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INTERVAL OSCILLATION OF SECOND-ORDER EMDEN-FOWLER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. Employing Riccati techniques and the integral averaging method, we establish interval oscillation criteria for the second-order Emden-Fowler neutral delay differential equation

$$[|x'(t)|^{\gamma-1}x'(t)]' + q_1(t)|y(t-\sigma)|^{\alpha-1}y(t-\sigma) + q_2(t)|y(t-\sigma)|^{\beta-1}y(t-\sigma) = 0,$$

where $t \geq t_0$ and $x(t) = y(t) + p(t)y(t-\tau)$. The criteria obtained here are different from most known criteria in the sense that they are based on information only on a sequence of subintervals of $[t_0,\infty)$, rather than on the whole half-line. In particular, two interesting examples that illustrate the importance of our results are included.

1. Introduction

Consider the second-order Emden-Fowler neutral delay differential equation

$$[|x'(t)|^{\gamma-1}x'(t)]' + q_1(t)|y(t-\sigma)|^{\alpha-1}y(t-\sigma) + q_2(t)|y(t-\sigma)|^{\beta-1}y(t-\sigma) = 0, \quad (1.1)$$

where $t \geq t_0$ and $x(t) = y(t) + p(t)y(t-\tau)$. In what follows we assume that

- (A1) τ and σ are nonnegative constants, α, β and γ are positive constants with $0 < \alpha < \gamma < \beta$;
- (A2) $q_1, q_2 \in \mathbf{C}([t_0,\infty),\mathbb{R}^+)$, $\mathbb{R}^+ = (0,\infty)$;
- (A3) $p \in \mathbf{C}([t_0,\infty),\mathbb{R})$, and $-1 < p_0 \leq p(t) \leq 1$, p_0 is a constant.

For any $\varphi \in \mathbf{C}([t_0-\theta,t_0],\mathbb{R})$, $\theta = \max\{\tau,\sigma\}$, (1.1) has a solution $y(t)$ extendable on $[t_0,\infty)$ satisfying the initial condition $y(t) \equiv \varphi(t)$ for $[t_0-\theta,t_0]$; see, e.g., Hale [6]. Our attention is restricted to those solutions $y = y(t)$ of (1.1) which exist on some half-line $[t_y,\infty)$ with $\sup\{|y(t)| : t \geq T\} > 0$ for any $T \geq t_y$, and satisfy (1.1). As usual, a nontrivial solution of (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Finally, (1.1) is called oscillatory if all its solutions are oscillatory. We say that (1.1) satisfies the superlinear condition if $q_1(t) \equiv 0$ and it satisfies the sublinear condition if $q_2(t) \equiv 0$.

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We note that second order neutral delay differential equation are used in many fields such as vibrating masses attached to an elastic bar and some variational problems, etc., see Hale [6].

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and/or nonoscillation of second order linear and nonlinear neutral delay differential equations (see, for example, the monographs [1, 3, 5] and the references therein). Recently, the results of Atkinson [2] and Belohorec [4] for the Emden-Fowler equation

$$y''(t) + q(t)|y(t)|^{\gamma-1}y(t) = 0, \quad q \in \mathbf{C}([t_0,\infty),\mathbb{R}) \text{ and } \gamma > 0 \tag{1.2}$$

have been extended to the second order neutral delay differential equation

$$[y(t) + p(t)y(t-\tau)]'' + q(t)f((y-\sigma)) = 0 \tag{1.3}$$

by Wong [13] under the assumption that the nonlinear function f satisfies the sublinear condition

$$0 < \int_{0^+}^{\varepsilon} \frac{du}{f(u)}, \quad \int_{0^-}^{-\varepsilon} \frac{du}{f(u)} < \infty \quad \text{for all } \varepsilon > 0,$$

as well as the superlinear condition

$$0 < \int_{\varepsilon}^{\infty} \frac{du}{f(u)}, \quad \int_{-\varepsilon}^{-\infty} \frac{du}{f(u)} < \infty \quad \text{for all } \varepsilon > 0.$$

Also it will be of great interest to find some oscillation criteria for special case for (1.3), even for the Emden-Fowler neutral delay differential equation

$$[y(t) + p(t)y(t-\tau)]'' + q(t)|y(t-\sigma)|^{\gamma-1}y(t-\sigma) = 0, \quad \gamma > 0. \tag{1.4}$$

This problem was posed by [13, Remark d]. As an positive answer to it, Saker [10], Saker and Manojlovič [11], and Xu and Liu [14] have given some oscillation criteria for (1.1), (1.3) and (1.4). As we know, the results obtained in [10, 11, 13, 14] involve the integral of the functions q, q_1, q_2 and hence require the information of those functions on the the entire half-linear $[t_0,\infty)$. As pointed out in Kong [8], oscillation is an interval property, that is, it is more reasonable to investigate solutions on an infinite set of bounded intervals. Therefore, the problem is to find oscillation criteria which use only the information about the involved functions on these intervals; outside of these intervals the behavior of the functions is irrelevant. Such type of criteria are referred to as interval oscillation criteria. The first beautiful interval criteria in this direction was due to Kong [8], who gave some interval criteria for the oscillation of the second order linear ordinary differentia equation

$$(r(t)y'(t))'(t) + q(t)y(t) = 0. \tag{1.5}$$

Recently, Yang et al [15] extended Kong-type interval criteria to certain neutral differential equations.

Motivated by the ideas of Kong [8] and Philos [9], in this paper, by using Riccati technique and the integral averaging method, we will establish some interval oscillation criteria for (1.1), that is, criteria given by the behavior of (1.1) only on a sequence of subintervals of $[t_0,\infty)$ (see Theorems 2.2–2.6 for details) rather than the whole half-line. Our theorems essentially improve some known results in [10, 14]. In particular, two interesting examples that illustrate the importance of our results are also included.

2. Main results

In this section, we shall establish Kong-type interval oscillation criteria for (1.1) under the cases when $0 \le p(t) \le 1$ and $-1 < p_0 \le p(t) \le 0$. It will be convenient to make the following notations in the remainder of this paper. Define

$$\mu = \min\big\{\frac{\beta-\alpha}{\beta-\gamma}, \frac{\beta-\alpha}{\gamma-\alpha}\big\}, \quad k = \frac{1}{(1+\gamma)^{1+\gamma}}$$
$$Q_1(t) = \mu[1-p(t-\sigma)]^\gamma [q_1^{\beta-\gamma}(t) q_2^{\gamma-\alpha}(t)]^{1/(\beta-\alpha)},$$
$$Q_2(t) = \mu[q_1^{\beta-\gamma}(t) q_2^{\gamma-\alpha}(t)]^{1/(\beta-\alpha)}.$$

In the sequel, we say the a function $H = H(t,s)$ belongs to a function class $\mathcal{H}$, denoted by $H \in \mathcal{H}$, if $H \in \mathbf{C}(D, [0,\infty))$, where $D = \{(t,s) : -\infty < s \le t < \infty\}$, and H satisfies

(H1) $H(t,t) = 0$, $H(t,s) > 0$ for $t > s$;

(H2) H has partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that
$$\frac{\partial H}{\partial t}(t,s) = h_1(t,s)\sqrt{H(t,s)} \quad \text{and} \quad \frac{\partial H}{\partial s}(t,s) = -h_2(t,s)\sqrt{H(t,s)},$$
where $h_1, h_2 \in L_{loc}(D,\mathbb{R})$.

The following Lemma will be useful for establishing oscillation criteria for (1.1) whose proof can be found in [12].

Lemma 2.1. *Let A_0, A_1, $A_2 \in \mathbf{C}([t_0,\infty),\mathbb{R})$ with $A_2 > 0$, and $w \in \mathbf{C}^1([t_0,\infty),\mathbb{R})$. If there exist interval $(a,b) \subset [t_0,\infty)$ such that*
$$w'(s) \le -A_0(s) + A_1(s)w(s) - A_2(s)|w(s)|^{(\gamma+1)/\gamma}, \quad s \in (a,b), \tag{2.1}$$
then for any $c \in (a,b)$,
$$\begin{aligned} &\frac{1}{H(c,a)}\int_a^c \Big[H(s,a)A_0(s) - \frac{k\,\gamma^\gamma}{(A_2(s))^\gamma}|\phi_1(s,a)|^{\gamma+1}\Big]ds \\ &+ \frac{1}{H(b,c)}\int_c^b \Big[H(b,s)A_0(s) - \frac{k\,\gamma^\gamma}{(A_2(s))^\gamma}|\phi_2(b,s)|^{\gamma+1}\Big]ds \le 0 \end{aligned} \tag{2.2}$$
for every $H \in \mathcal{H}$, where
$$\phi_1(s,a) = \frac{h_1(s,a)\sqrt{H(s,a)} + A_1(s)H(s,a)}{(H(s,a))^{\gamma/(\gamma+1)}},$$
$$\phi_2(b,s) = \frac{-h_2(b,s)\sqrt{H(b,s)} + A_1(s)H(b,s)}{(H(b,s))^{\gamma/(\gamma+1)}}.$$

Theorem 2.2. *Suppose that there exist interval $(a,b) \subset [t_0,\infty)$, constant $c \in (a,b)$, and functions $H \in \mathcal{H}$, $\rho \in \mathbf{C}^1([t_0,\infty),\mathbb{R}^+)$, such that one of the following two conditions is satisfied:*

(C1) *$0 \le p(t) \le 1$, and*
$$\begin{aligned} &\frac{1}{H(c,a)}\int_a^c \rho(s)\big[H(s,a)Q_1(s) - k|\phi_1(s,a)|^{\gamma+1}\big]ds \\ &+ \frac{1}{H(b,c)}\int_c^b \rho(s)\big[H(b,s)Q_1(s) - k|\phi_2(b,s)|^{\gamma+1}\big]ds > 0; \end{aligned} \tag{2.3}$$

(C2) $-1 < p_0 \le p(t) \le 0$, *and*

$$\begin{aligned} &\frac{1}{H(c,a)}\int_a^c \rho(s)\Big[H(s,a)Q_2(s) - k|\phi_1(s,a)|^{\gamma+1}\Big]ds \\ &+ \frac{1}{H(b,c)}\int_c^b \rho(s)\Big[H(b,s)Q_2(s) - k|\phi_2(b,s)|^{\gamma+1}\Big]ds > 0, \end{aligned} \tag{2.4}$$

where

$$\phi_1(s,a) = \frac{h_1(s,a)\sqrt{H(s,a)} + (\rho'(s)/\rho(s))H(s,a)}{(H(s,a))^{\gamma/(\gamma+1)}},$$

$$\phi_2(b,s) = \frac{-h_2(b,s)\sqrt{H(b,s)} + (\rho'(s)/\rho(s))H(b,s)}{(H(b,s))^{\gamma/(\gamma+1)}}.$$

Then every solution of (1.1) *has at least one zero in* (a,b) .

Proof. Case (C1): Let $y(t)$ be a nonoscillatory solution of (1.1). Without loss of generality we may assume that $y(t) > 0$ for $t \ge t_0$. (The case of $y(t) < 0$ can be considered similarly). Furthermore, we may suppose that exists a $t_1 \ge t_0$ such that

$$y(t) > 0, \quad y(t-\tau) > 0, \quad y(t-\sigma) > 0 \quad \text{for } t \ge t_1. \tag{2.5}$$

As in [15, Lemma 1 (1)], for some $T_0 \ge t_1 + \tau + \sigma$, we have immediately that

$$x(t) > 0, \quad x'(t) > 0, \quad x''(t) > 0 \quad \text{for } t \ge T_0 - \tau - \sigma. \tag{2.6}$$

Using these inequalities and noting that $x(t) \ge y(t)$, we obtain

$$y(t) = x(t) - p(t)x(t-\tau) \ge [1-p(t)]x(t).$$

Thus, for all $t \ge T_0$,

$$y(t-\sigma) \ge [1-p(t-\sigma)]x(t-\sigma).$$

Then (1.1) implies that for $t \ge T_0$,

$$[|x'(t)|^{\gamma-1}x'(t)]' + q_1(t)[1-p(t-\sigma)]^{\alpha}x^{\alpha}(t-\sigma) + q_2(t)[1-p(t-\sigma)]^{\beta}x^{\beta}(t-\sigma) \le 0. \tag{2.7}$$

Define

$$w(t) = \rho(t)\frac{(x'(t))^{\gamma}}{x^{\gamma}(t-\sigma)}. \tag{2.8}$$

Differentiating $w(t)$, and using (2.7), we get

$$\begin{aligned} w'(t) \le{}& -\rho(t)\{q_1(t)[1-p(t-\sigma)]^{\alpha}x^{\alpha-\gamma}(t-\sigma) \\ &+ q_2(t)[1-p(t-\sigma)]^{\beta}x^{\beta-\alpha}(t-\sigma)\} + \frac{\rho'(t)}{\rho(t)}w(t) - \gamma\rho(t)\Big(\frac{x'(t)}{x(t-\sigma)}\Big)^{\gamma+1}, \end{aligned} \tag{2.9}$$

since $x'(t) < x'(t-\sigma)$. By Young's inequality [7, Theorem 61], we have

$$\begin{aligned} &\frac{\beta-\gamma}{\beta-\alpha}q_1(t)[1-p(t-\sigma)]^{\alpha}x^{\alpha-\gamma}(t-\sigma) + \frac{\gamma-\alpha}{\beta-\alpha}q_2(t)[1-p(t-\sigma)]^{\beta}x^{\beta-\alpha}(t-\sigma) \\ &\ge [1-p(t-\sigma)]^{\gamma}[q_1^{\beta-\gamma}(t)q_2^{\gamma-\alpha}(t)]^{1/(\beta-\alpha)}, \end{aligned}$$

and consequently,

$$q_1(t)[1-p(t-\sigma)]^{\alpha}x^{\alpha-\gamma}(t-\sigma) + q_2(t)[1-p(t-\sigma)]^{\beta}x^{\beta-\alpha}(t-\sigma) \ge Q_1(t), \tag{2.10}$$

Combining this inequality with (2.9), we get

$$w'(t) \le -\rho(t)Q_1(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\gamma}{\rho^{1/\gamma}(t)}|w(t)|^{(\gamma+1)/\gamma} \quad \text{for } t \ge T_0. \tag{2.11}$$

Comparing the above inequality and (2.1), we find that

$$A_0(t) = \rho(t)Q_1(t), \quad A_1(t) = \frac{\rho'(t)}{\rho(t)}, \quad A_2(t) = \frac{\gamma}{\rho^{1/\gamma}(t)}.$$

Applying Lemma 2.1 to (2.11), we see that inequality (2.3) fails to hold, hence $y(t)$ has at least one zero in (a, b).

Case (C2): Let $y(t)$ be a nonoscillatory solution of (1.1). Without loss of generality we may assume that $y(t) > 0$ for $t \ge t_0$. Furthermore, as in [15, Lemma 1(2)], we suppose that there exists a $t_1 > t_0$ such that (2.5) holds. Then, for some $T_0 \ge t_1 + \tau + \sigma$, we still have (2.6) holds for $t \ge T_0$. Note that $y(t-\sigma) \ge x(t-\sigma)$ for $t \ge T_0$, (1.1) changes into

$$[|x'(t)|^{\gamma-1}x'(t)]' + q_1(t)x^{\alpha}(t-\sigma) + q_2(t)x^{\beta}(t-\sigma) \le 0, \quad t \ge T_0. \tag{2.12}$$

Consider the function $w(t)$ also defined by (2.8), as in the proof of (2.11), we obtain

$$w'(t) \le -\rho(t)Q_2(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\gamma}{\rho^{1/\gamma}(t)}|w(t)|^{(\gamma+1)/\gamma}. \tag{2.13}$$

The rest of proof is similar to that of case $(C1)$ and hence is omitted. □

If the conditions of Theorem 2.2 hold for a sequence $\{(a_n, b_n)\}$ of intervals such that $\lim_{n\to\infty} a_n = \infty$, then we may conclude that (1.1) is oscillatory. That is, the following theorem is established.

Theorem 2.3. *For each $T \ge t_0$, if there exist $H \in \mathcal{H}$, $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$, and constants a, b, $c \in \mathbb{R}$ with $T \le a < c < b$, such that the conditions of Theorem 2.2 are satisfied, then* (1.1) *is oscillatory.*

Theorem 2.4. *Suppose that there exist $H \in \mathcal{H}$, $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$, and for each $\tau \ge t_0$, such that one of the following conditions is satisfied:*

(C3) $0 \le p(t) \le 1$, *and*

$$\limsup_{t\to\infty} \int_\tau^t \rho(s)\Big[H(s,\tau)Q_1(s) - k|\phi_1(s,\tau)|^{\gamma+1}\Big]ds > 0, \tag{2.14}$$

$$\limsup_{t\to\infty} \int_\tau^t \rho(s)\Big[H(t,s)Q_1(s) - k|\phi_2(t,s)|^{\gamma+1}\Big]ds > 0; \tag{2.15}$$

(C4) $-1 < p_0 \le p(t) \le 0$, *and*

$$\limsup_{t\to\infty} \int_\tau^t \rho(s)\Big[H(s,\tau)Q_2(s) - k|\phi_1(s,\tau)|^{\gamma+1}\Big]ds > 0, \tag{2.16}$$

$$\limsup_{t\to\infty} \int_\tau^t \rho(s)\Big[H(t,s)Q_2(s) - k|\phi_2(t,s)|^{\gamma+1}\Big]ds > 0, \tag{2.17}$$

where ϕ_1 and ϕ_2 are defined in Theorem 2.2.

Then (1.1) *is oscillatory.*

Proof. We only prove case (C3). The proof of case (C4) is similar. For any $T \ge t_0$, let $a = T$. In (2.14) we choose $\tau = a$, then there exists $c > a$ such that

$$\int_a^c \rho(s)\Big[H(s,a)Q_1(s) - k|\phi_1(s,a)|^{\gamma+1}\Big]ds > 0. \tag{2.18}$$

In (2.15) we choose $\tau = c$. Then there exists $b > c$ such that

$$\int_c^b \rho(s)\Big[H(b,s)Q_1(s) - k|\phi_2(b,s)|^{\gamma+1}\Big]ds > 0. \tag{2.19}$$

From this inequality and (2.18), we obtain (2.3). The conclusion thus comes from Theorem 2.3 (C1). The proof is complete. □

With an appropriate choice of functions H, one can derive from Theorem 2.3 a number of oscillation criteria for (1.1). For the case where $H := H(t-s) \in \mathcal{H}$, we have that $h_1(t-s) \equiv h_2(t-s)$ and denote them by $h(t-s)$. The subclass of $\mathcal{H}$ containing such $H(t-s)$ is denoted by $\mathcal{H}_0$. Applying Theorem 2.3 to $\mathcal{H}_0$, we get the following results.

Theorem 2.5. *For each $T \ge t_0$, if there exist $H \in \mathcal{H}_0$, $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$, and constants $a, c \in \mathbb{R}$ with $T \le a < c$, such that one of the following conditions is satisfied:*

(C5) $0 \le p(t) \le 1$, *and*

$$\int_a^c H(s-a)[\rho(s)Q_1(s) + \rho(2c-s)Q_1(2c-s)]ds > k\,\Theta(a,c); \tag{2.20}$$

(C6) $-1 < p_0 \le p(t) \le 0$, *and*

$$\int_a^c H(s-a)[\rho(s)Q_2(s) + \rho(2c-s)Q_1(2c-s)]ds > k\,\Theta(a,c), \tag{2.21}$$

where

$$\begin{aligned}\Theta(a,c) = \int_a^c \frac{1}{(H(s-a))^\gamma}\Big\{\rho(s)\Big|h(s-a)\sqrt{H(s-a)} + \frac{\rho'(s)}{\rho(s)}H(s-a)\Big|^{\gamma+1} \\ + \rho(2c-s)\Big| - h(s-a)\sqrt{H(s-a)} + \frac{\rho'(2c-s)}{\rho(2c-s)}H(s-a)\Big|^{\gamma+1}\Big\}ds.\end{aligned}$$

Then (1.1) *is oscillatory.*

Let

$$H(t,s) = (t-s)^\lambda, \quad (t,s) \in D, \tag{2.22}$$

where $\lambda > \max\{1,\gamma\}$ is a constant. Then $H \in \mathcal{H}_0$ and

$$h_1(t-s) = h_2(t-s) = h(t-s) = \lambda(t-s)^{\frac{\lambda-2}{2}}.$$

By Theorem 2.4, we obtain the following oscillation criteria of Kamenev's type.

Theorem 2.6. *For each $\tau \ge t_0$ and for some $\lambda > \max\{1,\gamma\}$. Then* (1.1) *is oscillatory provided that one of the following conditions is satisfied:*

(C7) $0 \le p(t) \le 1$, *and*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-\gamma}} \int_\tau^t (s-\tau)^\lambda Q_1(s)ds > \frac{k\,\lambda^{\gamma+1}}{\lambda-\gamma}, \tag{2.23}$$

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-\gamma}} \int_\tau^t (t-s)^\lambda Q_1(s)ds > \frac{k\,\lambda^{\gamma+1}}{\lambda-\gamma}; \tag{2.24}$$

(C8) $-1 < p_0 \le p(t) \le 0$, *and*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-\gamma}} \int_\tau^t (s-\tau)^\lambda Q_2(s)ds > \frac{k\,\lambda^{\gamma+1}}{\lambda-\gamma}, \tag{2.25}$$

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-\gamma}} \int_\tau^t (t-s)^\lambda Q_2(s)ds > \frac{k\,\lambda^{\gamma+1}}{\lambda-\gamma}. \tag{2.26}$$

Proof. We only prove Case (C7). The proof of Case (C8) is similar. Taking $H(t,s)$ as in (2.22) and $\rho(t) \equiv 1$ for $t \ge t_0$, we get

$$\int_\tau^t \rho(s)|\phi_1(s,\tau)|^{\gamma+1}ds = \lambda^{\gamma+1}\int_\tau^t (s-\tau)^{\lambda-(\gamma+1)}ds = \frac{\lambda^{\gamma+1}}{\lambda-\gamma}(t-\tau)^{\lambda-\gamma},$$
$$\int_\tau^t \rho(s)|\phi_2(t,s)|^{\gamma+1}ds = \lambda^{\gamma+1}\int_\tau^t (t-s)^{\lambda-(\gamma+1)}ds = \frac{\lambda^{\gamma+1}}{\lambda-\gamma}(t-\tau)^{\lambda-\gamma}.$$

In view of the fact $\lambda > \gamma$, it follows that

$$\lim_{t\to\infty} \frac{1}{t^{\lambda-\gamma}} \int_\tau^t \rho(s)|\phi_1(s,\tau)|^{\gamma+1}ds = \frac{\lambda^{\gamma+1}}{\lambda-\gamma}, \tag{2.27}$$

$$\lim_{t\to\infty} \frac{1}{t^{\lambda-\gamma}} \int_\tau^t \rho(s)|\phi_2(t,s)|^{\gamma+1}ds = \frac{\lambda^{\gamma+1}}{\lambda-\gamma}, \tag{2.28}$$

From (2.23) and (2.27), we have

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{t^{\lambda-\gamma}} \int_\tau^t \rho(s)[H(s,\tau)Q_1(s) - k|\phi_1(s,\tau)|^{\gamma+1}]ds\\ &= \limsup_{t\to\infty} \frac{1}{t^{\lambda-\gamma}} \int_\tau^t (s-\tau)^\lambda Q_1(s)ds - \frac{k\,\lambda^{\gamma+1}}{\lambda-\gamma} > 0.\end{aligned}$$

i.e., (2.14) holds. Similarly, (2.24) and (2.28) imply that (2.15) holds, By Theorem 2.4 (C3), (1.1) is oscillatory. The proof is complete. □

3. Examples

In final section, we will show the application of our oscillation criteria by two examples.

Example 3.1. Consider the equation

$$\big[|x'(t)|^{\gamma-1}x'(t)\big]' + q_1(t)|y(t-2)|^{\alpha-1}y(t-2) + q_2(t)|y(t-2)|^{\beta-1}y(t-2) = 0, \tag{3.1}$$

where $t \ge 2$ and $x(t) = y(t) + p_0 y(t-1)$, $-1 < p_0 < 1$, $0 < \alpha < \gamma < \beta$ with $\gamma = (\alpha+\beta)/2$ and $q_1, q_2 \in \mathbf{C}([t_0,\infty),\mathbb{R}^+)$ with

$$q_1(t) = q_2(t) = \begin{cases} \eta(t-3n), & 3n < t \le 3n+1,\\ \eta(-t+3n+2), & 3n+1 < t < 3n+2,\\ q_0(t), & 3n+2 \le t \le 3n+3,\end{cases}$$

for $n \in \{1,2,\dots\}$, where $q_0(t)$ is any positive continuous function which makes $q_1(t)$ a continuous function, and

$$\eta > \begin{cases} \dfrac{k(\lambda+2)\lambda^{\gamma+1}}{2(1-p_0)^\gamma(\lambda-\gamma)}, & 0 \le p_0 < 1,\\[2ex] \dfrac{k(\lambda+2)\lambda^{\gamma+1}}{2(\lambda-\gamma)}, & -1 < p_0 \le 0,\end{cases}$$

is a constant for fixed $\lambda > \max\{1, \gamma\}$. Now, we consider the following two cases:

Case 1: $0 \le p_0 < 1$. Note that $Q_1(t) = 2(1-p_0)^\gamma q_1(t)$. Let $a = 3n$, $c = 3n+1$, $H(t,s) = (t-s)^\lambda$ and $\rho(t) \equiv 1$. Then

$$\begin{aligned}
&\int_a^c (s-a)^\lambda [Q_1(s) + Q_1(2c-s)]ds - 2k\lambda^{\gamma+1} \int_a^c (s-a)^{\lambda-\gamma-1} ds \\
&\ge 4\eta(1-p_0)^\gamma \int_{3k}^{3k+1} (s-3k)^{\gamma+1} ds - 2k\lambda^{\gamma+1} \int_{3k}^{3k+1} (s-3k)^{\lambda-(\gamma+1)} ds \\
&= \frac{4\eta(1-p_0)^\gamma}{\lambda+2} - \frac{2k\lambda^{\gamma+1}}{\lambda-\gamma} > 0,
\end{aligned}$$

i.e., (2.20) holds.

Case 2: $-1 < p_0 \le 0$. Note that $Q_2(t) \ge 2q_1(t)$. The rest of proof is similar to that of Case 1. Thus, (3.1) is oscillatory by Theorem 2.5.

However, the known results such as in [10, 11, 13, 14, 15] do not apply to (3.1).

Example 3.2. Consider the neutral delay differential equation

$$[|x'(t)|^{\gamma-1}x'(t)]' + q_1(t)|y(t-2)|^{\alpha-1}y(t-2) + q_2(t)|y(t-2)|^{\beta-1}y(t-2) = 0, \quad (3.2)$$

where $t \ge 2$ and $x(t) = y(t) + p_0 y(t-1)$, $-1 < p_0 < 1$, $0 < \alpha < \gamma < \beta$ with $\gamma = (\alpha+\beta)/2$, and q_1, $q_2 \in \mathbf{C}([t_0,\infty), \mathbb{R}^+)$ with $q_1(t)q_2(t) \ge (\varepsilon/t^{\gamma+1})^2$ with

$$\varepsilon > \begin{cases} \dfrac{k\gamma_0^{\gamma+1}}{2(1-p_0)^\gamma}, & 0 \le p_0 < 1, \\[2ex] \dfrac{k\gamma_0^{\gamma+1}}{2}, & -1 < p_0 \le 0, \end{cases}$$

where $\gamma_0 = \max\{1, \gamma\}$. Let $\rho(t) = t^{-(\gamma+1)}$. The following two cases will be considered.

Case 1: $0 \le p_0 < 1$. Note that $Q_1(t) \ge 2\varepsilon\,(1-p_0)^\gamma t^{-(\gamma+1)}$. Let $\lambda > \gamma_0$. Then

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t^{\lambda-\gamma}} \int_\tau^t (s-\tau)^\lambda Q_1(s) ds &\ge 2\varepsilon(1-p_0)^\gamma \lim_{t\to\infty} \frac{1}{t^{\lambda-\gamma}} \int_\tau^t \frac{(s-\tau)^\lambda}{s^{\gamma+1}} ds \\
&= \frac{2\varepsilon(1-p_0)^\gamma}{\lambda-\gamma} \lim_{t\to\infty} \frac{(t-\tau)^\lambda}{t^\lambda} \\
&= \frac{2\varepsilon(1-p_0)^\gamma}{\lambda-\gamma}.
\end{aligned}$$

For any $\varepsilon > k\ \gamma_0^{\gamma+1}/(2(1-p_0)^\gamma)$, there exists $\lambda > \gamma_0$ such that

$$\frac{2\varepsilon(1-p_0)^\gamma}{\lambda-\alpha} > \frac{k\,\lambda^{\gamma+1}}{\lambda-\alpha}.$$

This means that (2.23) holds. By [8, Lemma 3.1], (2.24) holds for the same λ. Applying Theorem 2.6 (C7), we find that (3.2) is oscillatory.

Case 2: $-1 < p_0 \le 0$. Note that $Q_2(t) \ge 2\varepsilon\, t^{-(\gamma+1)}$. The rest of proof is similar to that of Case 1. Hence, Theorem 2.6 (C8) holds. Thus, (3.2) is oscillatory .

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