>q_0,$$

one can easily obtain that for $q_0>C^{\frac{1}{1-\alpha}}$ inequality (2.14) takes place for $q_0<q_1\le\left(\frac{1}{C}\right)^{\frac{1}{\alpha}}q_0^{\frac{1}{\alpha}}$. Consequently it follows that $\tilde w$ cannot attain its positive maximum in $\overline{P}\setminus\Gamma$. Note that if $\alpha\le 0$ then the validity of (2.14) does not depend on q_1.

Now let us show that $\tilde w\big|_{\Gamma}\le 0$. Consider two possible cases: $\tau_0<2l$ and $\tau_0\ge 2l$. First let $\tau_0<2l$. For $t=0$:

$$\tilde w(0,x,y)=e^{-t}(u_0(x)-u_0(y)-h(x-y))\le e^{-t}(K(x-y)-h'(\tau^*)(x-y))\le 0,$$

where $\tau^*\in[0,\tau_0]$, due to the fact that $h'\ge q_0\ge K$. Obviously $\tilde w(t,x,y)\big|_{x=y}=0$ and when $x-y=\tau_0$ we have $\tilde w=e^{-t}(u(t,x)-u(t,y)-h(\tau_0))\le 0$ due to (2.6). Denote by $Q_1=\{(t,x):0<t\le T,-l<x<-l+\tau_0,y=-l\}$, $Q_2=\{(t,y):0<t\le T,l-\tau_0<y<l,x=l\}$. Estimate the normal derivative of $\tilde w$ on Q_1 and Q_2 using boundary conditions (1.2) and the fact that $h'\ge q_0>0$

$$\begin{gathered}-\tilde w_y(t,x,-l)=e^{-t}(u_y(t,-l)-h'(x+l))=-e^{-t}h'(x+l)<0,\\ \tilde w_x(t,l,y)=e^{-t}(u_x(t,l)-h'(l-y))=-e^{-t}h'(l-y)<0.\end{gathered}$$

Thus the function $\tilde w(t,x,y)$ cannot attain its positive maximum neither on Q_1 nor on Q_2 since $-\partial/\partial y$ and $\partial/\partial x$ are here outward normal derivatives with respect to P. Consequently, $\tilde w\big|_{\Gamma}\le 0$ and hence $\tilde w(t,x,y)\le 0$ in $\overline{P}$.

The case when $\tau_0\ge 2l$ can be treated similarly. The only difference is the absence of the boundary $x-y=\tau_0$. We put $\widetilde{Q}_1=\{(t,x):0<t\le T,-l<x\le l,y=-l\}$, $\widetilde{Q}_2=\{(t,y):0<t\le T,-l<y<l,x=l\}$ (note that the line $x=l,y=-l$ belongs to $\widetilde{Q}_1$). Consequently, $\tilde w\big|_{\Gamma}\le 0$ and hence $\tilde w(t,x,y)\le 0$ in $\overline{P}$. It means that

$$u(t,x)-u(t,y)\le h(x-y)\quad\text{in}\quad\overline{P}.\tag{2.15}$$

Treating similarly the function $\tilde{v}(t,x,y) = u(t,y) - u(t,x)$ one can easily see that for $\tilde{w}_1(t,x,y) = e^{-t}(\tilde{v}(t,x,y) - h(x-y))$ we have

$$\widetilde{L}_1(\tilde{w}_1) \geq e^{-t}[f_2(t,x,u(t,x),u_x(t,x)) - f_2(t,y,u(t,y),u_y(t,y))] \quad \text{in} \quad P.$$

Suppose that $\tilde{w}_1$ attains its positive maximum at $(\tilde{t}_1, \tilde{x}_1, \tilde{y}_1) \in \overline{P} \setminus \Gamma$. Consequently it should be $\widetilde{L}_1(\tilde{w}_1)\big|_{(\tilde{t}_1,\tilde{x}_1,\tilde{y}_1)} < 0$. On the other hand, we have

$$u(\tilde{t}_1, \tilde{y}_1) > u(\tilde{t}_1, \tilde{x}_1), \quad u_x(\tilde{t}_1, \tilde{x}_1) = u_y(\tilde{t}_1, \tilde{y}_1) = -h'(\tilde{x}_1 - \tilde{y}_1) < 0.$$

Using inequalities (1.19) - (1.21) we obtain in the same way that $\widetilde{L}_1(\tilde{w}_1) \geq 0$. From this contradiction it follows that $\tilde{w}_1$ cannot attain its positive maximum in $\overline{P} \setminus \Gamma$.

Consider $\tilde{w}_1$ on Γ. One can easily see that all considerations concerning the estimate of the function $\tilde{w}$ on the boundary Γ can be done without any changes in estimate of $\tilde{w}_1$. Thus we have that

$$u(t,y) - u(t,x) \leq h(x-y) \quad \text{in} \quad \overline{P}. \tag{2.16}$$

Combining (2.16) with (2.15) we get

$$|u(t,x) - u(t,y)| \leq h(x-y) \quad \text{in } \overline{P}.$$

In view of the symmetry of the variables x, y in the same manner we examine the case $y > x$. As a result we have that for

$$0 \leq t \leq T, \quad |x| \leq l, \quad |y| \leq l, \quad 0 < |x-y| \leq \tau_0$$

the inequality

$$\Big|\frac{u(t,x) - u(t,y)}{x-y}\Big| \leq \frac{h(|x-y|) - h(0)}{|x-y|}$$

holds and as a consequence we have

$$|u_x(t,x)| \leq h'(0) = q_1 = C_1.$$

Theorem 1.1 is proved. □

Let us pass to problem (1.1), (1.3), (1.5).

Corollary 2.1. *Let $u(t,x)$ be a classical solution of* (1.1), (1.3), (1.5) *and all conditions of Theorem* 1.1 *are fulfilled. Then in $\overline{Q}_T$ the inequality*

$$|u_x(t,x)| \leq C_2$$

holds, where the constant C_2 depends only on $\operatorname{osc}(u)$, *N_1, N_2, ψ, C, α, where $N_i = \sup|\sigma_i|$ (the supremum is taken over the set $[0,T] \times [-M,M]$).*

Proof. The proof of this corollary differs from the proof of Theorem 1.1 only in the selection of q_0 and in analyzing of the behavior of $\tilde{w}(t,x,y)$ on bounds Q_1 ($\widetilde{Q}_1$) and Q_2 ($\widetilde{Q}_2$). We select the quantity q_0 so that

$$q_0 > max\left\{C^{\frac{1}{1-\alpha}}, K, N_1, N_2\right\}. \tag{2.17}$$

and follow the proof of [22, Lemma 2, and Corollary 1.2]. Corollary 2.1 is proved. □

Consider now problem (1.1), (1.4), (1.5).

Corollary 2.2. . *Let $u(t,x)$ be a classical solution of* (1.1), (1.4), (1.5) *and all conditions of Theorem 1.1 are fulfilled. Suppose in addition that condition* (1.12) *is fulfilled and $u_0(\pm l) = 0$. Then in $\overline{Q}_T$ the following inequality*

$$|u_x(t,x)| \le C_3$$

holds, where the constant C_3 depends only on $\operatorname{osc}(u)$ *and ψ, C, α.*

Proof. The proof of this corollary differs from the proof of Theorem 1.1 only in analyzing of the behavior of $w(t,x,y)$ on Q_1 ($\tilde{Q}_1$) and Q_2 ($\tilde{Q}_2$) (see the proof of [22, Lemma 3 and Corollary 1.3]). Corollary 2.2 is proved. □

Remark 2.3. One can easily see that from Theorem 1.1 and Corollary 2.1 it immediately follows that conditions (1.19)-(1.21) are sufficient for the nonexistence of the gradient blow-up of a bounded solution in the interior of Q_T for problems (1.1), (1.2), (1.5) and (1.1), (1.3), (1.5). Concerning problem (1.1), (1.4), (1.5), one has to impose condition (1.12) supplementary to (1.19)-(1.21) in order to obtain the nonexistence of the gradient blow-up of a bounded solution in $\overline{Q}_T$. Obviously, if we suppose that $f_2(t,x,0,p) = 0$, then condition (1.12) is a simple consequence of the strict monotonicity of f_2 in u (see condition (1.20)) Thus if $f_2(t,x,0,p) = 0$, then the nonexistence of the gradient blow-up in $\overline{Q}_T$ for problem (1.1), (1.4), (1.5) follows from (1.19)-(1.21) and we do not need condition (1.12).

Remark 2.4. Note that if f_1 satisfies Bernstein-Nagumo condition (1.8) then for every q_0 there always exists $q_1 > q_0$ such that (1.14) takes place. Thus if $\alpha \le 0$ in (1.21), then inequality (2.14) does not depend on q_1 and we can always construct $h(\tau)$ that satisfies (2.6) (due to the divergence of the integral in (1.14) in this case).

Remark 2.5. Put $f_1 = 0$. In this case $\psi(p) = 0$ and $h'' = 0$. From (2.6) it follows that $h = \frac{osc\,u}{\tau_0}\tau$, $h' = \frac{osc\,u}{\tau_0}$. Thus we have that $q_0 = q_1 = \frac{osc\,u}{\tau_0}$. Consequently (2.14) takes the form

$$q_0 \ge C q_0^{\alpha} \tag{2.18}$$

that is fulfilled for $q_0 \ge C^{\frac{1}{1-\alpha}}$ and $\alpha < 1$. Obviously, when $\alpha = 1$, then in order to obtain the gradient a priori estimate one has to suppose that $C \le 1$. One can easily check that even in the case where $f_1 = 0$ (2.18) holds for $q_0 \le C^{\frac{1}{1-\alpha}}$ if $\alpha > 1$. Thus we can prove Theorem 1.1 with $\alpha > 1$ only for $K \le C^{\frac{1}{1-\alpha}}$. Note that condition (1.21) can be generalized in the following way

$$\max_{\mathbf{V}} \frac{K_1(t,x,y,u_1,p)}{\gamma(t,x,u_1,u_2,p)} \le \Psi(|p|),$$

where $\Psi(\rho)$ is a nondecreasing positive function. As a consequence condition (2.14) appears as

$$q_0 \ge \Psi(q_1).$$

Remark 2.6. Let us give some simple examples of functions that satisfy (1.19)-(1.21) and at the same time do not satisfy (1.10), (1.11). Easy calculations show that, for example, the functions

$$f_2(t,x,u,p) = x|p|^{\mu} - u|p|^{\nu}, \quad \forall \quad \mu,\nu \quad \text{such that } \mu - \nu \le 1,$$

$$f_2(t,x,u,p) = -ue^{(x+c)p^{\alpha}}, \quad \alpha < 1,\ c > l,$$

where α is such that p^α is defined, satisfy (1.19)-(1.21) and do not satisfy (1.10), (1.11). Moreover, in the case when $f_2 = -ue^{(x+c)p^\alpha}$, $\alpha < 1$, $c > l$, problem (1.4), (1.5) as well as problems (1.2), (1.5) and (1.3), (1.5), for example, for the equation

$$u_t = a(t,x,u,u_x)u_{xx} + f_2, \quad a > 0, \tag{2.19}$$

have a global classical solution for any Lipschitz continuous initial data. When $f_2(t,x,u,p) = x|p|^\mu - u|p|^\nu$, $\mu - \nu \le 1$, problems (2.19), (1.2), (1.5) and (2.19), (1.3), (1.5) have a global classical solution for any Lipschitz continuous initial data.

3. Gradient estimates in the special case

In this section we obtain global a priori estimates of the gradient of classical solutions for boundary-value problems for equation (1.1) where $f_2 = X(t,x)U(u)H(p)$. One can easily see that in this case conditions (1.19), (1.20) take the form

$$|f_2(t,x,u,p) - f_2(t,y,u,p)| \le K_2|U(u)||H(p)||x-y|$$

for $t \in [0,T]$, $x,y \in [-l,l]$, $0 < x-y < \tau_0$, $|u| \le M$, $p \in [-q_1,-q_0] \cup [q_0,q_1]$, where K_2 is the Lipschitz constant of $X(t,x)$ with respect to x,

$$\begin{aligned} f_2(t,x,u_1,p) - f_2(t,x,u_2,p) &= X(t,x)H(p)(U(u_1) - U(u_2)) \\ &\ge \gamma_1 X(t,x)H(p)(u_2-u_1) \ge \gamma_0(u_2-u_1) \end{aligned}$$

for $t \in [0,T]$, $x \in [-l,l]$, $|u_1|,|u_2| \le M$, $u_2 > u_1$, $p \in [-q_1,-q_0] \cup [q_0,q_1]$, where $\gamma_1, \gamma_0 > 0$. Recall that without loss of generality we assume that $U(u(t,x))$ is a strictly decreasing function. The last assumption means that $X(t,x)H(p) > 0$. Put $\gamma_2 = \min_{t,x \in [0,T]\times[-l,l]} |X(t,x)| > 0$. Obviously if $f_2(t,x,u,p) = X(t,x)U(u)H(p)$, then condition (1.21) is fulfilled with $C = \frac{K_2|U(-M)|}{\gamma_1\gamma_2}$, $\alpha = 0$ and as a consequence can be dropped.

Proof of Theorem 1.2. The proof differs from the proof of the previous theorem only in the choice of the quantity q_0. In the general case q_0 depends on q_1 and α (see (2.14)). We will show now that if $f_2(t,x,u,p) = X(t,x)U(u)H(p)$, then q_0 is independent of q_1 and α. Following the proof of Theorem 1.1 we arrive to (2.12) that appears in the form

$$\begin{aligned} \widetilde{L}_1(\tilde{w}) \ge e^{-t_1}[&-K_2|U(u(t_1,y_1))||H(h'(x_1-y_1))|(x_1-y_1) \\ &+ \gamma_1 X(t_1,x_1)H(h'(x_1-y_1))(u(t_1,x_1)) - u(t_1,y_1))]. \end{aligned} \tag{3.1}$$

Due to (1.23) we have that $\gamma_1 X(t,x)H(p) > 0$ and consequently

$$\gamma_1 X(t,x)H(p) = \gamma_1|X(t,x)||H(p)|. \tag{3.2}$$

Thus from (3.1), (3.2) we obtain that (recall that $U(u(t,x))$ is a strictly decreasing function and $|u| \le M$)

$$\begin{aligned} \widetilde{L}_1(\tilde{w}) \ge e^{-t_1}|H(h'(x_1-y_1))|[&-K_2|U(-M)|(x_1-y_1) \\ &+ \gamma_1|X(t_1,x_1)|(u(t_1,x_1) - u(t_1,y_1))]. \end{aligned} \tag{3.3}$$

Due to the fact that $u(t_1,x_1)) - u(t_1,y_1) \ge h'(\xi)(x_1-y_1)$, from (3.3) it follows (recall that $h'(\xi) \ge q_0$)

$$\begin{aligned} \widetilde{L}_1(\tilde{w}) &\ge e^{-t_1}|H(h'(x_1-y_1))|(x_1-y_1)[-K_2|U(-M)| + \gamma_1|X(t_1,x_1)|h'(\xi)] \\ &\ge e^{-t_1}|H(h'(x_1-y_1))|(x_1-y_1)[-K_2|U(-M)| + \gamma_1|X(t_1,x_1)|q_0]. \end{aligned} \tag{3.4}$$

To obtain the contradiction with $\widetilde{L}_1(\tilde{w})\big|_{(t_1,x_1,y_1)} < 0$ we have to show that

$$-K_2|U(-M)| + \gamma_1|X(t_1,x_1)|q_0 \geq 0. \tag{3.5}$$

Obviously this is the case when

$$q_0 \geq \frac{K_2|U(-M)|}{\gamma_1\gamma_2}. \tag{3.6}$$

From this contradiction we conclude that $\tilde{w}$ cannot attain its positive maximum in $\bar{P} \setminus \Gamma$. Similarly one can prove that $\tilde{w}_1 = e^{-t}(u(t,y) - u(t,x) - h(x-y))$ cannot attain its positive maximum in $\bar{P} \setminus \Gamma$. Further without any changes we follow the proof of Theorem 1.1. Note here that in (2.7) one has to suppose that $q_0 \geq \max\{K, \frac{K_2|U(-M)|}{\gamma_1\gamma_2}\}$. Theorem 1.2 is proved. □

Remark 3.1. In the case of problems (1.1), (1.3), (1.5) and (1.1), (1.4), (1.5) one can easily formulate results that are similar to those of Corollaries 2.1 and 2.2. The proofs of these results is an easy compilation of the proofs of Theorems 1.1, 1.2, Corollary 2.1 and of Theorems 1.1, 1.2, Corollary 2.2 respectively.

Remark 3.2. From the mentioned above it follows that the strict monotonicity of $f_2(t,x,u,p) = X(t,x)U(u)H(p)$ in u, coupled with the Lipschitz continuity in x, guarantee the nonexistence of the gradient blow-up of a bounded solution in the interior of Q_T for problems (1.1), (1.2), (1.5) and (1.1), (1.3), (1.5). If additionally $f_2(t,x,0,p) = 0$ then the strict monotonicity of $f_2(t,x,u,p)$ in u, coupled with the Lipschitz continuity in x, guarantee the nonexistence of the gradient blow-up of a bounded solution in $\overline{Q}_T$ for problem (1.1), (1.4), (1.5). Concerning problem (1.1), (1.4), (1.5) in the case where $f_2(t,x,0,p) \neq 0$, one has to impose condition (1.12) supplementary to (1.22), (1.23) in order to obtain the nonexistence of the gradient blow-up of a bounded solution in $\overline{Q}_T$.

Remark 3.3. Let us give some simple examples of functions that satisfy (1.22), (1.23) and at the same time do not satisfy (1.10), (1.11). Easy calculations show that, for example, the functions

$$f_2(t,x,u,p) = g(x)up^{2m}, \quad f_2(t,x,u,p) = g(x)u^3|p|^\nu, \quad f_2(t,x,u,p) = g(x)ue^p,$$

where $g(x) < 0$ is an arbitrary Lipschitz continuous function, $m > 1$ is an integer number, $\nu > 2$ is a real number, satisfy (1.22), (1.23) and do not satisfy (1.10), (1.11). Moreover, in the case where f_2 has one of the above representations, problems (2.19), (1.2), (1.5), (2.19), (1.3), (1.5) and (2.19), (1.4), (1.5) have a global classical solution for any Lipschitz continuous initial data.

References

[1] S. Angenent, M. Fila, *Interior gradient blow-up in a semilinear parabolic equation*, Diff. and Integral Equations, 9 (1996), N.5, 865-877.

[2] K. Asai, N. Ishimura, *On the interior derivative blow-up for the curvature evolution of capillary surfaces*, Proc. Amer. Math. Soc., 126 (1998), N3, 835 - 840.

[3] J. P. Bartier, Ph. Souplet, *Gradient bounds for solutions of semilinear parabolic equations without Bernstein's quadratic condition*, Comp. Rend. Math., 338(7) (2004), 533 - 538.

[4] S. N. Bernstein, *Sur les equations du calcul des variations*, Ann. Sci. Ecole Norm. Sup., 29 (1912), 431 - 485.

[5] S. N. Bernstein, *On the equations of calculus of variation*, Sobr. soch. M.: Izd-vo AN SSSR (1960) T.3, p. 191-242 [Russian].

[6] A. Constantin, J. Escher, *Global solutions for quasilinear parabolic problems, J. Evol. Equ.*, 2 (2002), n. 1, 97 - 111.
[7] M. G. Crandall, H. Ishii and P.L.Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. 27(1992), 1-67.
[8] B. Kawohl, N. Kutev, *Comparison principle and Lipschitz regularity for viscosity solutions of some classes of nonlinear partial differential equations*, Funk. Ekvac., 43 (2000), 241 - 253.
[9] T. Dlotko, *Examples of parabolic problems with blowing-up derivatives*, J. Math. Anl. Appl. 154 (1991) 226 - 237.
[10] M. Fila, G. Lieberman, *Derivative blow-up and beyond for quasilinear parabolic equations*, Diff. Integ. Equations, 7 (1994), No 3/4, 811-822.
[11] A. F. Filippov, *On condition for the existence of a solution of a quasilinear parabolic equation*, Dokl. Akad. Nauk SSSR 141 (1961), 568-570(Russian). English trasl. Soviet Math. Dokl. 2(1961), 1517-1519.
[12] Y. Giga, *Interior blow-up for quasilinear parabolic equations*, Discrete and Continuous Dynamical Systems 1 (1995), 449-461.
[13] S. N. Kruzhkov, *Quasilinear parabolic equations and systems with two independent variables*, Trudy Sem. Petrovsk. 5(1979), 217 - 272 [Russian. English translation in "Topics in Modern Math.", Consultant Bureau, New York, 1985].
[14] S.N. Kruzhkov, *Nonlinear parabolic equations in two independent variables*, Transact. Moscow Math. Soc. 16 (1968), 355 - 373.
[15] N. V. Krylov, *Nonlinear elliptic and parabolic equations of second order*, Mathematics and its Applications (Soviet Series), 7. D. Reidel Publishing Co., Dordrecht, 1987, 462.
[16] N. Kutev, *Global solvability and boundary gradient blow up for one dimensional parabolic equations*, Progress in PDE: Elliptic and Parabolic problems (C.Bandle, et al., eds.), Longman, 1992, 176 - 181.
[17] O. A. Ladyzhenskaja, V. A. Solonnikov, N. N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, Amer. Math. Soc. Transl. (2), vol. 23, 1968.
[18] G. M. Lieberman, *Second Order Parabolic Equations*, World Scientific Publishing Co., Inc., River Edge, N J, 1996.
[19] M. Nagumo, *Uber die Differentialgleichung* $y'' = f(t, y, y')$, Proc. Phys. Math. Soc. Japan 19 (1937) 861-866.
[20] J. Serrin, *The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables*, Philos. Trans. Roy. Soc. London Ser. A 264, 413-496 (1969).
[21] Ph. Souplet, *Gradient blow-up for multidimensional nonlinear parabolic equation with general boundary conditions*, Dif. Int. Equat. 15 (2002), N2, 237 - 256.
[22] Al. S. Tersenov, Ar. S. Tersenov, *On the Bernstein-Nagumo's condition in the theory of nonlinear parabolic equations*, J. Reine Angew. Math. 572 (2004), 197 - 217.
[23] M.P. Vishnevskii, T. I. Zelenyak, M. M. Lavrentiev, *Behavior of solutions to parabolic equations for large time value*, Sib. Mat. J. 36 (1995), N.3, 435 - 453.

Aris S. Tersenov
Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus
Tel.:+357 22892560, Fax:+357 22892550
E-mail address: aterseno@ucy.ac.cy