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ASYMPTOTIC SHAPE OF SOLUTIONS TO THE PERTURBED SIMPLE PENDULUM PROBLEMS

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Abstract. We consider the positive solution of the perturbed simple pendulum problem

$$u''(r) + \frac{N-1}{r} u'(r) - g(u(t)) + \lambda \sin u(r) = 0,$$

with $0 < r < R$, $u'(0) = u(R) = 0$. To understand well the shape of the solution u_λ when $\lambda \gg 1$, we establish the leading and second terms of $\|u_\lambda\|_q$ $(1 \le q < \infty)$ with the estimate of third term as $\lambda \to \infty$. We also obtain the asymptotic formula for $u'_\lambda(R)$ as $\lambda \to \infty$.

1. Introduction

We consider the perturbed simple pendulum problem

$$u''(r) + \frac{N-1}{r} u'(r) - g(u(t)) + \lambda \sin u(r) = 0, \quad 0 < r < R, \tag{1.1}$$

$$u(r) > 0, \quad 0 \le r < R, \tag{1.2}$$

$$u'(0) = u(R) = 0, \tag{1.3}$$

where $N \ge 2$, $R > 0$ is a constant and $\lambda > 0$ is a parameter. We assume the following conditions:

- (A1) $g \in C^{m,\gamma}(\mathbb{R})$ $(m \ge 1, 0 < \gamma < 1)$ and $g(u) > 0$ for $u > 0$.
- (A2) $g(0) = g'(0) = 0$.
- (A3) $g(u)/u$ is strictly increasing for $0 < u < \pi$.

A typical example of $g(u)$ is $g(u) = |u|^{m-1}u$ $(m > 1)$ and $g(u)$ is regarded as the nonlinear self-interaction term of the simple pendulum equation. The following properties (P1) and (P2) are well-known and easy to show (cf. [1, 2, 4]).

- (P1) For a given $\lambda \in \mathbb{R}$, (1.1)–(1.3) has a unique solution $u_\lambda \in C^3([0, R])$ if and only if $\lambda > \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in $B_R = \{|x| < R\} \subset \mathbb{R}^N$ with Dirichlet zero boundary condition.
- (P2) $\|u_\lambda\|_\infty < \pi$ and $u_\lambda \to \pi$ uniformly on any compact interval in $[0, R)$ as $\lambda \to \infty$.

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Therefore, we see from (P2) that u_λ is almost flat inside $[0, R)$. The purpose of this paper is to understand well the asymptotic behavior of u_λ as $\lambda \to \infty$ not from a local point of view, but from a viewpoint of total shape of u_λ in $[0, R]$. To this end, we establish the precise asymptotic formula for $\|u_\lambda\|_q$ $(1 \le q < \infty)$ as $\lambda \to \infty$. Here $\|u\|_q := |S^{N-1}| \int_0^R r^{N-1}|u(r)|^q dr$ and $|S^{N-1}|$ is the measure of $S^{N-1} = \{|x| = R\}$.

Singularly perturbed equations have been investigated by many authors. We refer to [3, 5, 6, 7, 9] and the references therein. In particular, one of the main concern in this area is to investigate asymptotic shapes of the corresponding solutions.

As for the pointwise behavior of the solution u_λ of (1.1)–(1.3) as $\lambda \to \infty$, there are some known results. Let us consider the case $N = 1$ in the interval $(-R, R)$ and $g \equiv 0$. We denote by $u_{0,\lambda}$ the unique solution associated with given $\lambda \gg 1$. Then it is known (cf. [8]) that as $\lambda \to \infty$

$$\|u_{0,\lambda}\|_\infty = \pi - 8(1 + o(1))e^{-\sqrt{\lambda}(1+o(1))R}. \tag{1.4}$$

We remark that the second term in the righthand side of the equation decays *exponentially* as $\lambda \to \infty$. Furthermore, when $N \ge 2$, $g(u) \not\equiv 0$ and satisfies (A.1)–(A.3), the following asymptotic formula has been obtained in [11].

Theorem 1.1 ([11]). *Let an arbitrary $0 \le r < R$ be fixed. Then the following asymptotic formula for the solution u_λ of (1.1)–(1.3) holds as $\lambda \to \infty$.*

$$u_\lambda(r) = \pi - \sum_{k=1}^{m+1} \frac{b_k}{\lambda^k} + o\Big(\frac{1}{\lambda^{m+1}}\Big), \tag{1.5}$$

where $b_1 = g(\pi)$ and b_k $(k = 2, 3, \dots, m+1)$ are constants determined by

$$\{g^{(j)}(\pi)\}_{j=0}^{k-1}.$$

In particular, we see from (1.5) that

$$\|u_\lambda\|_\infty = \pi - \sum_{k=1}^{m+1} \frac{b_k}{\lambda^k} + o\Big(\frac{1}{\lambda^{m+1}}\Big). \tag{1.6}$$

Theorem 1.1 gives us the precise pointwise information about u_λ inside $[0, R)$ as $\lambda \to \infty$. However, if we consider the asymptotic behavior of $\|u_\lambda\|_q$ as $\lambda \to \infty$ $(1 \le q < \infty)$ for the better understanding of the *total shape* of u_λ in $[0, R]$, then it is natural that $\|u_\lambda\|_q$ is affected by both the interior behavior of u_λ and the behavior near the boundary. Therefore, it is expected that the asymptotic formula for $\|u_\lambda\|_q$ $(1 \le q < \infty)$ is different from (1.6).

Now we state our main results. Let $G(u) := \int_0^u g(s)ds$.

Theorem 1.2. *Let $1 \le q < \infty$ be fixed. Then for any fixed $0 < \delta < 1/4$, the following asymptotic formula holds as $\lambda \to \infty$:*

$$\|u_\lambda\|_q = |B_R|^{1/q}\Big(\pi - \frac{NC_0}{\pi^{q-1}qR}\lambda^{-1/2} + O\big(\lambda^{-(1/2+\delta)}\big)\Big), \tag{1.7}$$

where $|B_R|$ is the volume of B_R and

$$C_0 = \int_0^\pi \frac{\pi^q - \theta^q}{\sqrt{2(1 + \cos\theta)}} d\theta.$$

Theorem 1.3. *The following asymptotic formula holds as $\lambda \to \infty$:*

$$|u_\lambda'(R)| = 2\sqrt{\lambda} - \frac{2(N-1)}{R} + o(1). \tag{1.8}$$

We see from Theorems 1.1, 1.2, 1.3 that the second term of $\|u_\lambda\|_q$ as $\lambda \to \infty$ is mainly affected by the slope of the boundary layer $u_\lambda'(R)$. It should also be mentioned that for the case $N = 1$, the exact third term of $\|u_\lambda\|_1$ has been obtained in [10].

We briefly explain the difficulty to treat the case $N \ge 2$. To prove Theorem 1.2, we calculate $\|u_\lambda\|_q$ which is affected both the behavior of u_λ inside $[0, R)$ and near the boundary. Moreover, (1.1) contains the term $(N-1)u_\lambda'(r)/r$, which is quite difficult to treat and does not appear when $N = 1$. Therefore, the calculation to obtain the remainder estimate in (1.7) is quite delicate and complicated. This is the reason why we need the restriction $0 < \delta < 1/4$.

2. Proof of Theorem 1.2

In what follows, C denotes various positive constants independent of $\lambda \gg 1$. We begin with the fundamental properties of u_λ. It is well known that

$$u_\lambda(0) = \|u_\lambda\|_\infty, \quad u_\lambda'(r) < 0 \quad (0 < r \le R). \tag{2.1}$$

Multiply (1.1) by u_λ'. Then for $r \in [0, R]$,

$$\big\{u_\lambda''(r) + \frac{N-1}{r}u_\lambda'(r) + \lambda \sin u_\lambda(r) - g(u_\lambda(r))\big\}u_\lambda'(r) = 0.$$

By (2.1), this implies that for $r \in [0, R]$,

$$\begin{aligned}
&\frac{1}{2}u_\lambda'(r)^2 + \int_0^r \frac{N-1}{s}u_\lambda'(s)^2 ds - \lambda \cos u_\lambda(r) - G(u_\lambda(r)) \equiv \quad \text{constant} \\
&= -\lambda \cos \|u_\lambda\|_\infty - G(\|u_\lambda\|_\infty) \quad (\text{put } r = 0) \\
&= \frac{1}{2}u_\lambda'(R)^2 + \int_0^R \frac{N-1}{s}u_\lambda'(s)^2 ds - \lambda \quad (\text{put } r = R).
\end{aligned} \tag{2.2}$$

Let $M_\lambda := \inf\{\theta > 0 : \lambda \sin \theta = g(\theta)\}$. It is clear that $M_\lambda < \pi$ and $\lambda \sin \theta > g(\theta)$ for $0 < \theta < M_\lambda$. We know from [1] that $\|u_\lambda\|_\infty < M_\lambda$. Therefore, for $0 \le r \le R$, we have

$$\lambda \sin u_\lambda(r) > g(u_\lambda(r)). \tag{2.3}$$

In particular,

$$\xi_\lambda := \lambda \sin \|u_\lambda\|_\infty - g(\|u_\lambda\|_\infty) > 0. \tag{2.4}$$

Furthermore, for $[0, R]$, we put

$$I_\lambda(r) := \lambda(\cos u_\lambda(r) - \cos \|u_\lambda\|_\infty) + G(u_\lambda(r)) - G(\|u_\lambda\|_\infty), \tag{2.5}$$

$$II_\lambda(r) := \int_0^r \frac{N-1}{s}u_\lambda'(s)^2 ds. \tag{2.6}$$

Then for $r \in [0, R]$, by (2.2), we obtain

$$\frac{1}{2}u_\lambda'(r)^2 = I_\lambda(r) - II_\lambda(r). \tag{2.7}$$

We explain the basic idea of the proof of Theorem 1.2. The main part of the proof of Theorem 1.2 is to show the following Proposition 2.1.

Proposition 2.1. *Let an arbitrary $0 < \delta < 1/4$ be fixed. Then for $\lambda \gg 1$*

$$|B_R|\|u_\lambda\|_\infty^q - \|u_\lambda\|_q^q = \frac{N|B_R|C_0}{R}\lambda^{-1/2} + O(\lambda^{-(1/2+\delta)}). \tag{2.8}$$

Once Proposition 2.1 is proved, then we obtain Theorem 1.2 easily as follows.

Proof of Theorem 1.2. By Proposition 2.1, Theorem 1.1 and Taylor expansion, for $\lambda \gg 1$,

$$\begin{aligned}
\|u_\lambda\|_q &= \Big(|B_R|\|u_\lambda\|_\infty^q - \frac{N|B_R|C_0}{R}\lambda^{-1/2} + O(\lambda^{-(1/2+\delta)})\Big)^{1/q} \\
&= |B_R|^{1/q}\|u_\lambda\|_\infty \Big(1 - \frac{NC_0}{R\|u_\lambda\|_\infty^q}\lambda^{-1/2} + O(\lambda^{-(1/2+\delta)})\Big)^{1/q} \\
&= |B_R|^{1/q}\Big(\pi - g(\pi)\lambda^{-1} + o(\lambda^{-1})\Big)\Big(1 - \frac{NC_0}{qR\pi^q}\lambda^{-1/2} + O(\lambda^{-(1/2+\delta)})\Big) \\
&= |B_R|^{1/q}\Big(\pi - \frac{NC_0}{\pi^{q-1}qR}\lambda^{-1/2} + O(\lambda^{-(1/2+\delta)})\Big).
\end{aligned}$$

Thus the proof is complete. □

The basic idea to obtain Proposition 2.1 is as follows. In what follows, let an arbitrary $0 < \delta < 1/4$ be fixed. Let $0 < R_{\lambda,\delta} < R$ satisfy $u_\lambda(R_{\lambda,\delta}) = \|u_\lambda\|_\infty - \lambda^{-\delta}$. By (2.7), we have

$$\begin{aligned}
&|B_R|\|u_\lambda\|_\infty^q - \|u_\lambda\|_q^q \\
&= |S^{N-1}|\int_0^R r^{N-1}(\|u_\lambda\|_\infty^q - u_\lambda(r)^q)\frac{-u_\lambda'(r)}{\sqrt{2(I_\lambda(r) - II_\lambda(r))}}dr \\
&= |S^{N-1}|\int_0^{R_{\lambda,\delta}} r^{N-1}(\|u_\lambda\|_\infty^q - u_\lambda(r)^q)\frac{-u_\lambda'(r)}{\sqrt{2(I_\lambda(r) - II_\lambda(r))}}dr \\
&\quad + |S^{N-1}|\int_{R_{\lambda,\delta}}^R r^{N-1}(\|u_\lambda\|_\infty^q - u_\lambda(r)^q)\frac{-u_\lambda'(r)}{\sqrt{2(I_\lambda(r) - II_\lambda(r))}}dr \\
&:= A(\lambda) + B(\lambda).
\end{aligned} \tag{2.9}$$

Therefore, to show Proposition 2.1, we have only to estimate $A(\lambda)$ and $B(\lambda)$.

For $0 \le \theta \le \|u_\lambda\|_\infty$, we put

$$V_0 := 2\lambda(\cos\theta + 1), \tag{2.10}$$

$$V_1 := 2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty), \tag{2.11}$$

$$V_2 := 2(G(\theta) - G(\|u_\lambda\|_\infty)) - 2\int_0^{u_\lambda^{-1}(\theta)} \frac{N-1}{s}u_\lambda'(s)^2 ds. \tag{2.12}$$

By putting $\theta = u_\lambda(r)$, we have

$$A_\lambda = |S^{N-1}|\int_{\|u_\lambda\|_\infty - \lambda^{-\delta}}^{\|u_\lambda\|_\infty} \frac{u_\lambda^{-1}(\theta)^{N-1}(\|u_\lambda\|_\infty^q - \theta^q)}{\sqrt{V_1 + V_2}}d\theta, \tag{2.13}$$

$$B_\lambda = |S^{N-1}|\int_0^{\|u_\lambda\|_\infty - \lambda^{-\delta}} \frac{u_\lambda^{-1}(\theta)^{N-1}(\|u_\lambda\|_\infty^q - \theta^q)}{\sqrt{V_1 + V_2}}d\theta. \tag{2.14}$$

We estimate A_λ first by using the following Lemma.

Lemma 2.2 ([11]). *Assume that $0 < r_\lambda < R$ satisfies $u_\lambda(r_\lambda) \to \pi$ as $\lambda \to \infty$. Then for $0 \le r \le r_\lambda$ and $\lambda \gg 1$*

$$I_\lambda(r) = \xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(r)) + \frac{1}{2}(\lambda + g'(\pi))(1+o(1))(\|u_\lambda\|_\infty - u_\lambda(r))^2, \tag{2.15}$$

$$\begin{aligned} II_\lambda(r) \le \frac{N-1}{N}&\xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(r)) \\ &+ \frac{N-1}{2(N+1)}(\lambda + g'(\pi))(1+o(1))(\|u_\lambda\|_\infty - u_\lambda(r))^2. \end{aligned} \tag{2.16}$$

Furthermore, $\xi_\lambda = o\big(\lambda e^{-\sqrt{2\lambda(1+o(1))/(N+1)}r_0}\big)$ as $\lambda \to \infty$.

Lemma 2.3. *$A(\lambda) = O(\lambda^{-(1/2+\delta)})$ for $\lambda \gg 1$.*

Proof. By Lemma 2.2, for $0 \le r \le R_{\lambda,\delta}$ and $\lambda \gg 1$

$$\begin{aligned} \frac{1}{2}u_\lambda'(r)^2 &= I_\lambda(r) - II_\lambda(r) \\ &\ge \frac{1}{N}\xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(r)) + \frac{1}{N+1}(1+o(1))(g'(\pi)+\lambda)(\|u_\lambda\|_\infty - u_\lambda(r))^2 \\ &\ge C\lambda(\|u_\lambda\|_\infty - u_\lambda(r))^2. \end{aligned} \tag{2.17}$$

This implies that for $\|u_\lambda\|_\infty - \lambda^{-\delta} \le \theta \le \|u_\lambda\|_\infty$,

$$V_1 + V_2 \ge C\lambda(\|u_\lambda\|_\infty - \theta)^2. \tag{2.18}$$

By this and (2.13), we obtain

$$\begin{aligned} A(\lambda) &\le |S^{N-1}|R^{N-1}\int_{\|u_\lambda\|_\infty - \lambda^{-\delta}}^{\|u_\lambda\|_\infty} \frac{\|u_\lambda\|_\infty^q - \theta^q}{\sqrt{\lambda(\|u_\lambda\|_\infty - \theta)^2}}d\theta \\ &\le \frac{C}{\sqrt{\lambda}}\int_{\|u_\lambda\|_\infty - \lambda^{-\delta}}^{\|u_\lambda\|_\infty} \frac{\|u_\lambda\|_\infty^q - \theta^q}{\|u_\lambda\|_\infty - \theta}d\theta \\ &= O\big(\lambda^{-(1/2+\delta)}\big). \end{aligned}$$

Thus the proof is complete. □

We next estimate $B(\lambda)$. We put

$$K_1 := |S^{N-1}|\int_0^{\|u_\lambda\|_\infty - \lambda^{-\delta}} \frac{u_\lambda^{-1}(\theta)^{N-1}(\|u_\lambda\|_\infty^q - \theta^q)}{\sqrt{2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty)}}d\theta \tag{2.19}$$

and $K_2 := B(\lambda) - K_1$. Once we obtain the estimates of K_1 and K_2, then the proof of Proposition 2.1 is complete.

To calculate K_2, we need the following lemma.

Lemma 2.4. *For $\lambda \gg 1$,*

$$\int_0^R \frac{N-1}{s}u_\lambda'(s)^2 ds \le C\sqrt{\lambda}. \tag{2.20}$$

Proof. Let an arbitrary $0 < \epsilon \ll 1$ be fixed. Then

$$\int_0^R \frac{N-1}{r}u_\lambda'(r)^2 dr = \int_0^{R-\epsilon} \frac{N-1}{r}u_\lambda'(r)^2 dr + \int_{R-\epsilon}^R \frac{N-1}{r}u_\lambda'(r)^2 dr. \tag{2.21}$$

By Lemma 2.2 and Theorem 1.1,

$$\begin{aligned}\int_0^{R-\epsilon}\frac{N-1}{r}u_\lambda'(r)^2dr &= II_\lambda(R-\epsilon)\\ &\le C\xi_\lambda(\|u_\lambda\|_\infty-u_\lambda(R-\epsilon))+C\lambda(\|u_\lambda\|_\infty-u_\lambda(R-\epsilon))^2\\ &\le C\lambda^{-1}.\end{aligned}\tag{2.22}$$

By (2.7) and putting $\theta=u_\lambda(r)$, for $\lambda\gg 1$, we have

$$\begin{aligned}\int_0^R r^{N-1}u_\lambda'(r)^2dr &\le R^{N-1}\int_0^R(-u_\lambda'(r))\sqrt{2\lambda(\cos u_\lambda(r)-\cos\|u_\lambda\|_\infty)}dr\\ &\le C\int_0^{\|u_\lambda\|_\infty}\sqrt{2\lambda(\cos\theta-\cos\|u_\lambda\|_\infty)}d\theta\le C\sqrt{\lambda}.\end{aligned}$$

By this, for $\lambda\gg 1$, we obtain

$$\begin{aligned}\int_{R-\epsilon}^R\frac{N-1}{r}u_\lambda'(r)^2dr &\le\frac{N-1}{(R-\epsilon)^N}\int_{R-\epsilon}^R r^{N-1}u_\lambda'(r)^2dr\\ &\le C\int_0^R r^{N-1}u_\lambda'(r)^2dr\le C\sqrt{\lambda}.\end{aligned}\tag{2.23}$$

By this and (2.22), we obtain our conclusion. □

Lemma 2.5. $K_2=O(\lambda^{-(1/2+\delta)})$ *for* $\lambda\gg 1$.

Proof. Let an arbitrary $0<\epsilon\ll 1$ be fixed. Since $V_2\le 0$, by (2.11), (2.12), (2.14) and (2.19), for $\lambda\gg 1$,

$$\begin{aligned}K_2 &= |S^{N-1}|\int_0^{\|u_\lambda\|_\infty-\lambda^{-\delta}}u_\lambda^{-1}(\theta)^{N-1}(\|u_\lambda\|_\infty^q-\theta^q)\Big\{\frac{1}{\sqrt{V_1+V_2}}-\frac{1}{\sqrt{V_1}}\Big\}d\theta\\ &\le CR^{N-1}\int_0^{\|u_\lambda\|_\infty-\lambda^{-\delta}}(\|u_\lambda\|_\infty^q-\theta^q)\frac{|V_2|}{\sqrt{V_1}\sqrt{V_1+V_2}(\sqrt{V_1+V_2}+\sqrt{V_1})}d\theta\\ &\le CR^{N-1}\int_{\|u_\lambda\|_\infty-\epsilon}^{\|u_\lambda\|_\infty-\lambda^{-\delta}}(\|u_\lambda\|_\infty^q-\theta^q)\frac{|V_2|}{(V_1+V_2)^{3/2}}d\theta\\ &\quad+CR^{N-1}\int_0^{\|u_\lambda\|_\infty-\epsilon}(\|u_\lambda\|_\infty^q-\theta^q)\frac{|V_2|}{(V_1+V_2)^{3/2}}d\theta\\ &= K_{2,1}+K_{2,2}.\end{aligned}\tag{2.24}$$

We know from Lemma 2.4 that $|V_2|\le C\sqrt{\lambda}$. By this, Lemma 2.2 and the same estimate as (2.18), we obtain

$$\begin{aligned}K_{2,1} &\le CR^{N-1}\int_{\|u_\lambda\|_\infty-\epsilon}^{\|u_\lambda\|_\infty-\lambda^{-\delta}}(\|u_\lambda\|_\infty^q-\theta^q)\frac{|V_2|}{(V_1+V_2)^{3/2}}d\theta\\ &\le C\int_{\|u_\lambda\|_\infty-\epsilon}^{\|u_\lambda\|_\infty-\lambda^{-\delta}}(\|u_\lambda\|_\infty^q-\theta^q)\frac{\sqrt{\lambda}}{(\sqrt{\lambda}(\|u_\lambda\|_\infty-\theta)^2)^{3/2}}d\theta\\ &\le C\lambda^{-1}\int_{\|u_\lambda\|_\infty-\epsilon}^{\|u_\lambda\|_\infty-\lambda^{-\delta}}\frac{1}{(\|u_\lambda\|_\infty-\theta)^2}d\theta\\ &\le O\left(\lambda^{-1+\delta}\right)=o\big(\lambda^{-(1/2+\delta)}\big).\end{aligned}\tag{2.25}$$

We note that $0 < \delta < 1/4$. By (2.11), (2.12) and Lemma 2.4, for $0 \le \theta \le \|u_\lambda\|_\infty - \epsilon$,

$$V_1 + V_2 \ge C\lambda - C - C\sqrt{\lambda} \ge C\lambda. \tag{2.26}$$

By this, Lemma 2.4 and (2.24), we obtain

$$K_{2,2} \le CR^{N-1} \int_0^{\|u_\lambda\|_\infty - \epsilon} \frac{\sqrt{\lambda}}{\lambda^{3/2}} d\theta \le C\lambda^{-1}. \tag{2.27}$$

By (2.24), (2.25) and (2.27), we obtain our conclusion. Thus the proof is complete. $\square$

We next calculate K_1. We put $K_1 := L_1 + L_2$, where

$$L_1 := |S^{N-1}|R^{N-1} \int_0^\pi \frac{\pi^q - \theta^q}{\sqrt{2\lambda(\cos\theta + 1)}} d\theta = \frac{N|B_R|C_0}{R} \lambda^{-1/2} \tag{2.28}$$

and $L_2 := K_1 - L_1$. All we have to do is to calculate L_2. To do this, we put $L_2 := D_1 + D_2 + D_3 + D_4$, where

$$D_1 := -|S^{N-1}| \int_0^{\|u_\lambda\|_\infty - \lambda^{-\delta}} \frac{(R^{N-1} - u_\lambda^{-1}(\theta)^{N-1})(\|u_\lambda\|_\infty^q - \theta^q)}{\sqrt{2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty)}} d\theta, \tag{2.29}$$

$$D_2 := |S^{N-1}| \int_0^{\|u_\lambda\|_\infty - \lambda^{-\delta}} \frac{R^{N-1}(\|u_\lambda\|_\infty^q - \pi^q)}{\sqrt{2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty)}} d\theta, \tag{2.30}$$

$$\begin{aligned} D_3 := {}& |S^{N-1}| \int_0^{\|u_\lambda\|_\infty - \lambda^{-\delta}} \frac{R^{N-1}(\pi^q - \theta^q)}{\sqrt{2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty)}} d\theta \\ & - |S^{N-1}| \int_0^{\|u_\lambda\|_\infty - \lambda^{-\delta}} \frac{R^{N-1}(\pi^q - \theta^q)}{\sqrt{2\lambda(\cos\theta + 1)}} d\theta, \end{aligned} \tag{2.31}$$

$$D_4 := -|S^{N-1}| \int_{\|u_\lambda\|_\infty - \lambda^{-\delta}}^\pi \frac{R^{N-1}(\pi^q - \theta^q)}{\sqrt{2\lambda(\cos\theta + 1)}} d\theta. \tag{2.32}$$

The most essential term in L_2 is D_1. Therefore, we treat D_1 after the estimates of D_2, D_3 and D_4.

Lemma 2.6. $|D_4| \le C\lambda^{-(1/2+\delta)}$ *for* $\lambda \gg 1$.

Proof. Since $(\pi^q - \theta^q)/\sqrt{1 + \cos\theta}$ is bounded for $0 \le \theta \le \pi$, by Theorem 1.1 and (2.32),

$$\begin{aligned} |D_4| &\le C\lambda^{-1/2} \int_{\|u_\lambda\|_\infty - \lambda^{-\delta}}^\pi \frac{\pi^q - \theta^q}{\sqrt{\cos\theta + 1}} d\theta \\ &\le C\lambda^{-1/2}(\pi - \|u_\lambda\|_\infty + \lambda^{-\delta}) \le C\lambda^{-(1/2+\delta)}. \end{aligned}$$

Thus the proof is complete. $\square$

Lemma 2.7. $|D_2| \le C\lambda^{-3/2} \log\lambda$ *for* $\lambda \gg 1$.

Proof. Let an arbitrary $0 < \epsilon \ll 1$ be fixed. By Taylor expansion, we see that for $\|u_\lambda\|_\infty - \epsilon \le \theta \le \|u_\lambda\|_\infty - \lambda^{-\delta}$

$$\cos\theta - \cos\|u_\lambda\|_\infty \ge C(\|u_\lambda\|_\infty - \theta)^2. \tag{2.33}$$

By this and Theorem 1.1,

$$
\begin{aligned}
|D_2| &\le C\lambda^{-1}\int_{\|u_\lambda\|_\infty-\epsilon}^{\|u_\lambda\|_\infty-\lambda^{-\delta}} \frac{1}{\sqrt{2\lambda(\cos\theta-\cos\|u_\lambda\|_\infty)}}d\theta \\
&\quad + C\lambda^{-1}\int_0^{\|u_\lambda\|_\infty-\epsilon} \frac{1}{\sqrt{2\lambda(\cos\theta-\cos\|u_\lambda\|_\infty)}}d\theta \\
&\le C\lambda^{-3/2}\int_{\|u_\lambda\|_\infty-\epsilon}^{\|u_\lambda\|_\infty-\lambda^{-\delta}} \frac{1}{\|u_\lambda\|_\infty-\theta}d\theta + C\lambda^{-3/2} \\
&\le C\lambda^{-3/2}\log\lambda.
\end{aligned} \tag{2.34}
$$

Thus the proof is complete. ☐

Lemma 2.8. *$|D_3| \le C\lambda^{-7/2+2\delta}$ for $\lambda \gg 1$.*

Proof. Let an arbitrary $0 < \epsilon \ll 1$ be fixed. By (2.10), (2.11), (2.31), (2.33) and Theorem 1.1,

$$
\begin{aligned}
|D_3| &\le \int_0^{\|u_\lambda\|_\infty-\lambda^{-\delta}} (\pi^q-\theta^q)\left(\frac{1}{\sqrt{V_1}}-\frac{1}{\sqrt{V_0}}\right)d\theta \\
&\le C\int_0^{\|u_\lambda\|_\infty-\lambda^{-\delta}} \frac{1+\cos\|u_\lambda\|_\infty}{\sqrt{V_0}\sqrt{V_1}(\sqrt{V_0}+\sqrt{V_1})}d\theta \\
&\le C\lambda^{-2}\int_{\|u_\lambda\|_\infty-\epsilon}^{\|u_\lambda\|_\infty-\lambda^{-\delta}} \frac{1}{(2\lambda(\cos\theta-\cos\|u_\lambda\|_\infty))^{3/2}}d\theta \\
&\quad + C\lambda^{-2}\int_0^{\|u_\lambda\|_\infty-\epsilon} \frac{1}{(2\lambda(\cos\theta-\cos\|u_\lambda\|_\infty))^{3/2}}d\theta \\
&= C\lambda^{-7/2}\int_{\|u_\lambda\|_\infty-\epsilon}^{\|u_\lambda\|_\infty-\lambda^{-\delta}} \frac{1}{(\|u_\lambda\|_\infty-\theta)^3}d\theta + C\lambda^{-7/2} \\
&\le C\lambda^{-7/2+2\delta}.
\end{aligned} \tag{2.35}
$$

Thus the proof is complete. ☐

Now we calculate D_1. To do this, we need additional two lemmas.

Lemma 2.9. *For $\lambda \gg 1$*

$$
u_\lambda'(R)^2 = 4\lambda + O(\sqrt{\lambda}). \tag{2.36}
$$

Proof. By (2.2), Theorem 1.1 and Lemma 2.4, we obtain

$$
\begin{aligned}
\frac{1}{2}u_\lambda'(R)^2 &= \lambda(1-\cos\|u_\lambda\|_\infty) - \int_0^R \frac{N-1}{s}u_\lambda'(s)^2ds - G(\|u_\lambda\|_\infty) \\
&= 2\lambda + O(\sqrt{\lambda}).
\end{aligned}
$$

Thus the proof is complete. ☐

Lemma 2.10. *For $\lambda \gg 1$*

$$
R - R_{\lambda,\delta} \le C\lambda^{-1/2+\delta}. \tag{2.37}
$$

Proof. Since $u_\lambda(r)$ is decreasing for $0 \le r \le R$ by (2.1), for $R_{\lambda,\delta} \le r \le R$

$$\cos u_\lambda(r) \ge \cos(\|u_\lambda\|_\infty - \lambda^{-\delta}) = -1 + \frac{1}{2}(1+o(1))\lambda^{-2\delta}. \tag{2.38}$$

By this, (2.2) and Lemmas 2.4 and 2.9, for $R_{\lambda,\delta} \le r \le R$,

$$\begin{aligned} \frac{1}{2}u_\lambda'(r)^2 &= \frac{1}{2}u_\lambda'(R)^2 + \int_r^R \frac{N-1}{s}u_\lambda'(s)^2 ds + \lambda(\cos u_\lambda(r) - 1) + G(u_\lambda(r)) \\ &\ge \frac{1}{2}(4\lambda + O(\sqrt{\lambda})) + \lambda(-2 + \frac{1}{2}(1+o(1))\lambda^{-2\delta}) \\ &= \frac{1}{2}\lambda^{1-2\delta}(1+o(1)). \end{aligned} \tag{2.39}$$

Note that $0 < \delta < 1/4$. By this, for $\lambda \gg 1$, we obtain

$$C\lambda^{(1-2\delta)/2}(R - R_{\lambda,\delta}) \le \int_{R_{\lambda,\delta}}^R -u_\lambda'(r)dr \le u(R_{\lambda,\delta}) < \pi.$$

This implies our conclusion. $\square$

Lemma 2.11. $|D_1| \le C\lambda^{-(1/2+\delta)}$ *for* $\lambda \gg 1$.

Proof. It is easy to see that for $0 \le \theta \le \|u_\lambda\|_\infty - \lambda^{-\delta}$,

$$\frac{\|u_\lambda\|_\infty^q - \theta^q}{\sqrt{\cos\theta - \cos\|u_\lambda\|_\infty}} \le C. \tag{2.40}$$

By this and Lemma 2.10,

$$\begin{aligned} |D_1| &= |S^{N-1}| \int_0^{\|u_\lambda\|_\infty - \lambda^{-\delta}} \frac{(R^{N-1} - u_\lambda^{-1}(\theta)^{N-1})(\|u_\lambda\|_\infty^q - \theta^q)d\theta}{\sqrt{2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty)}} \\ &\le C(R - R_{\lambda,\delta})\lambda^{-1/2}\int_0^{\|u_\lambda\|_\infty - \lambda^{-\delta}} d\theta \\ &= C\lambda^{-1+\delta} \le C\lambda^{-(1/2+\delta)}. \end{aligned}$$

We note here that $0 < \delta < 1/4$. Thus the proof is complete. $\square$

By Lemmas 2.3, 2.5 2.6, 2.7, 2.8, 2.11, we obtain Proposition 2.1. Thus the proof is complete.

3. Proof of Theorem 1.3

To prove Theorem 1.3, we have only to improve Lemma 2.4.

Lemma 3.1. *For* $\lambda \gg 1$

$$\int_0^R \frac{N-1}{r}u_\lambda'(r)^2 dr = \frac{4(N-1)}{R}\sqrt{\lambda} + o(\sqrt{\lambda}). \tag{3.1}$$

Proof. Let an arbitrary $0 < \epsilon \ll 1$ be fixed. We consider (2.21). Then by (2.7) and Theorem 1.1, for $\lambda \gg 1$

$$\begin{aligned}
\int_{R-\epsilon}^{R} \frac{N-1}{r} u_\lambda'(r)^2 dr &\le \frac{N-1}{R-\epsilon} \int_{R-\epsilon}^{R} u_\lambda'(r)^2 dr \\
&\le \frac{N-1}{R-\epsilon} \int_{R-\epsilon}^{R} \sqrt{2\lambda(\cos u_\lambda(r) - \cos \|u_\lambda\|_\infty)}(-u_\lambda'(r)) dr \\
&= \frac{N-1}{R-\epsilon} \sqrt{\lambda} \int_0^{u_\lambda(R-\epsilon)} \sqrt{2(\cos\theta - \cos\|u_\lambda\|_\infty)} d\theta \\
&= \frac{N-1}{R-\epsilon} \sqrt{\lambda}(1+o(1)) \int_0^{\pi} \sqrt{2(\cos\theta + 1)} d\theta \\
&= \frac{4(N-1)}{R-\epsilon} \sqrt{\lambda}(1+o(1)).
\end{aligned} \tag{3.2}$$

By the same argument as that just above, we obtain

$$\int_{R-\epsilon}^{R} \frac{N-1}{r} u_\lambda'(r)^2 dr \ge \frac{4(N-1)}{R} \sqrt{\lambda}(1+o(1)). \tag{3.3}$$

Since $0 < \epsilon \ll 1$ is arbitrary, by (2.21), (2.22), (3.2) and (3.3), we obtain (3.1). Thus the proof is complete. □

Proof of Theorem 1.3. By Theorem 1.1, Lemma 3.1 and (2.2), for $\lambda \gg 1$,

$$\begin{aligned}
\frac{1}{2} u_\lambda'(R)^2 &= \lambda(1 - \cos\|u_\lambda\|_\infty) - \int_0^R \frac{N-1}{s} u_\lambda'(s)^2 ds - G(\|u_\lambda\|_\infty) \\
&= \lambda\Big(2 - \frac{1}{2}(1+o(1)) g(\pi)^2 \lambda^{-2}\Big) - \frac{4(N-1)}{R}\sqrt{\lambda} + o(\sqrt{\lambda}) \\
&= 2\lambda - \frac{4(N-1)}{R}\sqrt{\lambda} + o(\sqrt{\lambda}).
\end{aligned}$$

By this, we obtain Theorem 1.3. Thus the proof is complete. □

References

[1] P. Clément and G. Sweers, *Existence and multiplicity results for a semilinear elliptic eigenvalue problem*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 14 (1987), 97–121.

[2] D. G. Figueiredo, *On the uniqueness of positive solutions of the Dirichlet problem* $-\triangle u = \lambda \sin u$, Pitman Res. Notes in Math. **122** (1985), 80–83.

[3] F. A. Howes, *Singularly perturbed semilinear elliptic boundary value problems*, Comm. in Partial Differential Equations **4** (1979), 1–39.

[4] F. A. Howes, *Boundary-interior layer interactions in nonlinear singular perturbation theory*, Mem. Amer. Math. Soc. **15**, No. 203 (1978).

[5] R. E. O'Malley Jr., *Phase-plane solutions to some singular perturbation problems*, J. Math. Anal. Appl. **54** (1976), 449–466.

[6] R. E. O'Malley Jr., *Singular perturbation methods for ordinary differential equations*, Springer, New York, 1989.

[7] D. H. Sattinger, "Topics in stability and bifurcation theory", Lect. Notes in Math. **309**, Springer, New York, 1973.

[8] T. Shibata, *Precise spectral asymptotics for the Dirichlet problem* $-u''(t) + g(u(t)) = \lambda \sin u(t)$, J. Math. Anal. Appl. **267** (2002), 576–598.

[9] T. Shibata, *Asymptotic expansion of the boundary layers of the perturbed simple pendulum problems*, J. Math. Anal. Appl. **283** (2003), 431-439.

[10] T. Shibata, *Asymptotic shapes of solutions to nonlinear eigenvalue problems*, Electronic J. Differential Equations 2005 (2005), no. 37, 1–16.
[11] T. Shibata, *Asymptotic expansion of solutions to nonlinear elliptic eigenvalue problems*, Proc. Amer. Math. Soc. 133 (2005), 2597–2604.

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