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DYNAMIC FRICTIONAL CONTACT FOR ELASTIC VISCOPLASTIC MATERIAL

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ABSTRACT. Using a general theory for evolution inclusions, existence and uniqueness theorems are obtained for weak solutions to a frictional dynamic contact problem for elastic visco-plastic material. An existence theorem in the case where the friction coefficient is discontinuous is also presented.

1. INTRODUCTION

The purpose of this paper is to consider a model involving frictional contact between an elastic visco-plastic material and a foundation. The balance of momentum and initial conditions are of the form

$$\ddot{\mathbf{u}} = \operatorname{div}(\sigma) + \mathbf{f} \quad \text{for} \quad (t, \mathbf{x}) \in (0, T) \times \Omega, \tag{1.1}$$

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \tag{1.2}$$

$$\dot{\mathbf{u}}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}), \tag{1.3}$$

where for convenience, in the top balance of momentum equation, the density has been taken to equal 1.

The domain Ω is a bounded open subset of $\mathbb{R}^d$ for $d = 2$ or 3 having Lipschitz boundary consisting of the union of three disjoint sets, Γ_C, Γ_0, and Γ_N, any of which could be empty. Dirichlet conditions for $\mathbf{u}$ will be given on Γ_0, and on Γ_N, the traction density will be specified, while on Γ_C are the complicated contact conditions involving friction. The following will be needed to describe these.

Let $\mathbf{n}$ be the unit outward normal to $\partial\Omega$. Then $u_n, \mathbf{u}_T, \sigma_T$, and σ_n are defined by the following.

$$u_n = \mathbf{u} \cdot \mathbf{n},$$

$$\mathbf{u}_T = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n},$$

$$\sigma_n = \sigma_{ij} n_j n_i,$$

$$\sigma_{Ti} = \sigma_{ij} n_j - \sigma_n n_i.$$

Then on Γ_C the boundary conditions are of the form

$$\sigma_n = -p((u_n - g)_+) C_n, \tag{1.4}$$

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$$|\sigma_T| \leq F((u_n - g)_+)\mu(|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|), \tag{1.5}$$

$$|\sigma_T| < F((u_n - g)_+)\mu(|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|) \quad \text{implies } \dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T = \mathbf{0}, \tag{1.6}$$

$$|\sigma_T| = F((u_n - g)_+)\mu(|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|) \quad \text{implies } \dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T = -\lambda\sigma_T. \tag{1.7}$$

Here g is a non negative function in $L^\infty(\Gamma_C)$ which represents the gap between the foundation and Ω, C_n is a positive function in $L^\infty(\Gamma_C)$, $\dot{\mathbf{U}}_T \in L^\infty(0, T; (L^2(\Gamma_C))^d)$, λ is non negative, and μ is a bounded positive function which is Lipschitz continuous. The dependence of μ on $\mathbf{x}$ is suppressed in the interest of simpler notation. The following lemma gives a way to simplify the above boundary conditions.

Lemma 1.1. *σ_T satisfies* (1.5)–(1.7) *if and only if*

$$\sigma_T \in -F((u_n - g)_+)\mu(|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|)\partial\eta(\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T) \tag{1.8}$$

where $\eta(\mathbf{x}) \equiv |\mathbf{x}|$.

Proof. Suppose first (1.5)-(1.7) and $\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T \neq \mathbf{0}$. Then from (1.7) and (1.6),

$$|\sigma_T| = F((u_n - g)_+)\mu(|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|)$$

and $\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T = -\lambda\sigma_T$. Thus $\lambda = \frac{|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|}{F((u_n - g)_+)\mu(|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|)}$ and so

$$\sigma_T = \frac{-F\big((u_n - g)_+\big)\mu(|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|)}{|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|}(\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T).$$

Now $\frac{\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T}{|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|} = \partial\eta(\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T)$ where $\eta(\mathbf{x}) \equiv |\mathbf{x}|$. Therefore, (1.8) holds.

Next suppose $\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T \neq \mathbf{0}$ and (1.8) holds. Since $\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T \neq \mathbf{0}$,

$$\partial\eta(\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T) = \frac{\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T}{|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|}$$

and so

$$\sigma_T = -F((u_n - g)_+)\mu(|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|)\frac{\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T}{|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|} = -\lambda(\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T).$$

Also since, in this case, $|\partial\eta(\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T)| = 1$, it follows

$$|\sigma_T| = F\big((u_n - g)_+\big)\mu(|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|).$$

Now suppose $\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T = \mathbf{0}$ and (1.8) holds. Then if $\mathbf{z} \in \partial\eta(\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T)$ such that equality holds, it follows $|\mathbf{z}| \leq 1$ and so $|\sigma_T| \leq F((u_n - g)_+)\mu(|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|)$.

Finally suppose $\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T = \mathbf{0}$ and (1.5) - (1.7). If $\sigma_T = \mathbf{0}$, let $\mathbf{z} = \mathbf{0}$ and then $\mathbf{0} \in \partial\eta(\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T)$ and

$$\sigma_T = -F((u_n - g)_+)\mu(|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|)\mathbf{0}$$

If $\sigma_T \neq \mathbf{0}$, then let

$$\mathbf{z} = \frac{\sigma_T}{-F((u_n - g)_+)\mu(|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|)} \in \partial\eta(\mathbf{0})$$

thanks to (1.5). Then

$$\sigma_T = -F((u_n - g)_+)\mu(|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|)\mathbf{z}$$

and so (1.8) holds. This proves the lemma. □

On Γ_0, the boundary condition is

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_0 \tag{1.9}$$

and on Γ_N,

$$\sigma \mathbf{n} = \mathbf{f}_n \quad \text{on } \Gamma_N. \tag{1.10}$$

Systems like the above model dynamic friction contact problems [11], [9], [4] [3]. The condition (1.4) is the normal compliance contact condition. It says the normal component of the traction force density is dependent on the normal penetration of the body into the foundation surface. See [11] for a discussion of the physical significance of this condition. Conditions (1.5) -(1.7) model friction. These conditions indicate the tangential part of the traction force density is bounded by a function determined by the normal force or penetration. No sliding takes place until $|\sigma_T|$ reaches this bound, $F((u_n - g)_+)\mu(0)$. When this occurs, the tangential force density has a direction opposite the relative tangential velocity (1.7). The dependence of the friction coefficient on the magnitude of the slip velocity, $|\dot{\mathbf{u}}_T - \dot{\mathbf{U}}_T|$ is interesting and so it has been included. It is assumed μ is a Lipschitz continuous function although the Lipschitz constant may be arbitrarily large. Of course in elementary physics, one allows this function to have two values, one if sliding occurs and another larger value if no sliding occurs. This will be considered later as a limit as the Lipschitz constant converges to ∞. All the functions may be assumed to depend on $\mathbf{x}$ but this dependence is often suppressed for the sake of simpler notation.

The material coming into contact with the foundation is an elastic-visco-plastic material for the stress, σ satisfying the following constitutive relation which was studied for a different kind of contact problem in [16].

$$\sigma(t) \equiv \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{E}\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\sigma(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon(\mathbf{u}(s)))ds \tag{1.11}$$

The physical explanation of this constitutive relation is described in this reference. This general form for the stress is the main new item in this paper. If $\mathcal{G}$ is equal to 0, the stress is like the one considered in [11], [8], [2] or [7].

For the sake of simplicity, I will consider the more general equation,

$$\sigma(t) \equiv \mathcal{A}\varepsilon(\mathbf{v}(t)) + \mathcal{E}\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\sigma(s), \varepsilon(\mathbf{v}(s)), \varepsilon(\mathbf{u}(s)))ds. \tag{1.12}$$

As in [16], the following is assumed on the functions, $\mathcal{A}$, $\mathcal{E}$, and $\mathcal{G}$.

$$\mathcal{A} : \Omega \times \mathfrak{S}^d \to \mathfrak{S}^d. \tag{1.13}$$

There exists $L_{\mathcal{A}} > 0$ such that

$$|\mathcal{A}(\mathbf{x}, \varepsilon_1) - \mathcal{A}(\mathbf{x}, \varepsilon_2)|_{\mathfrak{S}^d} \le L_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2|_{\mathfrak{S}^d} \tag{1.14}$$

for all $\varepsilon_1, \varepsilon_2 \in \mathfrak{S}^d$, a.e. $\mathbf{x} \in \Omega$. There exists $m_{\mathcal{A}} > 0$ such that

$$(\mathcal{A}(\mathbf{x}, \varepsilon_1) - \mathcal{A}(\mathbf{x}, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \ge m_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2|^2_{\mathfrak{S}^d} \tag{1.15}$$

for all $\varepsilon_1, \varepsilon_2 \in \mathfrak{S}^d$, a.e. $\mathbf{x} \in \Omega$.

For any $\varepsilon \in \mathfrak{S}^d, \mathbf{x} \to \mathcal{A}(\mathbf{x}, \varepsilon)$ is measurable on Ω and the mapping $\mathbf{x} \to \mathcal{A}(\mathbf{x}, \mathbf{0})$ is in $\mathcal{H}$.

$$\mathcal{E} : \Omega \times \mathfrak{S}^d \to \mathfrak{S}^d. \tag{1.16}$$

For any $\varepsilon \in \mathfrak{S}^d, \mathbf{x} \to \mathcal{A}(\mathbf{x}, \varepsilon)$ is measurable on Ω and the mapping $\mathbf{x} \to \mathcal{A}(\mathbf{x}, \mathbf{0})$ is in $\mathcal{H}$.

There exists $L_{\mathcal{E}} > 0$ such that

$$\|\mathcal{E}(\mathbf{x},\varepsilon_1) - \mathcal{E}(\mathbf{x},\varepsilon_2)\| \leq L_{\mathcal{E}}(\|\varepsilon_1 - \varepsilon_2\|) \tag{1.17}$$

for all $\varepsilon_1, \varepsilon_2 \in \mathfrak{S}^d, a.e.\ \mathbf{x} \in \Omega$.

For any $\varepsilon \in \mathfrak{S}^d, \mathbf{x} \to \mathcal{E}(\mathbf{x},\varepsilon)$ is measurable on Ω and the mapping $\mathbf{x} \to \mathcal{E}(\mathbf{x},\mathbf{0})$ is in $\mathcal{H}$.

$$\mathcal{G} : \Omega \times \mathfrak{S}^d \times \mathfrak{S}^d \times \mathfrak{S}^d \to \mathfrak{S}^d. \tag{1.18}$$

There exists $L_{\mathcal{G}} > 0$ such that

$$\|\mathcal{G}(\mathbf{x},\varepsilon_1',\varepsilon_1,\sigma_1) - \mathcal{G}(\mathbf{x},\varepsilon_2',\varepsilon_2,\sigma_2)\| \leq L_{\mathcal{G}}(\|\varepsilon_1' - \varepsilon_2'\| + \|\varepsilon_1 - \varepsilon_2\| + \|\sigma_1 - \sigma_2\|) \tag{1.19}$$

for all $\varepsilon_1', \varepsilon_2', \varepsilon_1, \varepsilon_2, \sigma_1, \sigma_2 \in \mathfrak{S}^d$, a.e. $\mathbf{x} \in \Omega$.

For any $\varepsilon', \varepsilon, \sigma \in \mathfrak{S}^d, \mathbf{x} \to \mathcal{G}(\mathbf{x},\varepsilon',\varepsilon,\sigma)$ is measurable on Ω and the mapping

$$\mathbf{x} \to \mathcal{G}(\mathbf{x},\mathbf{0},\mathbf{0},\mathbf{0})$$

is in $\mathcal{H}$. Also assume the following on p and F. The functions p and F are increasing and

$$\delta^2 r - K \leq p(r) \leq K(1+r), \quad r \geq 0, \tag{1.20}$$

$$p(r) = 0,\ r < 0,$$

$$F(r) \leq K(1+r) \quad r \geq 0, \tag{1.21}$$

$$F(r) = 0 \quad \text{if } r < 0,$$

$$|\mu(r_1) - \mu(r_2)| \leq \operatorname{Lip}(\mu)|r_1 - r_2|, \quad \|\mu\|_\infty \leq C, \tag{1.22}$$

and for $a = F, p$, and $r_1, r_2 \geq 0$,

$$|a(r_1) - a(r_2)| \leq K|r_1 - r_2|. \tag{1.23}$$

To allow for dependence on $\mathbf{x}$ of the functions, p and F,

$$\begin{gathered} \mathbf{x} \to p(\mathbf{x},r) \quad \text{is measurable on } \Gamma_C \\ \mathbf{x} \to F(\mathbf{x},r) \quad \text{is measurable on } \Gamma_C. \end{gathered} \tag{1.24}$$

However, this dependence on $\mathbf{x}$ will be usually ignored for the sake of simpler notation.

With the above conventions and definitions, the following existence theorem will be obtained.

Theorem 1.2. *Let Ω be a bounded open set in $\mathbb{R}^d$ having Lipschitz boundary. Then there exists a weak solution to the partial differential equation given by* (1.1) - (1.3)*, the boundary conditions given by* (1.4) - (1.10) *with the constitutive equation for σ given in* (1.11) *under the conditions given in* (1.13) - (1.24)*.*

The plan is to show the conditions of a fundamental existence theorem, presented in the next section are satisfied. First here are some function spaces and definitions of the same sort used in [16].

$\mathfrak{S}^d$ denotes the space of second order symmetric tensors on $\mathbb{R}^d$ with the usual Frobenius inner product,

$$A \cdot B \equiv A_{ij}B_{ij} = \operatorname{trace}(AB^T).$$

In which the repeated index summation convention is used as will be the case whenever convenient. It is always assumed Ω is a bounded open set having Lipschitz boundary. Define the following spaces.

$$\mathcal{H} \equiv \{\sigma = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$

with the norm and inner product given by

$$\|\sigma\|_{\mathcal{H}}^2 \equiv \int_\Omega \sigma_{ij}\sigma_{ij}dx, \quad (\sigma,\tau)_{\mathcal{H}} \equiv \int_\Omega \sigma_{ij}\tau_{ij}dx.$$

Also define

$$H_1 \equiv \{\mathbf{u} = (u_i) : \varepsilon(\mathbf{u}) \in \mathcal{H}\},$$

with an inner product given by

$$(\mathbf{u},\mathbf{v})_{H_1} \equiv (\mathbf{u},\mathbf{v})_{L^2(\Omega)^d} + (\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathcal{H}}$$

2. A fundamental existence theorem

The monograph, [12] describes the theory and application of set valued pseudomonotone maps. The definition given there is as follows.

Definition 2.1. *$A : V \to \mathcal{P}(V')$, for V a reflexive real Banach space and V' the space of linear functionals, is pseudomonotone if the following conditions hold.*

(1) *The set Au is non empty, bounded, closed, and convex for all $u \in V$.*
(2) *If F is a finite dimensional subspace of V, $u \in F$, and if U is a weakly open set in V' such that $U \supseteq Au$, then there exists $\delta > 0$ such that if $v \in B(u,\delta) \cap F$, then $Av \subseteq U$.*
(3) *If $u_i \rightharpoonup u$ in V and if $u_i^* \in Au_i$ is such that*
$$\limsup_{i\to\infty} \langle u_i^*, u_i - u\rangle_V \le 0, \tag{2.1}$$
then for each $v \in V$ there exists $u^*(v) \in Au$ such that
$$\liminf_{i\to\infty} \langle u_i^*, u_i - v\rangle_V \ge \langle u^*(v), u - v\rangle_V. \tag{2.2}$$

As a special case, the above is implied if the following simpler conditions hold.

Definition 2.2. $A : V \to \mathcal{P}(V')$, for V a reflexive real Banach space and V' the space of bounded linear functionals, is pseudomonotone if the following conditions hold.

(1) The set Au is non empty, bounded, closed, and convex for all $u \in V$ and the set,
$$\{u^* : u^* \in Au \text{ for } u \in B\}$$
for B a bounded set is bounded. Simply stated, A is bounded.
(2) If $u_i \rightharpoonup u$ in V and if $u_i^* \in Au_i$ is such that
$$\limsup_{i\to\infty} \langle u_i^*, u_i - u\rangle_V \le 0, \tag{2.3}$$
then for each $v \in V$ there exists $u^*(v) \in Au$ such that
$$\liminf_{i\to\infty} \langle u_i^*, u_i - v\rangle_V \ge \langle u^*(v), u - v\rangle_V. \tag{2.4}$$

The existence theorems in this paper are obtained from reducing to a situation in which the following theorem can be applied. [7]

Theorem 2.3. *Let V be a real Banach space and let H be a real Hilbert space containing V such that V is dense in H. Identify H and H'. Suppose $p \ge 2$, and define the space of solutions as follows:*

$$X \equiv \left\{u \in L^p(0,T;V) : u' \in L^{p'}(0,T;V')\right\}$$
$$\|u\|_X \equiv \|u\|_{L^p(0,T;V)} + \|u'\|_{L^{p'}(0,T;V')}$$

where the derivative is taken in the sense of V' valued distributions,

$$u'(\phi) \equiv -\int_0^T \phi'(t)u(t)dt$$

for all $\phi \in C_c^\infty(0,T)$, the space of test functions having compact support in $(0,T)$. Then suppose

$$A : X \to \mathcal{P}(X')$$

is pseudomonotone and for $\mathcal{V} \equiv L^p(0,T;V)$,

$$A : \mathcal{V} \to \mathcal{P}(\mathcal{V}')$$

is bounded and coercive in the sense that

$$\lim_{\|u\|_{\mathcal{V}} \to \infty, u \in X} \frac{\inf\{\langle u^*, u\rangle : u^* \in Au\}}{\|u\|_{\mathcal{V}}} = \infty$$

Also let $f \in \mathcal{V}'$. Then there exists a solution to the initial value problem,

$$u' + Au \ni f, \quad u(0) = u_0 \in H.$$

In the problem considered in this paper, $\mathcal{V}_t$ will equal $L^2(0,t;V)$ where V is a closed subspace of H_1 described above which also contains the functions, $C_c^\infty(\Omega)^d$. Specifically,

$$V \equiv \{\mathbf{u} \in H_1 : \mathbf{u} = \mathbf{0} \text{ on } \Gamma_0\}.$$

If no subscript is placed on $\mathcal{V}$ it will mean $t = T$. The Hilbert space mentioned in the above will be $L^2(\Omega)^d$

3. The abstract formulation

In the formula for $\sigma(t)$ given in (1.11) , denote by $\mathbf{v}$ the function, $\dot{\mathbf{u}}$. Then in terms of these functions,

$$\sigma(t) \equiv \mathcal{A}\varepsilon(\mathbf{v}(t)) + \mathcal{E}\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\sigma(s), \varepsilon(\mathbf{v}(s)), \varepsilon(\mathbf{u}(s)))ds$$

where

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s)ds \tag{3.1}$$

and it will always be assumed that $\mathbf{u}_0 \in V$.

I will also denote by K a constant which is larger than all the Lipschitz constants which could occur. Thus, for each $\mathbf{v} \in \mathcal{V}$, σ is a fixed point of the operator,

$$\Psi(\mathbf{v})\sigma(t) \equiv \mathcal{A}\varepsilon(\mathbf{v}(t)) + \mathcal{E}\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\sigma(s), \varepsilon(\mathbf{v}(s)), \varepsilon(\mathbf{u}(s)))ds \tag{3.2}$$

Lemma 3.1. *For each $\mathbf{v} \in \mathcal{V}$, there exists a unique fixed point $\sigma(t) \in L^2(0,T;\mathcal{H})$ for $\Psi(\mathbf{v})$. Also, letting σ_i be the fixed point corresponding to $\mathbf{v}_i$,*

$$|\sigma_1(t) - \sigma_2(t)|_{\mathcal{H}} \le K|\varepsilon(\mathbf{v}_1)(t) - \varepsilon(\mathbf{v}_2(t))|_{\mathcal{H}} + K\int_0^t |\varepsilon(\mathbf{v}_1(s)) - \varepsilon(\mathbf{v}_2(s))|_{\mathcal{H}}ds \tag{3.3}$$

It also follows there exist constants δ, C and K such that

$$(\sigma, \varepsilon(\mathbf{v}))_{\mathcal{H}} \ge \delta^2|\varepsilon(\mathbf{v}(t))|^2_{\mathcal{H}} - C - K\int_0^t |\varepsilon(\mathbf{v}(s))|^2_{\mathcal{H}}ds. \tag{3.4}$$

Letting σ_i be the fixed point corresponding to $\mathbf{v}_i$, there exist constants, δ, K such that

$$\begin{aligned}&(\sigma_1(t)-\sigma_2(t),\varepsilon(\mathbf{v}_1(t))-\varepsilon(\mathbf{v}_2(t)))_{\mathcal{H}}\\&\geq \delta^2|\varepsilon(\mathbf{v}_1(t))-\varepsilon(\mathbf{v}_2(t))|^2_{\mathcal{H}}-K\int_0^t|\varepsilon(\mathbf{v}_1(s))-\varepsilon(\mathbf{v}_2(s))|^2_{\mathcal{H}}ds.\end{aligned}\tag{3.5}$$

Proof. Consider the equivalent norm on $L^2(0,T;\mathcal{H})$,

$$\|\sigma\|^2_\lambda\equiv\int_0^T e^{-\lambda t}\|\sigma(t)\|^2dt$$

I will show if λ is large enough, $\Psi(\mathbf{v})$ is a contraction map. Let $\sigma_i\in L^2(0,T;\mathcal{H})$, $i=1,2$.

$$\begin{aligned}&\|\Psi(\mathbf{v})\sigma_1-\Psi(\mathbf{v})\sigma_2\|^2_\lambda\\&\equiv\int_0^T e^{-\lambda t}\Big\|\int_0^t\mathcal{G}(\sigma_1(s),\varepsilon(\mathbf{v}(s)),\varepsilon(\mathbf{u}(s)))-\mathcal{G}(\sigma_2(s),\varepsilon(\mathbf{v}(s)),\varepsilon(\mathbf{u}(s)))ds\Big\|^2dt\\&\leq K\int_0^T e^{-\lambda t}t\int_0^t\|\sigma_1(s)-\sigma_2(s)\|^2dsdt\\&=K\int_0^T\|\sigma_1(s)-\sigma_2(s)\|^2\int_s^T te^{-\lambda t}dtds\\&=\int_0^T\|\sigma_1(s)-\sigma_2(s)\|^2e^{-\lambda s}\int_s^T te^{\lambda(s-t)}dtds\\&\leq T(\frac{K}{\lambda})\int_0^T\|\sigma_1(s)-\sigma_2(s)||^2e^{-\lambda s}ds\\&=\frac{TK}{\lambda}\|\sigma_1(s)-\sigma_2(s)\|^2_\lambda\end{aligned}$$

Thus there exists a unique fixed point for $\Psi(\mathbf{v})$ as claimed.

Now consider (3.3). From the description of σ in (1.12), it follows there is a suitable constant, K such that

$$\begin{aligned}|\sigma_1(t)-\sigma_2(t)|_{\mathcal{H}}&\leq K(|\varepsilon(\mathbf{v}_1(t))-\varepsilon(\mathbf{v}_2(t))|_{\mathcal{H}}+\int_0^t|\varepsilon(\mathbf{v}_1(s))-\varepsilon(\mathbf{v}_2(s))|_{\mathcal{H}}ds)\\&\quad+K\int_0^t|\sigma_1(s)-\sigma_2(s)|_{\mathcal{H}}ds\end{aligned}$$

and now the desired result follows from Gronwall's inequality and adjusting constants.

Consider (3.4). First, it follows from the description of σ in (1.12) and the assumption that $\mathbf{x}\to\mathcal{G}(\mathbf{x},\mathbf{0},\mathbf{0},\mathbf{0})$ is in $\mathcal{H}$ that for some constants, C and K depending on the Lipschitz constants for $\mathcal{G}$ and the initial data,

$$\begin{aligned}|\sigma(t)|^2_{\mathcal{H}}&\leq K\Big(|\varepsilon(\mathbf{v}(t))|^2_{\mathcal{H}}+C+\int_0^t|\varepsilon(\mathbf{v}(s))|^2_{\mathcal{H}}ds\Big)\\&\quad+C+K\Big(\int_0^t|\sigma(s)|^2_{\mathcal{H}}+|\varepsilon(\mathbf{v}(s))|^2_{\mathcal{H}}ds\Big)\end{aligned}$$

and after adjusting the constants and using Gronwall's inequality,

$$|\sigma(t)|^2_{\mathcal{H}}\leq C+K|\varepsilon(\mathbf{v}(t))|^2_{\mathcal{H}}+K\int_0^t|\varepsilon(\mathbf{v}(s))|^2_{\mathcal{H}}ds.\tag{3.6}$$

Now from (1.12) and the assumptions on $\mathcal{A}, \mathcal{E}$, $\mathcal{G}$, there exist constants δ, C, K such that

$$\begin{aligned}(\sigma, \varepsilon(\mathbf{v}))_{\mathcal{H}} \geq \delta^2 |\varepsilon(\mathbf{v}(t))|_{\mathcal{H}}^2 - K|\varepsilon(\mathbf{u}(t))|_{\mathcal{H}}^2 - K\int_0^t |\sigma(s)|_{\mathcal{H}}^2 ds \\ - K\int_0^t |\varepsilon(\mathbf{v}(s))|_{\mathcal{H}}^2 ds - K\int_0^t |\varepsilon(\mathbf{u}(s))|_{\mathcal{H}}^2 ds - C\end{aligned}$$

and so, adjusting these constants, yields

$$(\sigma, \varepsilon(\mathbf{v}))_{\mathcal{H}} \geq \delta^2 |\varepsilon(\mathbf{v}(t))|_{\mathcal{H}}^2 - K\int_0^t |\varepsilon(\mathbf{v}(s))|_{\mathcal{H}}^2 ds - K\int_0^t |\sigma(s)|_{\mathcal{H}}^2 ds - C.$$

Now from (3.6), a further adjusting of constants yields

$$(\sigma(t), \varepsilon(\mathbf{v}(t)))_{\mathcal{H}} \geq \delta^2 |\varepsilon(\mathbf{v}(t))|_{\mathcal{H}}^2 - K\int_0^t |\varepsilon(\mathbf{v}(s))|_{\mathcal{H}}^2 ds - C$$

Finally, let σ_i correspond to $\mathbf{v}_i$. Then from the properties of $\mathcal{A}, \mathcal{E}, \mathcal{G}$, it follows there exists a constant, K such that

$$\begin{aligned}|\sigma_1(t) - \sigma_2(t)|_{\mathcal{H}} \leq K\Big(|\varepsilon(\mathbf{v}_1(t)) - \varepsilon(\mathbf{v}_2(t))|_{\mathcal{H}} + \int_0^t |\varepsilon(\mathbf{v}_1(s)) - \varepsilon(\mathbf{v}_2(s))|_{\mathcal{H}} ds\Big) \\ + K\int_0^t |\sigma_1(s) - \sigma_2(s)|_{\mathcal{H}} ds\end{aligned}$$

and so by Gronwall's inequality, it follows that after adjusting the constant,

$$|\sigma_1(t) - \sigma_2(t)|_{\mathcal{H}} \leq K\Big(|\varepsilon(\mathbf{v}_1(t)) - \varepsilon(\mathbf{v}_2(t))|_{\mathcal{H}} + \int_0^t |\varepsilon(\mathbf{v}_1(s)) - \varepsilon(\mathbf{v}_2(s))|_{\mathcal{H}} ds\Big)$$

which implies that on adjusting the constant again,

$$|\sigma_1(t) - \sigma_2(t)|_{\mathcal{H}}^2 \leq K\Big(|\varepsilon(\mathbf{v}_1(t)) - \varepsilon(\mathbf{v}_2(t))|_{\mathcal{H}}^2 + \int_0^t |\varepsilon(\mathbf{v}_1(s)) - \varepsilon(\mathbf{v}_2(s))|_{\mathcal{H}}^2 ds\Big). \quad (3.7)$$

Now

$$\begin{aligned}&(\sigma_1(t) - \sigma_2(t), \varepsilon(\mathbf{v}_1(t)) - \varepsilon(\mathbf{v}_2(t)))_{\mathcal{H}} \\ &\geq \delta^2 |\varepsilon(\mathbf{v}_1(t)) - \varepsilon(\mathbf{v}_2(t))|_{\mathcal{H}}^2 - K\int_0^t |\varepsilon(\mathbf{v}_1(s)) - \varepsilon(\mathbf{v}_2(s))|_{\mathcal{H}}^2 ds \\ &\quad - K\int_0^t |\sigma_1(s) - \sigma_2(s)|_{\mathcal{H}}^2 ds\end{aligned}$$

which, from (3.7), is greater than or equal to

$$\begin{aligned}&\delta^2 |\varepsilon(\mathbf{v}_1(t)) - \varepsilon(\mathbf{v}_2(t))|_{\mathcal{H}}^2 - K\int_0^t |\varepsilon(\mathbf{v}_1(s)) - \varepsilon(\mathbf{v}_2(s))|_{\mathcal{H}}^2 ds \\ &- K\int_0^t \int_0^s |\varepsilon(\mathbf{v}_1(r)) - \varepsilon(\mathbf{v}_2(r))|_{\mathcal{H}}^2 dr ds\end{aligned}$$

and adjusting the constants, is greater than or equal to

$$\delta^2 |\varepsilon(\mathbf{v}_1(t)) - \varepsilon(\mathbf{v}_2(t))|_{\mathcal{H}}^2 - K\int_0^t |\varepsilon(\mathbf{v}_1(s)) - \varepsilon(\mathbf{v}_2(s))|_{\mathcal{H}}^2 ds.$$

This proves the lemma. □

For the rest of this article, σ will be this unique fixed point satisfying

$$\sigma(t) \equiv \mathcal{A}\varepsilon(\mathbf{v}(t)) + \mathcal{E}\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\sigma(s), \varepsilon(\mathbf{v}(s)), \varepsilon(\mathbf{u}(s)))ds. \tag{3.8}$$

Recall

$$V \equiv \{\mathbf{u} \in H_1 : \mathbf{u} = \mathbf{0} \text{ on } \Gamma_0\}$$

and $\mathbf{f} \in \mathcal{V}'$ is defined by

$$\langle \mathbf{f}, \mathbf{w} \rangle \equiv \int_0^T (\mathbf{f}_b, \mathbf{w})_{L^2(\Omega)} dt + \int_0^T (\mathbf{f}_n, \mathbf{w})_{L^2(\Gamma_N)} dt \tag{3.9}$$

where $\mathbf{f}_b \in L^2(0, T; L^2(\Omega)^d)$ and $\mathbf{f}_n \in L^2(0, T; L^2(\Omega)^d)$. Also, I will continue to use the convention that $\mathbf{v} = \dot{\mathbf{u}}$ as described above.

Now let $\mathbf{w} \in V$ and consider the term, $\operatorname{div}(\sigma)$. Then from the boundary conditions,

$$\begin{aligned}
\int_\Omega \operatorname{div}(\sigma) \cdot \mathbf{w} dx &= -\int_\Omega \sigma \cdot \varepsilon(\mathbf{w}) dx + \int_{\Gamma_N} \mathbf{f}_n \cdot \mathbf{w} d\alpha + \int_{\Gamma_C} \sigma \mathbf{n} \cdot \mathbf{w} d\alpha \\
&= -\int_\Omega \sigma \cdot \varepsilon(\mathbf{w}) dx + \int_{\Gamma_N} \mathbf{f}_n \cdot \mathbf{w} d\alpha + \int_{\Gamma_C} \sigma_n \mathbf{n} \cdot \mathbf{w} d\alpha \\
&\quad + \int_{\Gamma_C} \sigma_T \cdot \mathbf{w} d\alpha \\
&= -\int_\Omega \sigma \cdot \varepsilon(\mathbf{w}) dx + \int_{\Gamma_N} \mathbf{f}_n \cdot \mathbf{w} d\alpha \\
&\quad - \int_{\Gamma_C} p((u_n - g)_+) C_n \mathbf{n} \cdot \mathbf{w} d\alpha + \int_{\Gamma_C} \sigma_T \cdot \mathbf{w}_T d\alpha
\end{aligned}$$

Let γ_T denote the operator

$$\gamma_T \mathbf{w} \equiv (\gamma \mathbf{w})_T = \gamma \mathbf{w} - \gamma(\mathbf{w} \cdot \mathbf{n})\mathbf{n}$$

where γ is the trace map on the boundary. Thus γ_T gives the tangential value of $\mathbf{w}$ on $\partial\Omega$. Then from Lemma (1.1), the boundary condition for the friction on Γ_C is of the form

$$\sigma_T \in -\gamma_T^* F((u_n - g)_+) \mu(|\mathbf{v}_T - \dot{\mathbf{U}}_T|) \partial\eta(\mathbf{v}_T - \dot{\mathbf{U}}_T)$$

where $\eta(\mathbf{x}) \equiv |\mathbf{x}|$ for $\mathbf{x} \in \mathbb{R}^d$. Now define an operator, $\Sigma : \mathcal{V} \to \mathcal{V}'$ as

$$\langle \Sigma \mathbf{v}, \mathbf{w} \rangle_{\mathcal{V}} \equiv \int_0^T \int_\Omega \sigma \cdot \varepsilon(\mathbf{w}) dx dt$$

where σ satisfies (3.8). Thus from Lemma (3.1)

$$\begin{aligned}
\langle \Sigma \mathbf{v}, \mathbf{v} \rangle_{\mathcal{V}} &\geq \int_0^T (\delta^2 |\varepsilon(\mathbf{v}(t))|^2_{\mathcal{H}} - C - K \int_0^t |\varepsilon(\mathbf{v}(s))|^2_{\mathcal{H}}) ds \\
&= \delta^2 \int_0^T |\varepsilon(\mathbf{v}(t))|^2_{\mathcal{H}} dt - K \int_0^T \int_0^t |\varepsilon(\mathbf{v}(s))|^2_{\mathcal{H}} ds\, dt - C
\end{aligned} \tag{3.10}$$

after adjusting the constants.

Also let $Q : \mathcal{V} \to \mathcal{P}(\mathcal{V})$ be defined by saying that $\mathbf{v}^* \in Q\mathbf{v}$ means there exists $\mathbf{z} \in L^\infty(0, T; L^\infty(\Gamma_C)^d)$ such that for all $\mathbf{w} \in \mathcal{V}$,

$$\int_0^T \int_{\Gamma_C} \mathbf{z} \cdot \mathbf{w}_T d\alpha\, dt \leq \int_0^T \int_{\Gamma_C} |\mathbf{v}_T - \dot{\mathbf{U}}_T + \mathbf{w}_T| - |\mathbf{v}_T - \dot{\mathbf{U}}_T| d\alpha\, dt. \tag{3.11}$$

and

$$\langle \mathbf{v}^*, \mathbf{w}\rangle = \int_0^T \int_{\Gamma_C} F((u_n - g)_+)\mu(|\mathbf{v}_T - \dot{\mathbf{U}}_T|)\mathbf{z} \cdot \mathbf{w}_T d\alpha\, dt \tag{3.12}$$

The following lemma will be useful later.

Lemma 3.2. *In case the function F is bounded, there exists a constant, K such that if $\mathbf{v}_i^* \in Q\mathbf{v}_i$ for $i = 1, 2$,*

$$\langle \mathbf{v}_1^* - \mathbf{v}_2^*, \mathbf{v}_1 - \mathbf{v}_2\rangle_{\mathcal{V}_t} \geq -K \int_0^t ||\mathbf{v}_1(s) - \mathbf{v}_2(s)||_U^2 ds$$

where U is a Sobolev space with the property that V embeds compactly into U and the trace map from U to $L^2(\Gamma_C)^d$ is continuous.

Proof. Let $(\mathbf{v}_i, \mathbf{z}_i)$ for $i = 1, 2$ be such that $\mathbf{v}_i^* \in Q\mathbf{v}_i$ is given by (3.12). Then

$$\begin{aligned}
&\langle \mathbf{v}_1^* - \mathbf{v}_2^*, \mathbf{v}_1 - \mathbf{v}_2\rangle_{\mathcal{V}_t} \\
&= \int_0^t \int_{\Gamma_C} \Big[F((u_{1n} - g)_+)\mu(|\mathbf{v}_{1T} - \dot{\mathbf{U}}_T|)\mathbf{z}_1 \\
&\quad - F((u_{2n} - g)_+)\mu(|\mathbf{v}_{2T} - \dot{\mathbf{U}}_T|)\mathbf{z}_2\Big] \cdot (\mathbf{v}_1 - \mathbf{v}_2)_T d\alpha\, ds \\
&= \int_0^t \int_{\Gamma_C} F((u_{1n} - g)_+)\mu(|\mathbf{v}_{1T} - \dot{\mathbf{U}}_T|)(\mathbf{z}_1 - \mathbf{z}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2)_T d\alpha\, ds \\
&\quad + \int_0^t \int_{\Gamma_C} \Big(F((u_{1n} - g)_+)\mu(|\mathbf{v}_{1T} - \dot{\mathbf{U}}_T|) \\
&\quad - F((u_{2n} - g)_+)\mu(|\mathbf{v}_{2T} - \dot{\mathbf{U}}_T|)\Big)\mathbf{z}_2 \cdot (\mathbf{v}_1 - \mathbf{v}_2)_T d\alpha\, ds
\end{aligned} \tag{3.13}$$

Now the expression in the first term of the right hand side is nonnegative because of (3.11). Therefore, $\langle \mathbf{v}_1^* - \mathbf{v}_2^*, \mathbf{v}_1 - \mathbf{v}_2\rangle_{\mathcal{V}}$ is bounded below by the expression in (3.13). The absolute value of this is bounded above by an expression of the form

$$\begin{aligned}
&K \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_U \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_U ds + \\
&K \int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_U^2 ds.
\end{aligned}$$

Now adjusting the constants, this is dominated by an expression of the form

$$K \int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_U^2 ds$$

which proves the lemma. □

Finally, define $P : \mathcal{V}_t \to \mathcal{V}_t'$ by

$$\langle P\mathbf{u}, \mathbf{w}\rangle_{\mathcal{V}} \equiv \int_0^t \int_{\Gamma_C} p((u_n - g)_+)C_n \mathbf{n} \cdot \mathbf{w} d\alpha.$$

Thus, letting $J'(r) \equiv C_n p(r_+)$,

$$\begin{aligned}
\langle P\mathbf{u}, \mathbf{v}\rangle_{\mathcal{V}} &= \int_0^t \int_{\Gamma_C} p((u_n - g)_+)C_n v_n d\alpha \\
&= \int_{\Gamma_C} (J(u_n(t) - g) - J(u_{0n} - g))d\alpha \geq C
\end{aligned}$$

where C depends on u_{0n}. Also from the assumptions on p, for each $\varepsilon > 0$ there exists a constant, K_ε such that

$$|\langle P\mathbf{u}_1 - P\mathbf{u}_2, \mathbf{v}_1 - \mathbf{v}_2\rangle_{\mathcal{V}_t}| \leq \varepsilon \int_0^t \|\mathbf{v}_1 - \mathbf{v}_2\|_V^2 ds + K_\varepsilon \int_0^t \int_0^s \|\mathbf{v}_1 - \mathbf{v}_2\|_V^2 dr ds \quad (3.14)$$

4. Existence and uniqueness for an abstract formulation

It follows an abstract formulation of the above initial boundary value problem, (1.1) - (1.10) where the stress is given by (3.8) is

$$\mathbf{v}' + \Sigma\mathbf{v} + Q\mathbf{v} + P\mathbf{u} \ni \mathbf{f} \text{ in } \mathcal{V}', \quad \mathbf{v}(0) = \mathbf{v}_0 \in L^2(\Omega)^d. \quad (4.1)$$

Solutions to this abstract inclusion are the weak solutions of Theorem 1.2. From now on a prime will denote the weak time derivative in the sense of V' valued distributions.

Theorem 4.1. *Let $\mathbf{u}_0 \in V$ and let $\mathbf{v}_0 \in L^2(\Omega)^d$. Then there exists a solution to* (4.1)*. This solution satisfies*

$$\mathbf{v} \in \mathcal{V}, \quad \mathbf{v} \in C(0, T; L^2(\Omega)^d).$$

Proof. I will not consider (4.1) directly. Instead, it will be reformulated in terms of a new dependent variable, $\mathbf{v}_\lambda$ defined by

$$e^{\lambda t}\mathbf{v}_\lambda(t) \equiv \mathbf{v}(t)$$

because the problem in terms of this new dependent variable will satisfy the hypotheses of the Theorem 2.3 stated above.

Then with this definition, $\mathbf{v}$ is a solution of (4.1) if and only if $\mathbf{v}_\lambda$ is a solution of

$$\begin{gathered}\mathbf{v}_\lambda' + \lambda\mathbf{v}_\lambda + e^{-\lambda(\cdot)}\Sigma(e^{\lambda(\cdot)}\mathbf{v}_\lambda) + e^{-\lambda(\cdot)}Q(e^{\lambda(\cdot)}\mathbf{v}_\lambda) + e^{-\lambda(\cdot)}P(\mathbf{u}) \ni e^{-\lambda(\cdot)}\mathbf{f} \quad \text{in } \mathcal{V}', \\ \mathbf{v}_\lambda(0) = \mathbf{v}_0.\end{gathered} \quad (4.2)$$

From (3.4) and the definition of Σ,

$$\begin{aligned}
&\langle e^{-\lambda(\cdot)}\Sigma(e^{\lambda(\cdot)}\mathbf{v}_\lambda), \mathbf{v}_\lambda\rangle_{\mathcal{V}} \\
&= \langle e^{-2\lambda(\cdot)}\Sigma(e^{\lambda(\cdot)}\mathbf{v}_\lambda), e^{\lambda(\cdot)}\mathbf{v}_\lambda\rangle_{\mathcal{V}} \\
&\geq \int_0^T e^{-2\lambda t}\Big(e^{2\lambda t}\delta^2|\varepsilon(\mathbf{v}_\lambda(t))|_{\mathcal{H}}^2 - C - K\int_0^t e^{2\lambda s}|\varepsilon(\mathbf{v}_\lambda(s))|_{\mathcal{H}}^2 ds\Big)dt \\
&\geq \delta^2\|\varepsilon(\mathbf{v}_\lambda)\|_{L^2(0,T;\mathcal{H})}^2 - C - K\int_0^T\int_0^t e^{-2\lambda(t-s)}|\varepsilon(\mathbf{v}_\lambda(s))|_{\mathcal{H}}^2 ds dt \\
&\geq \delta^2\|\varepsilon(\mathbf{v}_\lambda)\|_{L^2(0,T;\mathcal{H})}^2 - C - K\int_0^T\int_s^T e^{-2\lambda(t-s)}|\varepsilon(\mathbf{v}_\lambda(s))|_{\mathcal{H}}^2 dt ds \\
&\geq \delta^2\|\varepsilon(\mathbf{v}_\lambda)\|_{L^2(0,T;\mathcal{H})}^2 - C - K\frac{1}{2\lambda}\int_0^T |\varepsilon(\mathbf{v}_\lambda(s))|_{\mathcal{H}}^2 ds \\
&\geq \delta^2\|\mathbf{v}_\lambda\|_{\mathcal{V}}^2 - \delta^2|\mathbf{v}_\lambda|_{L^2(0,T;L^2(\Omega)^d)}^2 - C - K\frac{1}{2\lambda}\|\mathbf{v}_\lambda\|_{\mathcal{V}}^2
\end{aligned}$$

Thus, if λ is large enough, the above expression is greater than or equal to

$$(\delta^2/2)\|\mathbf{v}_\lambda\|_{\mathcal{V}}^2 - C - \delta^2|\mathbf{v}_\lambda|_{L^2(0,T;L^2(\Omega)^d)}^2.$$

From the assumptions on F,μ,

$$
\begin{aligned}
&|\langle e^{-\lambda(\cdot)}Q(e^{\lambda(\cdot)}\mathbf{v}_\lambda),\mathbf{v}_\lambda\rangle_{\mathcal{V}}| \\
&\leq C\int_0^T e^{-\lambda t}\|\mathbf{u}(t)\|_V\|\mathbf{v}_\lambda(t)\|_V dt \\
&\leq C+\int_0^T e^{-\lambda t}\int_0^t \|\mathbf{v}_\lambda(s)\|ds\|\mathbf{v}_\lambda(t)\|_V dt \\
&\leq C+(\delta^2/16)\|\mathbf{v}_\lambda\|_{\mathcal{V}}^2+C_\delta\int_0^T \Big(e^{-\lambda t}\int_0^t\|\mathbf{v}_\lambda(s)\|ds\Big)^2 dt \\
&\leq C+(\delta^2/16)\|\mathbf{v}_\lambda\|_{\mathcal{V}}^2+C_\delta\int_0^T e^{-\lambda t}t\int_0^t \|\mathbf{v}_\lambda(s)||^2 ds dt \\
&= C+(\delta^2/16)\|\mathbf{v}_\lambda\|_{\mathcal{V}}^2+C_\delta\int_0^T\int_s^T e^{-\lambda t}t dt\|\mathbf{v}_\lambda(s)\|^2 ds \\
&\leq C+(\delta^2/16)\|\mathbf{v}_\lambda\|_{\mathcal{V}}^2+C_\delta T(\frac{1}{\lambda})\int_0^T\|\mathbf{v}_\lambda(s)\|^2 ds \\
&\leq C+(\delta^2/8)\|\mathbf{v}_\lambda\|_{\mathcal{V}}^2
\end{aligned}
$$

provided λ is large enough. Next using the growth condition for p and adjusting the constants as the computation proceeds,

$$
\begin{aligned}
|\langle e^{-\lambda(\cdot)}P(\mathbf{u}),\mathbf{v}_\lambda\rangle_{\mathcal{V}}| &= |\int_0^T e^{-\lambda t}\int_{\Gamma_C} p((u_n-g)_+)C_n v_{\lambda n} d\alpha\, dt| \\
&\leq C_n K\int_0^T e^{-\lambda t}\int_{\Gamma_C}(1+|u_n(t)|)|v_{\lambda n}(t)|d\alpha\, dt \\
&\leq (\delta^2/16)\|\mathbf{v}_\lambda\|_{\mathcal{V}}^2+C_\delta\int_0^T(e^{-\lambda t}\int_{\Gamma_C}(1+|u_n(t)|)d\alpha)^2 dt \\
&\leq (\delta^2/16)\|\mathbf{v}_\lambda\|_{\mathcal{V}}^2+C_\delta\int_0^T e^{-2\lambda t}\int_{\Gamma_C}(1+|u_n(t)|^2)d\alpha\, dt \\
&\leq (\delta^2/16)\|\mathbf{v}_\lambda\|_{\mathcal{V}}^2+C_\delta\int_0^T e^{-2\lambda t}(1+\|\mathbf{u}(t)\|_V^2)dt \\
&\leq (\delta^2/16)\|\mathbf{v}_\lambda\|_{\mathcal{V}}^2+C_\delta\int_0^T e^{-2\lambda t}\int_0^t\|\mathbf{v}(s)\|^2 ds dt+C_\delta/\lambda \\
&\leq (\delta^2/16)\|\mathbf{v}_\lambda\|_{\mathcal{V}}^2+C_\delta/\lambda\int_0^T||\mathbf{v}(s)\|^2 ds+C_\delta/\lambda \\
&\leq (\delta^2/8)\|\mathbf{v}_\lambda\|_{\mathcal{V}}^2+1
\end{aligned}
$$

whenever λ is large enough. Letting

$$
A\mathbf{v}_\lambda\equiv\lambda\mathbf{v}_\lambda+e^{-\lambda(\cdot)}\Sigma(e^{\lambda(\cdot)}\mathbf{v}_\lambda)+e^{-\lambda(\cdot)}Q(e^{\lambda(\cdot)}\mathbf{v}_\lambda)+e^{-\lambda(\cdot)}P(\mathbf{u})
$$

It follows (4.2) is of the form

$$
\mathbf{v}_\lambda'+A\mathbf{v}_\lambda\ni e^{-\lambda(\cdot)}\mathbf{f} \tag{4.3}
$$

and $A:\mathcal{V}\to\mathcal{P}(\mathcal{V})$ is coercive as described in Theorem 2.3 whenever λ is sufficiently large.

It is clear from the definition that A is bounded. I need to verify A is pseudomonotone and then the existence of a solution will follow from Theorem 2.3. Letting $\mathbf{u}_i$ correspond to $\mathbf{v}_i$ as described above and then $\mathbf{v}_{\lambda i}$ also be given as above, it follows from (3.5)

$$\begin{aligned}
&\Big\langle e^{-\lambda(\cdot)}\Sigma(e^{\lambda(\cdot)}\mathbf{v}_{\lambda 1}) + e^{-\lambda(\cdot)}P(\mathbf{u}_1) - (e^{-\lambda(\cdot)}\Sigma(e^{\lambda(\cdot)}\mathbf{v}_{\lambda 1}) + e^{-\lambda(\cdot)}P(\mathbf{u}_1)), \mathbf{v}_{\lambda 1} - \mathbf{v}_{\lambda 2}\Big\rangle_{\mathcal{V}}\\
&\geq \int_0^T \delta^2|\varepsilon(\mathbf{v}_{\lambda 1}(t)) - \varepsilon(\mathbf{v}_{\lambda 2}(t))|^2_{\mathcal{H}} - Ke^{-2\lambda t}\int_0^t e^{2\lambda s}|\varepsilon(\mathbf{v}_{\lambda 1}(s)) - \varepsilon(\mathbf{v}_{\lambda 2}(s))|^2_{\mathcal{H}} ds\,dt\\
&- K\int_0^T e^{-\lambda t}\int_{\Gamma_C}|u_{1n} - u_{2n}||v_{\lambda 1n} - v_{\lambda 2n}|d\alpha\,dt
\end{aligned}$$

Using the continuity of the trace maps, this dominates

$$\begin{aligned}
&\int_0^T \delta^2|\varepsilon(\mathbf{v}_{\lambda 1}(t)) - \varepsilon(\mathbf{v}_{\lambda 2}(t))|^2_{\mathcal{H}} - Ke^{-2\lambda t}\int_0^t e^{2\lambda s}|\varepsilon(\mathbf{v}_{\lambda 1}(s)) - \varepsilon(\mathbf{v}_{\lambda 2}(s))|^2_{\mathcal{H}} ds\,dt\\
&- C_\delta\int_0^T e^{-2\lambda t}\|\mathbf{u}_1 - \mathbf{u}_2\|^2_V dt - (\delta^2/2)\int_0^T \|\mathbf{v}_{\lambda 1} - \mathbf{v}_{\lambda 2}\|^2_V dt\\
&\geq \delta^2/2\int_0^T \|\mathbf{v}_{\lambda 1} - \mathbf{v}_{\lambda 2}\|^2_V - Ke^{-2\lambda t}\int_0^t e^{2\lambda s}|\varepsilon(\mathbf{v}_{\lambda 1}(s)) - \varepsilon(\mathbf{v}_{\lambda 2}(s))|^2_{\mathcal{H}} dsdt\\
&\quad - \delta^2/2\int_0^T |\mathbf{v}_{\lambda 1} - \mathbf{v}_{\lambda 2}|^2_{L^2(\Omega)^d} dt - C_\delta\int_0^T e^{-2\lambda t}\int_0^t e^{2\lambda s}\|\mathbf{v}_{\lambda 1} - \mathbf{v}_{\lambda 2}\|^2 dsdt\\
&\geq \delta^2/2\int_0^T \|\mathbf{v}_{\lambda 1} - \mathbf{v}_{\lambda 2}\|^2_V dt - \frac{K + C_\delta}{\lambda}\int_0^T \|\mathbf{v}_{\lambda 1} - \mathbf{v}_{\lambda 2}\|^2 dt\\
&\quad - \frac{\delta^2}{2}\int_0^T |\mathbf{v}_{\lambda 1} - \mathbf{v}_{\lambda 2}|^2_{L^2(\Omega)^d} dt
\end{aligned}$$

It follows that for all λ large enough, $B\mathbf{v}_\lambda$ given by

$$B\mathbf{v}_\lambda \equiv \lambda\mathbf{v}_\lambda + e^{-\lambda(\cdot)}\Sigma(e^{\lambda(\cdot)}\mathbf{v}_\lambda) + e^{-\lambda(\cdot)}P(\mathbf{u}) \tag{4.4}$$

is monotone and bounded as a map from $\mathcal{V}$ to $\mathcal{V}'$. It is also clearly hemicontinuous, meaning it is continuous on line segments. Therefore, this operator is pseudomonotone. Since $A = B + e^{-\lambda(\cdot)}Q$, it only remains to verify $e^{-\lambda(\cdot)}Q$ is also pseudomonotone.

This operator is clearly bounded. It only remains to verify the pseudomonotone condition as a map from the space of solutions, X described above to $\mathcal{P}(X)$. To do this, it is helpful to use the following two interesting Theorems found in Lions [10] and Seidman [14].

Theorem 4.2. *If $p \geq 1$, $q > 1$, and $W \subseteq U \subseteq Y$ where the inclusion map of W into U is compact and the inclusion map of U into Y is continuous, let*

$$S = \{\mathbf{u} \in L^p(0,T;W) : \mathbf{u}' \in L^q(0,T;Y) \text{ and } \|\mathbf{u}\|_{L^p(0,T;W)} + \|\mathbf{u}'\|_{L^q(0,T;Y)} < R\}$$

Then S is pre compact in $L^p(0,T;U)$.

Theorem 4.3. *Let W, U, and Y be as in Theorem 4.2 and let*

$$S = \{\mathbf{u} : \|\mathbf{u}(t)\|_W + \|\mathbf{u}'\|_{L^q(0,T;Y)} \leq R \text{ for } t \in [0,T]\}$$

for $q > 1$. Then S is pre compact in $C(0,T;U)$.

It suffices to verify that if $\mathbf{v}_k$ converges to $\mathbf{v}$ weakly in X and if $\mathbf{v}_k^* \in Q\mathbf{v}_k$ then for any $\mathbf{w} \in X$, there exists $\mathbf{v}^*(\mathbf{w}) \in Q\mathbf{v}$ such that

$$\liminf_{k\to\infty} \langle \mathbf{v}_k^*, \mathbf{v}_k - \mathbf{w}\rangle \geq \langle \mathbf{v}^*(\mathbf{w}), \mathbf{v} - \mathbf{w}\rangle.$$

This will imply the same is true of $e^{-\lambda(\cdot)}Qe^{\lambda(\cdot)}$ which will show $e^{-\lambda(\cdot)}Qe^{\lambda(\cdot)}$ is pseudomonotone on X. Suppose then that $\mathbf{v}_k$ converges weakly to $\mathbf{v}$ in X and that $\mathbf{v}_k^* \in Q\mathbf{v}_k$ and let $\mathbf{z}_k$ be the element of $\partial\eta(\mathbf{v}_T - \dot{\mathbf{U}}_T)$ for $\eta(\mathbf{x}) \equiv |\mathbf{x}|$ which satisfies

$$\langle \mathbf{v}_k^*, \mathbf{w}\rangle = \int_0^T \int_{\Gamma_C} F((u_{kn} - g)_+)\mu(|\mathbf{v}_{kT} - \dot{\mathbf{U}}_T|)\mathbf{z}_k \cdot \mathbf{w}_T d\alpha\, dt.$$

I need to verify that for all $\mathbf{w} \in X$ there exists $\mathbf{v}^*(\mathbf{w}) \in Q\mathbf{v}$, such that

$$\liminf_{k\to\infty} \langle \mathbf{v}_k^*, \mathbf{v}_k - \mathbf{w}\rangle \geq \langle \mathbf{v}^*(\mathbf{w}), \mathbf{v} - \mathbf{w}\rangle$$

Suppose this does not happen. Then there exists a sequence as described above and $\mathbf{w} \in X$ such that for every $\mathbf{v}^* \in Q\mathbf{v}$,

$$\liminf_{k\to\infty} \langle \mathbf{v}_k^*, \mathbf{v}_k - \mathbf{w}\rangle < \langle \mathbf{v}^*, \mathbf{v} - \mathbf{w}\rangle \tag{4.5}$$

Since $\{\mathbf{z}_k\}$ is bounded in $L^\infty(0,T;L^\infty(\Gamma_C)^d)$ it has a subsequence which converges weak $*$ to $\mathbf{z} \in L^\infty(0,T;L^\infty(\Gamma_C)^d)$. Let U be a Sobolev space such that V embeds compactly into U and the trace map from U to $L^2(\Gamma_C)^d$ is continuous. By Theorem 4.3 there is a further subsequence such that $\mathbf{u}_k$ converges strongly to $\mathbf{u}$ in $C(0,T;U)$ and by Theorem 4.2 there is a further subsequence such that $\mathbf{v}_k$ converges strongly to $\mathbf{v}$ in $L^2(0,T;U)$. Also $\mathbf{v}_k^*$ is bounded in $\mathcal{V}'$ so a further subsequence converges weak $*$ to $\mathbf{v}^*$. Now using this final subsequence,

$$\int_0^T \int_{\Gamma_C} \mathbf{z}_k \cdot \mathbf{w}_T d\alpha\, dt \leq \int_0^T \int_{\Gamma_C} |\mathbf{v}_{kT} - \dot{\mathbf{U}}_T + \mathbf{w}_T| - |\mathbf{v}_{kT} - \dot{\mathbf{U}}_T| d\alpha\, dt.$$

and so, passing to the limit yields

$$\int_0^T \int_{\Gamma_C} \mathbf{z} \cdot \mathbf{w}_T d\alpha\, dt \leq \int_0^T \int_{\Gamma_C} |\mathbf{v}_T - \dot{\mathbf{U}}_T + \mathbf{w}_T| - |\mathbf{v}_T - \dot{\mathbf{U}}_T| d\alpha\, dt.$$

and

$$\langle \mathbf{v}_k^*, \mathbf{v}_k - \mathbf{w}\rangle = \int_0^T \int_{\Gamma_C} F((u_{kn} - g)_+)\mu(|\mathbf{v}_{kT} - \dot{\mathbf{U}}_T|)\mathbf{z}_k \cdot (\mathbf{v}_k - \mathbf{w})_T d\alpha\, dt$$

so passing to a limit in this expression yields

$$\langle \mathbf{v}^*, \mathbf{v} - \mathbf{w}\rangle = \int_0^T \int_{\Gamma_C} F((u_n - g)_+)\mu(|\mathbf{v}_T - \dot{\mathbf{U}}_T|)\mathbf{z} \cdot (\mathbf{v} - \mathbf{w})_T d\alpha\, dt$$

showing that

$$\lim_{k\to\infty} \langle \mathbf{v}_k^*, \mathbf{v}_k - \mathbf{w}\rangle = \langle \mathbf{v}^*, \mathbf{v} - \mathbf{w}\rangle$$

and that $\mathbf{v}^* \in Q\mathbf{v}$ contradicting (4.5). This shows A satisfies all the conditions of Theorem 2.3 and this proves Theorem 4.1. □

Next consider the question of uniqueness.

Theorem 4.4. *In the case that the function F is bounded, the solution to Theorem 4.1 is unique.*

Proof. Recall the abstract equation of Theorem 4.1 is

$$\mathbf{v}' + \Sigma\mathbf{v} + Q\mathbf{v} + P\mathbf{u} \ni \mathbf{f} \text{ in } \mathcal{V}', \quad \mathbf{v}(0) = \mathbf{v}_0 \in L^2(\Omega)^d$$

where the operators are defined above. Suppose $\mathbf{v}_i$ each are solutions for $i = 1, 2$ and denote by $\mathbf{v}_i^* \in Q\mathbf{v}_i$ that which makes the above inclusion an equality. Then it follows from Lemma (3.2) there exists a constant, K such that

$$\langle \mathbf{v}_1^* - \mathbf{v}_2^*, \mathbf{v}_1 - \mathbf{v}_2 \rangle_{\mathcal{V}_t} \geq -K \int_0^t ||\mathbf{v}_1(s) - \mathbf{v}_2(s)||_U^2 ds$$

where U is a Sobolev space such that the embedding of V into U is compact. From (3.14),

$$\langle P\mathbf{u}_1 - P\mathbf{u}_2, \mathbf{v}_1 - \mathbf{v}_2 \rangle_{\mathcal{V}_t} \geq -\varepsilon \int_0^t ||\mathbf{v}_1 - \mathbf{v}_2\|_V^2 ds - K_\varepsilon \int_0^t \int_0^s \|\mathbf{v}_1 - \mathbf{v}_2\|_V^2 dr\, ds$$

Also from (3.5) it follows

$$\langle \Sigma\mathbf{v}_1 - \Sigma\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2 \rangle_{\mathcal{V}_t} \geq \delta^2 \int_0^t \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_V^2 ds - K \int_0^t \int_0^s \|\mathbf{v}_1 - \mathbf{v}_2\|_V^2 drds.$$

Letting $\varepsilon < \delta^2/2$ and adjusting the constants, it follows

$$\begin{aligned} &\frac{1}{2}|\mathbf{v}_1(t) - \mathbf{v}_2(t)|_{L^2(\Omega)^d}^2 + \frac{\delta^2}{2} \int_0^t \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_V^2 ds \\ &\leq K_\varepsilon \int_0^t \int_0^s \|\mathbf{v}_1 - \mathbf{v}_2\|_V^2 drds + K \int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_U^2 ds \end{aligned} \tag{4.6}$$

Now the compactness of the embedding of V into U implies for every $\varepsilon > 0$ there exists a constant C_ε such that

$$\|\mathbf{w}\|_U^2 \leq \varepsilon ||\mathbf{w}\|_V^2 + C_\varepsilon |\mathbf{w}|_{L^2(\Omega)^d}^2.$$

Choosing ε small enough, (4.6) implies

$$\begin{aligned} &\frac{1}{2}|\mathbf{v}_1(t) - \mathbf{v}_2(t)|_{L^2(\Omega)^d}^2 + \frac{\delta^2}{4} \int_0^t \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_V^2 ds \\ &\leq K_\varepsilon \int_0^t \int_0^s \|\mathbf{v}_1 - \mathbf{v}_2\|_V^2 drds + K_\varepsilon \int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_{L^2(\Omega)^d}^2 ds \end{aligned}$$

and now the conclusion that $\mathbf{v}_1 = \mathbf{v}_2$ follows from Gronwall's inequality. This proves the theorem. □

5. The case of discontinuous μ

Assuming F is bounded, it is possible, as in [7, 9] to extend the existence part of the above results to the case where the coefficient of friction, μ is discontinuous. This is the situation discussed in every elementary physics book where static friction is greater than sliding friction. Specifically, assume the function μ, has a jump discontinuity at 0, becoming smaller when the sliding speed is positive. Because of the discontinuity of μ the definition of the friction operator, Q will be modified slightly as follows: $\mathbf{v}^* \in Q\mathbf{v}$ will mean

$$\langle \mathbf{v}^*, \mathbf{w} \rangle_{\mathcal{V}} \leq \int_0^T \int_{\Gamma_C} F((u_n - g)_+)\psi \cdot \big[|\mathbf{v}_T - \dot{\mathbf{U}}_T + \mathbf{w}_T| - |\mathbf{v}_T - \dot{\mathbf{U}}_T|\big] d\alpha\, dt$$

where ψ is in the graph of μ a.e. $(t,\mathbf{x})$ where μ is the coefficient of friction, assumed to be decreasing and with a jump at 0. The question of uniqueness is open but the existence theorem is the following.

Theorem 5.1. *Let $\mathbf{u}_0 \in V$, $\mathbf{v}_0 \in L^2(\Omega)^d$. Also let $\mu(0+) < \mu(0)$ and μ is decreasing and Lipschitz continuous on $(0,\infty)$. Then there exists a solution, $\mathbf{v}$, to the following problem.*

$$\begin{gathered}
\mathbf{v} \in \mathcal{V}, \quad \mathbf{v}' \in \mathcal{V}', \quad (u_n - g)_+ \in L^\infty(0,T;L^2(\Gamma_C)),\\
\mathbf{v}' + \Sigma\mathbf{v} + P(\mathbf{u}) + Q(\mathbf{v}) \ni \mathbf{f} \quad \text{in } \mathcal{V}',\\
\mathbf{v}(0) = \mathbf{v}_0, \quad \mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s)ds,
\end{gathered}$$

where $\mathbf{v}^* \in Q(\mathbf{v})$ means

$$\langle \mathbf{v}^*, \mathbf{w}\rangle_{\mathcal{V}} \le \int_0^T \int_{\Gamma_C} F((u_n-g)_+)\psi\big[|\mathbf{v}_T - \dot{\mathbf{U}}_T + \mathbf{w}_T| - |\mathbf{v}_T - \dot{\mathbf{U}}_T|\big] d\alpha\, dt$$

where for a.e. $(t,\mathbf{x})$,

$$\psi(t,\mathbf{x}) \in [\mu(0+), \mu(0)]$$

whenever $(\mathbf{v}_T - \dot{\mathbf{U}}_T)(t,\mathbf{x}) = 0$ and if $(\mathbf{v}_T - \dot{\mathbf{U}}_T)(t,\mathbf{x}) \neq 0$, then for a.e. $(t,\mathbf{x})$,

$$\psi(t,\mathbf{x}) = \mu(|\mathbf{v}_T - \dot{\mathbf{U}}_T|(t,\mathbf{x})).$$

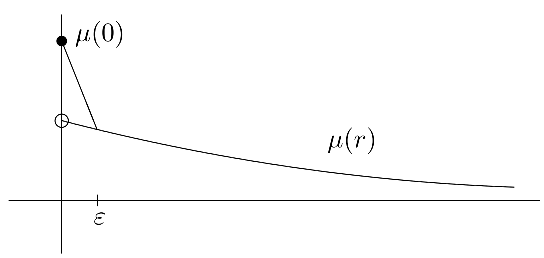
This solution is the weak limit in X of solutions to the friction contact problem in which the coefficient of friction is Lipschitz continuous.

Proof. In the following argument, it is assumed that whenever necessary, the functions involved are product measurable representatives.

Let $\mu_\varepsilon(r) = \mu(r)$ for all $r > \varepsilon$, μ_ε a decreasing function, and μ_ε is Lipschitz continuous. Thus for $r > 0$,

$$\lim_{\varepsilon\to 0} \mu_\varepsilon(r) = \mu(r).$$

See the following picture which describes the situation.



Then let Q_ε be defined as before but with μ_ε in place of μ. Thus by Theorems 4.1 and 4.4, there exists a unique solution, $\mathbf{v}_\varepsilon$ to the abstract problem

$$\mathbf{v}_\varepsilon' + \Sigma\mathbf{v}_\varepsilon + Q_\varepsilon\mathbf{v}_\varepsilon + P\mathbf{u}_\varepsilon \ni \mathbf{f} \text{ in } \mathcal{V}', \quad \mathbf{v}_\varepsilon(0) = \mathbf{v}_0 \in L^2(\Omega)^d.$$

As before, it is convenient to consider an equivalent problem in which the dependent variable, $\mathbf{v}_{\varepsilon\lambda}$ is defined by

$$\mathbf{v}_{\varepsilon\lambda}(t)e^{\lambda t} = \mathbf{v}_\varepsilon(t).$$

for λ large and positive. As before, this yields

$$\begin{gathered}\mathbf{v}_{\varepsilon\lambda}' + \lambda\mathbf{v}_{\varepsilon\lambda} + e^{-\lambda(\cdot)}\Sigma(e^{\lambda(\cdot)}\mathbf{v}_{\varepsilon\lambda}) + e^{-\lambda(\cdot)}Q_\varepsilon(e^{\lambda(\cdot)}\mathbf{v}_{\varepsilon\lambda}) + e^{-\lambda(\cdot)}P(\mathbf{u}_\varepsilon) \ni e^{-\lambda(\cdot)}\mathbf{f} \text{ in } \mathcal{V}',\\ \mathbf{v}_{\varepsilon\lambda}(0) = \mathbf{v}_0\end{gathered} \tag{5.1}$$

and the operator $B : \mathcal{V} \to \mathcal{V}'$ given in (4.4) is pseudomonotone, in fact monotone bounded and hemicontinuous while the operator A_ε of (4.3) defined by

$$\langle A_\varepsilon\mathbf{v}, \mathbf{v}\rangle_{\mathcal{V}} = \langle \lambda\mathbf{v} + e^{-\lambda(\cdot)}\Sigma(e^{\lambda(\cdot)}\mathbf{v}) + e^{-\lambda(\cdot)}Q_\varepsilon(e^{\lambda(\cdot)}\mathbf{v}) + e^{-\lambda(\cdot)}P(\mathbf{u}), \mathbf{v}\rangle_{\mathcal{V}}$$

is coercive. Furthermore, the coercivity is independent of ε in the sense that

$$\lim_{\|\mathbf{v}\|_{\mathcal{V}}\to\infty} \frac{\langle A_\varepsilon\mathbf{v}, \mathbf{v}\rangle_{\mathcal{V}}}{\|\mathbf{v}\|_{\mathcal{V}}} = \infty$$

independent of $\varepsilon > 0$.

Therefore, there exists a constant C independent of ε such that for $\mathbf{v}_{\varepsilon\lambda}$ the solution to (5.1),

$$\|\mathbf{v}_{\varepsilon\lambda}\|_{\mathcal{V}} \le C.$$

Since the various operators in (5.1) are bounded, it follows $\{\mathbf{v}_{\varepsilon\lambda}\}$ is also bounded in X, the space of solutions defined above. It follows there exists a subsequence, $\varepsilon \to 0$, still denoted by $\{\mathbf{v}_{\varepsilon\lambda}\}$ converging weakly to $\mathbf{v}_\lambda \in X$. Thus (5.1) is of the form

$$\mathbf{v}_{\varepsilon\lambda}' + B\mathbf{v}_{\varepsilon\lambda} + e^{-\lambda(\cdot)}Q_\varepsilon(e^{\lambda(\cdot)}\mathbf{v}_{\varepsilon\lambda}) \ni e^{-\lambda(\cdot)}\mathbf{f} \text{ in } \mathcal{V}',\ \mathbf{v}_{\varepsilon\lambda}(0) = \mathbf{v}_0 \tag{5.2}$$

where B is pseudomonotone on $\mathcal{V}$. Let $\mathbf{v}_{\varepsilon\lambda}^* \in e^{-\lambda(\cdot)}Q_\varepsilon(e^{\lambda(\cdot)}\mathbf{v}_{\varepsilon\lambda})$ be such that equality holds in the above inclusion. Letting $\mathbf{v}_\varepsilon \equiv e^{\lambda(\cdot)}\mathbf{v}_{\varepsilon\lambda}$ and $\mathbf{v}_\varepsilon^* \equiv e^{\lambda(\cdot)}\mathbf{v}_{\varepsilon\lambda}^*$, it follows

$$\mathbf{v}_\varepsilon^* \in Q_\varepsilon(\mathbf{v}_\varepsilon).$$

Then taking a further subsequence, it can be assumed $\mathbf{v}_\varepsilon^* \to \mathbf{v}^* \in X'$. Let $\mathbf{z}_\varepsilon$ be the element of $L^\infty(0,T;L^\infty(\Omega)^d)$ such that

$$\langle \mathbf{v}_\varepsilon^*, \mathbf{w}\rangle_{\mathcal{V}} = \int_0^T \int_{\Gamma_C} F((u_{\varepsilon n} - g)_+)\mu_\varepsilon(|\mathbf{v}_{\varepsilon T} - \dot{\mathbf{U}}_T|)\mathbf{z}_\varepsilon \cdot \mathbf{w}_T d\alpha\, dt \tag{5.3}$$

and for all $\mathbf{w} \in \mathcal{V}$,

$$\int_0^T \int_{\Gamma_C} \mathbf{z}_\varepsilon \cdot \mathbf{w}_T d\alpha\, dt \le \int_0^T \int_{\Gamma_C} |\mathbf{v}_{\varepsilon T} - \dot{\mathbf{U}}_T + \mathbf{w}_T| - |\mathbf{v}_{\varepsilon T} - \dot{\mathbf{U}}_T| d\alpha\, dt. \tag{5.4}$$

By Theorems 4.2 and 4.3, a subsequence satisfies

$$u_{\varepsilon n}(t) \to u_n(t)$$

uniformly in $L^2(\Gamma_C)$ and so

$$F((u_{\varepsilon n} - g)_+) \to F((u_n - g)_+) \text{ in } L^2(\Gamma_C)$$

uniformly which implies a subsequence converges pointwise *a.e.* Taking a further subsequence,

$$\mu_\varepsilon(|\mathbf{v}_{\varepsilon T} - \dot{\mathbf{U}}_T|) \to \psi \text{ weak } * \text{ in } L^\infty([0,T] \times \Omega).$$

Also, $\mathbf{v}_\varepsilon \to \mathbf{v}$ in $L^2(0,T;L^2(\Gamma_C)^d)$ and so a subsequence has the property that the convergence is also pointwise *a.e.* If $(\mathbf{v}_T - \dot{\mathbf{U}}_T)(t,\mathbf{x}) > 0$, then for all ε small enough, $(\mathbf{v}_{\varepsilon T} - \dot{\mathbf{U}}_T)(t,\mathbf{x})$ is bounded away from 0 also and so for such $(t,\mathbf{x})$,

$$\lim_{\varepsilon\to 0} \mu_\varepsilon(|\mathbf{v}_{\varepsilon T} - \dot{\mathbf{U}}_T|) = \mu(|\mathbf{v}_T - \dot{\mathbf{U}}_T|).$$

On the other hand, if $(\mathbf{v}_T - \dot{\mathbf{U}}_T)(t,\mathbf{x}) = 0$ then for η any small positive number it follows that for all ε small enough,

$$\mu_\varepsilon(|\mathbf{v}_{\varepsilon T} - \dot{\mathbf{U}}_T|(t,\mathbf{x})) \in [\mu(0+) - \eta, \mu(0)]$$

Thus if $E = (\mathbf{v}_T - \dot{\mathbf{U}}_T)^{-1}(0)$,

$$(\alpha \times m)(E)(\mu(0+) - \eta) \leq \int_0^T \int_{\Gamma_C} \mathcal{X}_E \psi d\alpha\, dt \leq (\alpha \times m)(E)\mu(0)$$

which requires $\psi(t,\mathbf{x}) \in [(\mu(0+) - \eta), \mu(0)]$ a.e. Since η is arbitrary, it follows that for these values of $(t,\mathbf{x}), \psi(t,\mathbf{x}) \in [\mu(0+), \mu(0)]$ a.e. Thus for a.e. $(t,\mathbf{x}), \psi(t,\mathbf{x})$ is in the graph of $(t,\mathbf{x}) \to \mu(|\mathbf{v}_T - \dot{\mathbf{U}}_T|)$.

Now consider (5.3). It follows

$$\begin{aligned}
\langle \mathbf{v}_\varepsilon^*, \mathbf{w}\rangle_{\mathcal{V}} &= \int_0^T \int_{\Gamma_C} F((u_{\varepsilon n} - g)_+)\mu_\varepsilon(|\mathbf{v}_{\varepsilon T} - \dot{\mathbf{U}}_T|)\mathbf{z}_\varepsilon \cdot \mathbf{w}_T d\alpha\, dt \\
&\leq \int_0^T \int_{\Gamma_C} F((u_{\varepsilon n} - g)_+)\mu_\varepsilon(|\mathbf{v}_{\varepsilon T} - \dot{\mathbf{U}}_T|) \\
&\quad \cdot \big[|\mathbf{v}_{\varepsilon T} - \dot{\mathbf{U}}_T + \mathbf{w}_T| - |\mathbf{v}_{\varepsilon T} - \dot{\mathbf{U}}_T|\big] d\alpha\, dt
\end{aligned} \tag{5.5}$$

Therefore, passing to the limit as $\varepsilon \to 0$ in the above, it follows

$$\langle \mathbf{v}^*, \mathbf{w}\rangle_{\mathcal{V}} \leq \int_0^T \int_{\Gamma_C} F((u_n - g)_+)\psi(|\mathbf{v}_T - \dot{\mathbf{U}}_T + \mathbf{w}_T| - |\mathbf{v}_T - \dot{\mathbf{U}}_T|) d\alpha\, dt \tag{5.6}$$

Note the strong convergence properties of $\{\mathbf{v}_{\varepsilon T}\}$ also imply

$$\limsup_{\varepsilon \to 0} \langle \mathbf{v}_\varepsilon^*, \mathbf{v}_\varepsilon - \mathbf{v}\rangle_{\mathcal{V}} \leq 0.$$

To see this consider (5.5) with $\mathbf{w}$ replaced with $\mathbf{v}_\varepsilon - \mathbf{v}$. However, you can also replace $\mathbf{w}$ with $\mathbf{v} - \mathbf{v}_\varepsilon$ and conclude

$$0 \geq \limsup_{\varepsilon \to 0} \langle \mathbf{v}_\varepsilon^*, \mathbf{v} - \mathbf{v}_\varepsilon\rangle_{\mathcal{V}} = -\liminf_{\varepsilon \to 0} \langle \mathbf{v}_\varepsilon^*, \mathbf{v}_\varepsilon - \mathbf{v}\rangle_{\mathcal{V}}$$

so that

$$0 \leq \liminf_{\varepsilon \to 0} \langle \mathbf{v}_\varepsilon^*, \mathbf{v}_\varepsilon - \mathbf{v}\rangle_{\mathcal{V}} \leq \limsup_{\varepsilon \to 0} \langle \mathbf{v}_\varepsilon^*, \mathbf{v}_\varepsilon - \mathbf{v}\rangle_{\mathcal{V}} \leq 0. \tag{5.7}$$

Now recall

$$\mathbf{v}_{\varepsilon\lambda}' + B\mathbf{v}_{\varepsilon\lambda} + \mathbf{v}_{\varepsilon\lambda}^* = e^{-\lambda(\cdot)}\mathbf{f} \text{ in } \mathcal{V}', \quad \mathbf{v}_{\varepsilon\lambda}(0) = \mathbf{v}_0 \tag{5.8}$$

where

$$\mathbf{v}_{\varepsilon\lambda}^* \in e^{-\lambda(\cdot)} Q_\varepsilon(e^{\lambda(\cdot)}\mathbf{v}_{\varepsilon\lambda})$$

Hence letting $\mathbf{v}_\lambda^* \equiv e^{-\lambda(\cdot)}\mathbf{v}^*$ where $\mathbf{v}^*$ is from (5.6) and $e^{-\lambda(\cdot)}\mathbf{v} \equiv \mathbf{v}_\lambda$, it follows $\mathbf{v}_\lambda^* \in e^{-\lambda(\cdot)}Q(e^{\lambda(\cdot)}\mathbf{v}_\lambda)$, where the following limits hold.

$$\begin{gathered}
\mathbf{v}_{\varepsilon\lambda}^* \to \mathbf{v}_\lambda^* \quad \text{weak } * \text{ in } \mathcal{V}' \\
\mathbf{v}_{\varepsilon\lambda} \to \mathbf{v}_\lambda \quad \text{weakly in } \mathcal{V} \\
\mathbf{v}_{\varepsilon\lambda}' \to \mathbf{v}_\lambda' \quad \text{weak } * \text{ in } \mathcal{V}'
\end{gathered}$$

Taking another subsequence if necessary, it can also be assumed

$$B\mathbf{v}_{\varepsilon\lambda} \to \mathbf{g} \quad \text{weak } * \text{ in } \mathcal{V}'.$$

Passing to a limit in (5.8),

$$\mathbf{v}_\lambda' + \mathbf{g} + \mathbf{v}_\lambda^* = e^{-\lambda(\cdot)}\mathbf{f} \text{ in } \mathcal{V}', \quad \mathbf{v}_\lambda(0) = \mathbf{v}_0$$

and it only remains to identify $\mathbf{g}$ with $B\mathbf{v}_\lambda$. From (5.8),

$$\Big(\langle \mathbf{v}'_{\varepsilon\lambda} - \mathbf{v}'_\lambda, \mathbf{v}_{\varepsilon\lambda} - \mathbf{v}_\lambda\rangle + \langle \mathbf{v}'_\lambda, \mathbf{v}_{\varepsilon\lambda} - \mathbf{v}_\lambda\rangle + \langle B\mathbf{v}_{\varepsilon\lambda}, \mathbf{v}_{\varepsilon\lambda} - \mathbf{v}_\lambda\rangle + \langle \mathbf{v}^*_{\varepsilon\lambda}, \mathbf{v}_{\varepsilon\lambda} - \mathbf{v}_\lambda\rangle\Big) \\ = \langle e^{-\lambda(\cdot)}\mathbf{f}, \mathbf{v}_{\varepsilon\lambda} - \mathbf{v}_\lambda\rangle$$

and so

$$\langle \mathbf{v}'_\lambda, \mathbf{v}_{\varepsilon\lambda} - \mathbf{v}_\lambda\rangle + \langle B\mathbf{v}_{\varepsilon\lambda}, \mathbf{v}_{\varepsilon\lambda} - \mathbf{v}_\lambda\rangle + \langle \mathbf{v}^*_{\varepsilon\lambda}, \mathbf{v}_{\varepsilon\lambda} - \mathbf{v}_\lambda\rangle \le \langle e^{-\lambda(\cdot)}\mathbf{f}, \mathbf{v}_{\varepsilon\lambda} - \mathbf{v}_\lambda\rangle.$$

Hence from (5.7),

$$\begin{aligned} &\limsup_{\varepsilon\to 0}(\langle \mathbf{v}'_\lambda, \mathbf{v}_{\varepsilon\lambda} - \mathbf{v}_\lambda\rangle + \langle B\mathbf{v}_{\varepsilon\lambda}, \mathbf{v}_{\varepsilon\lambda} - \mathbf{v}_\lambda\rangle + \langle \mathbf{v}^*_{\varepsilon\lambda}, \mathbf{v}_{\varepsilon\lambda} - \mathbf{v}_\lambda\rangle) \\ &= \limsup_{\varepsilon\to 0}\langle B\mathbf{v}_{\varepsilon\lambda}, \mathbf{v}_{\varepsilon\lambda} - \mathbf{v}_\lambda\rangle \le 0. \end{aligned}$$

which implies

$$\liminf_{\varepsilon\to 0}\langle B\mathbf{v}_{\varepsilon\lambda}, \mathbf{v}_{\varepsilon\lambda} - \mathbf{v}_\lambda\rangle \ge \langle B\mathbf{v}_\lambda, \mathbf{v}_\lambda - \mathbf{v}_\lambda\rangle = 0$$

and so $\lim_{\varepsilon\to 0}\langle B\mathbf{v}_{\varepsilon\lambda}, \mathbf{v}_{\varepsilon\lambda} - \mathbf{v}_\lambda\rangle = 0$. Since B is pseudomonotone, it follows that for all $\mathbf{w} \in \mathcal{V}$,

$$\begin{aligned} \liminf_{\varepsilon\to 0}\langle B\mathbf{v}_{\varepsilon\lambda}, \mathbf{v}_{\varepsilon\lambda} - \mathbf{w}\rangle &= \liminf_{\varepsilon\to 0}(\langle B\mathbf{v}_{\varepsilon\lambda}, \mathbf{v}_{\varepsilon\lambda} - \mathbf{v}_\lambda\rangle + \langle B\mathbf{v}_{\varepsilon\lambda}, \mathbf{v}_\lambda - \mathbf{w}\rangle) \\ &= \liminf_{\varepsilon\to 0}\langle B\mathbf{v}_{\varepsilon\lambda}, \mathbf{v}_\lambda - \mathbf{w}\rangle = \langle \mathbf{g}, \mathbf{v}_\lambda - \mathbf{w}\rangle \\ &\ge \langle B\mathbf{v}_\lambda, \mathbf{v}_\lambda - \mathbf{w}\rangle \end{aligned}$$

and so

$$\langle \mathbf{g}, \mathbf{v}_\lambda - \mathbf{w}\rangle \ge \langle B\mathbf{v}_\lambda, \mathbf{v}_\lambda - \mathbf{w}\rangle$$

and since $\mathbf{w}$ was arbitrary, this shows $\mathbf{g} = B\mathbf{v}_\lambda$. It follows there exists a solution to

$$\mathbf{v}'_\lambda + B\mathbf{v}_\lambda + e^{-\lambda(\cdot)}Qe^{\lambda(\cdot)}\mathbf{v}_\lambda = e^{-\lambda(\cdot)}\mathbf{f} \text{ in } \mathcal{V}', \quad \mathbf{v}_\lambda(0) = \mathbf{v}_0$$

which implies there exists a solution to

$$\mathbf{v}' + \Sigma\mathbf{v} + P(\mathbf{u}) + Q(\mathbf{v}) \ni \mathbf{f}, \quad \mathbf{v}(0) = \mathbf{v}_0.$$

This proves the theorem. □

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