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ENERGY ESTIMATE FOR WAVE EQUATIONS WITH COEFFICIENTS IN SOME BESOV TYPE CLASS

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Abstract. In this paper, we obtain an energy estimate for wave equations with coefficients satisfying Besov type conditions. We give an example of a wave equation with continuous and nowhere differentiable coefficients for which the L^2 estimate holds.

1. Introduction

Consider a wave equation on $[0,T]\times\mathbb{R}$:

$$Lu=\partial_t^2u-a(t)\partial_x^2u \tag{1.1}$$

with a positive coefficient $a(t)\geq\delta_0$ with $\delta_0>0$. It is well known that, if $a(t)$ is Lipschitz continuous, then we have the energy estimate

$$\sum_{0\leq j+k\leq 1}\|\partial_t^j\partial_x^k u(t,\cdot)\|\leq C\Big(\sum_{0\leq j+k\leq 1}\|\partial_t^j\partial_x^k u(0,\cdot)\|+\int_0^t\|Lu(s,\cdot)\|\,ds\Big) \tag{1.2}$$

(see for example [5, Ch. IX]). Here $\|\cdot\|$ denotes L^2 norm.

Colombini, De Giorgi and Spagnolo [2] (see also [4]) have shown that the estimate (1.2) is still valid if the coefficient $a(t)$ has a bounded variation, that is, in the integral form, there exists a constant $C\geq 0$ such that we have

$$\int_0^{T-\varepsilon}|a(t+\varepsilon)-a(t)|\,dt\leq C\varepsilon\quad(0<\varepsilon\leq T/2). \tag{1.3}$$

Furthermore, in the same paper, they have shown that if $a(t)$ satisfies

$$\int_0^{T-\varepsilon}|a(t+\varepsilon)-a(t)|\,dt\leq C\varepsilon(|\log\varepsilon|+1)\quad(0<\varepsilon\leq T/2) \tag{1.4}$$

with a constant $C\geq 0$, then the Cauchy problem for L is C^∞ well posed.

According to Yamazaki [9], we have the estimate (1.2) when $a(t)\in C^2((0,T])$ satisfies $|a(t)|+|ta'(t)|+|t^2a''(t)|\leq C$ on $(0,T]$ (see also [7]). Then we see that the estimate (1.2) is valid for L with some coefficient $a(t)$ whose total variation is not finite, for example $a(t)=2+\sin(\log t)$.

In this paper we introduce an integral version of the condition $|a(t)|+|ta'(t)|+|t^2a''(t)|\leq C$ so that the estimate (1.2) holds still for L with the coefficient $a(t)$

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satisfying such a condition. Namely we show the following. When the coefficient $a(t)$ is a bounded measurable function on $[0,T]$ and satisfies: there exists a constant $C \geq 0$ such that we have

$$\int_{\varepsilon}^{T-\varepsilon} |a(t+\varepsilon)+a(t-\varepsilon)-2a(t)|\,dt \leq C\varepsilon \quad (0<\varepsilon\leq T/2), \tag{1.5}$$

then the estimate (1.2) holds. Using the same method, we show also the following. The Cauchy problem for L is C^∞ well posed if the coefficient $a(t)$ is a bounded measurable function on $[0,T]$ and satisfies the following: There exists a constant $C \geq 0$ such that

$$\int_{\varepsilon}^{T-\varepsilon} |a(t+\varepsilon)+a(t-\varepsilon)-2a(t)|\,dt \leq C\varepsilon(|\log\varepsilon|+1) \quad (0<\varepsilon\leq T/2). \tag{1.6}$$

Note that the boundedness of $a(t)$ and the estimate (1.5) imply

$$\int_{0}^{T-\varepsilon} |a(t+\varepsilon)-a(t)|^2\,dt \leq C\varepsilon \quad (0<\varepsilon\leq T/2)$$

with some constant C. While from the boundedness of $a(t)$ and (1.6) we obtain

$$\int_{0}^{T-\varepsilon} |a(t+\varepsilon)-a(t)|^2\,dt \leq C\varepsilon(|\log\varepsilon|+1) \quad (0<\varepsilon\leq T/2)$$

with some constant C (see the next section).

We remark that Colombini, Del Santo and Reissig [1] (see also [6] and [7]) have shown that the Cauchy problem for L is C^∞ well posed when $a(t)$ satisfies $|a(t)|+|(t\log t)a'(t)|+|(t\log t)^2a''(t)| \leq C$ on $(0,T]$. For example the Cauchy problem for L with $a(t)=2+\sin(|\log t|^2)$ is C^∞ well posed but this function $a(t)$ does not satisfy the condition (1.6). Nonetheless we can find some positive function $a(t)$ which satisfies the estimate (1.6) with the right hand side replaced with $C\varepsilon(|\log\varepsilon|+1)^{1+\delta}$ ($\delta>0$), so that the Cauchy problem for L is not C^∞ well posed. Indeed Colombini and Lerner [3] have given an example of a positive function $a(t)$ such that $a(t)$ satisfies $\sup_{\varepsilon\in(0,1],t\in[0,1]} |a(t+\varepsilon)-a(t)|/(\varepsilon(|\log\varepsilon|+1)^{1+\delta}) < \infty$ (for any $\delta>0$) but the Cauchy problem on $[0,1]\times\mathbb{R}$ for $\partial_t^2 - a(t)\partial_x^2$ is not C^∞ well posed.

In the next section, in order to study properties of bounded functions that satisfying (1.5) or (1.6), we define the function spaces $Z_\gamma(I)$ and show some properties of functions in such spaces. Some properties of examples are discussed in the appendix. In the third section, we state and prove the main theorems.

We use the following notation. Let $L^2(\mathbb{R}^d)$ or L^2 denote the space of all square integrable functions on $\mathbb{R}^d$ with the norm $\|\cdot\|$ given by $\|f(\cdot)\|^2 = \int |f(x)|^2\,dx$. For $s\in\mathbb{R}$ let H^s denote the space that consists of functions $f(x)$ on $\mathbb{R}^d$ satisfying $\int (1+|\xi|^2)^s|\hat{f}(\xi)|^2\,d\xi < \infty$ where $\hat{f}(\xi)$ is the Fourier transform of $f(x)$ and $\|\cdot\|_s$ be its norm, that is, $\|f(\cdot)\|_s^2 = \int (1+|\xi|^2)^s|\hat{f}(\xi)|^2\,d\xi$. We set $H^\infty = \bigcap_{s\in\mathbb{R}} H^s$. For $X=H^s$, H^∞ or $C^\infty(\mathbb{R}^d)$, the space of indefinitely differentiable functions on $\mathbb{R}^d$, and $T>0$, we denote by $L^1([0,T],X)$ the space of X-valued integrable functions on $[0,T]$ and by $C^j([0,T],X)$ with an integer j the space of X-valued j-times continuously differentiable functions on $[0,T]$. We use also the standard notation of multi-index. We use C or C with some suffix in order to denote a non-negative constant that may be different line by line.

2. Space $Z_\gamma(I)$

Let $I=(t_0,t_1)\subset\mathbb{R}$ with $t_0<t_1$ and $\gamma\ge 0$. We say $f(t)\in Z_\gamma(I)$ if $f(t)$ is a bounded measurable function on the interval I and satisfies, with a constant $C\ge 0$,

$$\int_{t_0+\varepsilon}^{t_1-\varepsilon} |f(t+\varepsilon)+f(t-\varepsilon)-2f(t)|\,dt \le C\varepsilon(\log(\varepsilon^{-1}+1)+1+\gamma)^\gamma \tag{2.1}$$

for any $\varepsilon\in(0,(t_1-t_0)/2)$.

Here we remark that, when $\gamma=0$, (2.1) corresponds to the Besov $B^1_{1,\infty}$ estimate. We remark also that $s(\log(s^{-1}+1)+1+\gamma)^\gamma$ is increasing on $(0,\infty)$ when $\gamma\ge 0$. We set

$$\begin{aligned}\|f\|_{Z_\gamma(I)} = \|f\|_{L^\infty(I)} &+ \sup_{0<\varepsilon<d/2}\frac{1}{\varepsilon(\log(\varepsilon^{-1}+1)+1+\gamma)^\gamma}\\ &\times\int_{t_0+\varepsilon}^{t_1-\varepsilon}|f(t+\varepsilon)+f(t-\varepsilon)-2f(t)|\,dt\end{aligned}$$

where $I=(t_0,t_1)$ and $d=t_1-t_0$.

In the following we assume that functions in $Z_\gamma(I)$ are real valued. But we see that the properties discussed below are valid also for complex valued functions by considering the real part and the imaginary part separately.

From the boundedness of $f(t)$, we see that $f(t)\in Z_\gamma(I)$ satisfies

$$\begin{aligned}&\int_{t_0+\varepsilon}^{t_1-\varepsilon}(|f(t+\varepsilon)-f(t)|^2+|f(t-\varepsilon)-f(t)|^2)\,dt\\ &\le C\varepsilon(\log(\varepsilon^{-1}+1)+1+\gamma)^\gamma \quad (0<\varepsilon\le(t_1-t_0)/2).\end{aligned}\tag{2.2}$$

with the constant C depending only on $\|f(\cdot)\|_{Z_\gamma(I)}$. Indeed, since

$$|f(t+\varepsilon)-f(t)|^2=(f(t+\varepsilon)-f(t))f(t+\varepsilon)-(f(t+\varepsilon)-f(t))f(t),$$

we see that

$$\begin{aligned}J&=\int_{t_0+\varepsilon}^{t_1-\varepsilon}|f(t+\varepsilon)-f(t)|^2\,dt\\ &=\int_{t_0+2\varepsilon}^{t_1}(f(t)-f(t-\varepsilon))f(t)\,dt-\int_{t_0+\varepsilon}^{t_1-\varepsilon}(f(t+\varepsilon)-f(t))f(t)\,dt.\end{aligned}$$

Then we see that

$$J=-\int_{t_0+2\varepsilon}^{t_1-\varepsilon}(f(t+\varepsilon)-2f(t)+f(t-\varepsilon))f(t)\,dt+R,$$

where

$$R=\int_{t_1-\varepsilon}^{t_1}(f(t)-f(t-\varepsilon))f(t)\,dt-\int_{t_0+\varepsilon}^{t_0+2\varepsilon}(f(t+\varepsilon)-f(t))f(t)\,dt,$$

from which, taking account of (2.1) and the boundedness of $f(t)$, we obtain $|J|\le C\varepsilon(\log(\varepsilon^{-1}+1)+1+\gamma)^\gamma$. Similarly we obtain the estimate for the integral of second term. Hence we have (2.2). Since

$$\begin{aligned}&f(t+\varepsilon)g(t+\varepsilon)-2f(t)g(t)+f(t-\varepsilon)g(t-\varepsilon)\\ &=(f(t+\varepsilon)-2f(t)+f(t-\varepsilon))g(t)+f(t)(g(t+\varepsilon)-2g(t)+g(t-\varepsilon))\\ &\quad+(f(t+\varepsilon)-f(t))(g(t+\varepsilon)-g(t))+(f(t-\varepsilon)-f(t))(g(t-\varepsilon)-g(t)),\end{aligned}$$

we see from (2.2) and Schwarz's inequality that $f(t), g(t) \in Z_\gamma(I)$ implies that $f(t)g(t) \in Z_\gamma(I)$.

For $f(t) \in Z_\gamma(I)$, we consider an extension of $f(t)$ on $\mathbb{R}$ as a bounded measurable function so that its L^∞-norm is equal to $\|f\|_{L^\infty(I)}$. We still denote by $f(t)$ such an extension. Let $I = (t_0, t_1)$. Then for any $\varepsilon > 0$, we have

$$\int_{t_0}^{t_1} |f(t+\varepsilon) - 2f(t) + f(t-\varepsilon)|\,dt \le C\varepsilon(\log(\varepsilon^{-1}+1)+1+\gamma)^\gamma \tag{2.3}$$

$$\int_{t_0}^{t_1} (|f(t+\varepsilon) - f(t)|^2 + |f(t) - f(t-\varepsilon)|^2)\,dt \le C\varepsilon(\log(\varepsilon^{-1}+1)+1+\gamma)^\gamma. \tag{2.4}$$

where the constant C depends only on $\|f\|_{Z_\gamma(I)}$. Indeed if $\varepsilon \ge (t_1 - t_0)/2$, we see that the right hand side of (2.3) is not larger than $8\varepsilon\|f\|_{L^\infty(I)}$. While, in the case of $\varepsilon < (t_1 - t_0)/2$, we see that, on the right hand side of (2.3), the integral on the interval $[t_0+\varepsilon, t_1-\varepsilon]$ is not larger than $\varepsilon(\log(\varepsilon^{-1}+1)+1+\gamma)^\gamma\|f\|_{Z_\gamma(I)}$ and the integral on the remainder part is not larger than $8\varepsilon\|f\|_{L^\infty(I)}$. Hence we have (2.3). Similarly we obtain (2.4).

Now we consider the regularization of a function $f(t)$ in $Z_\gamma(I)$. We take the above mentioned extension $f(t)$. Let $\phi(s)$ be a smooth function on $\mathbb{R}$ satisfying $\phi(-s) = \phi(s)$, $\phi(s) \ge 0$, $\phi(s) = 0$ for $|s| \ge 1$ and $\int_{\mathbb{R}} \phi(s)\,ds = 1$. We denote by $f_\varepsilon(t)$ with $\varepsilon > 0$ the regularization of $f(t)$ given by

$$f_\varepsilon(t) = \frac{1}{\varepsilon}\int_{\mathbb{R}} \phi(\frac{t-s}{\varepsilon}) f(s)\,ds.$$

Then we have the following result.

Lemma 2.1.

$$\int_I |f_\varepsilon(t) - f(t)|\,dt \le C_1\varepsilon(\log(\varepsilon^{-1}+1)+1+\gamma)^\gamma \tag{2.5}$$

$$\int_I (|f_\varepsilon''(t)| + |f_\varepsilon'(t)|^2)\,dt \le C_2(\log(\varepsilon^{-1}+1)+1+\gamma)^\gamma/\varepsilon \tag{2.6}$$

where the constants C_1 and C_2 depend on $\|f\|_{Z_\gamma(I)}$ and $\phi(s)$ but not on the length of the interval I. Furthermore, for any function $F \in C^2(\mathbb{R})$, setting $h(t) = F(f_\varepsilon(t))$, we have

$$\int_I (|h''(t)| + |h'(t)|^2)\,dt \le C(\log(\varepsilon^{-1}+1)+1+\gamma)^\gamma/\varepsilon. \tag{2.7}$$

Here the constant C is also independent of the length of the interval I.

Proof. Since $f_\varepsilon(t) - f(t) = \int_{\mathbb{R}} \phi(s)(f(t-\varepsilon s) - f(t))\,ds$ and $\phi(-s) = \phi(s)$, we have

$$\begin{aligned} |f_\varepsilon(t) - f(t)| &= |\int_{\mathbb{R}} \frac{\phi(s)+\phi(-s)}{2}(f(t-\varepsilon s) - f(t))\,ds| \\ &= \frac{1}{2}|\int_{\mathbb{R}} \phi(s)(f(t+\varepsilon s) + f(t-\varepsilon s) - 2f(t))\,ds|, \end{aligned}$$

from which and from (2.3), we obtain

$$\int_I |f_\varepsilon(t) - f(t)|\,dt \le C\int_{\mathbb{R}} \phi(s)|s|\varepsilon(\log((|s|\varepsilon)^{-1}+1)+1+\gamma)^\gamma\,ds.$$

Since $s(\log(s^{-1}+1)+1+\gamma)^\gamma$ is increasing, the right hand side of the estimate above is not larger than $C\varepsilon(\log(\varepsilon^{-1}+1)+1+\gamma)^\gamma$. Similarly, $f''_\varepsilon(t) = \varepsilon^{-2}\int_{\mathbb{R}} \phi''(s)(f(t-\varepsilon s)-f(t))\,ds$ and $\phi''(-s)=\phi''(s)$ imply

$$\int_I |f''_\varepsilon(t)|\,dt \le C(\log(\varepsilon^{-1}+1)+1+\gamma)^\gamma/\varepsilon.$$

While it follows from $f'_\varepsilon(t) = \varepsilon^{-1}\int_{\mathbb{R}} \phi'(s)(f(t-\varepsilon s)-f(t))\,ds$, (2.3) and Schwarz's inequality that

$$|f'_\varepsilon(t)|^2 \le \varepsilon^{-2}\|\phi'(\cdot)\|_{L^1}\int_I |\phi'(s)||f(t-\varepsilon s)-f(t)|^2\,ds,$$

from which and from (2.4) we obtain the desired estimate of $\int_I |f'_\varepsilon(t)|^2\,dt$. Hence we have (2.6). We obtain (2.7) from (2.5) and (2.6). $\square$

Example 2.2. If $f(t)\in C^2((0,1/2])$ satisfies $|f(t)|+|f''(t)|t^2/|\log t|^\gamma \le C$ on $I=(0,1/2)$, then $f(t)$ belongs to $Z_\gamma(I)$. Indeed, if $\varepsilon<t<1-\varepsilon$, $f(t+\varepsilon)+f(t-\varepsilon)-2f(t)$ is equal to $\varepsilon^2(f''(t+\theta\varepsilon)+f''(t-\theta\varepsilon))/2$ with some $\theta\in(0,1)$. Then we have

$$|f(t+\varepsilon)+f(t-\varepsilon)-2f(t)| \le C\varepsilon^2|\log(t-\varepsilon)|^\gamma/(t-\varepsilon)^2 \quad (2\varepsilon\le t<1/2-\varepsilon),$$

from which we have

$$\int_{2\varepsilon}^{1/2-\varepsilon} |f(t+\varepsilon)+f(t-\varepsilon)-2f(t)|\,dt \le C\varepsilon|\log\varepsilon|^\gamma.$$

Then noting $|f(t)|\le C$, we see $f(t)\in Z_\gamma(I)$.

For example, let $h_\gamma(t)=\sin(|\log t|^{\gamma+1})$ with $\gamma\ge 0$. Then $h_\gamma(t)$ belongs to $Z_{2\gamma}((0,1/2))$. Indeed we have $h''_\gamma(t) = -(\gamma+1)^2\sin(|\log t|^{\gamma+1})|\log t|^{2\gamma}/t^2 + r(t)$ where $|r(t)|\le C|\log t|^\gamma/t^2$. We see also that $h_\gamma(t)\notin Z_\sigma((0,1/2))$ when $0\le\sigma<2\gamma$. Furthermore we see that

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon|\log\varepsilon|^{1+\gamma}}\int_0^{1/2-\varepsilon}|h_\gamma(t+\varepsilon)-h_\gamma(t)|\,dt > 0 \tag{2.8}$$

(see the appendix for detail). Thus we see that $h_{1/2}(t)$ belongs to $Z_1((0,1/2))$ but does not satisfy (1.4) with $T=1/2$.

Example 2.3. Here we show that the Weierstrass function

$$w_\gamma(t) = \sum_{n=1}^\infty 2^{-n}n^\gamma\cos 2^n t$$

with $\gamma\ge 0$, that is continuous and nowhere differentiable (see for example [8]), belongs to $Z_\gamma((0,2\pi))$. Indeed, for any $\varepsilon\in(0,1/2)$ we have $w_\gamma(t)=w_{\gamma,1,\varepsilon}(t)+w_{\gamma,2,\varepsilon}(t)$ where

$$w_{\gamma,1,\varepsilon}(t) = \sum_{1\le n\le\frac{|\log\varepsilon|}{\log 2}} 2^{-n}n^\gamma\cos 2^n t \quad\text{and}\quad w_{\gamma,2,\varepsilon}(t) = \sum_{n>\frac{|\log\varepsilon|}{\log 2}} 2^{-n}n^\gamma\cos 2^n t.$$

Since $|w''_{\gamma,1,\varepsilon}(t)|\le C\varepsilon^{-1}|\log\varepsilon|^\gamma$ and $|w_{\gamma,2,\varepsilon}(t)|\le C\varepsilon|\log\varepsilon|^\gamma$, then we see $|w_\gamma(t+\varepsilon)+w_\gamma(t-\varepsilon)-2w_\gamma(t)|\le C|\log\varepsilon|^\gamma\varepsilon$. Hence $w_\gamma(t)\in Z_\gamma((0,2\pi))$.

We remark that $w_0(t)$ satisfies (1.4). Indeed, in the expression above $w_0(t)=w_{0,1,\varepsilon}(t)+w_{0,2,\varepsilon}(t)$ we have $|w'_{0,1,\varepsilon}(t)|\le C|\log\varepsilon|$ and $|w_{0,2,\varepsilon}(t)|\le C\varepsilon$. Then we see $|w_0(t+\varepsilon)-w_0(t)|\le C|\log\varepsilon|\varepsilon$.

3. Main results

Let $a_{jk}(t)$ $(j,k=1,\dots,d)$ be a real-valued bounded measurable function on $(0,T)$ with $T>0$ satisfying $a_{kj}(t)=a_{jk}(t)$ and

$$\sum_{j,k=1}^{d} a_{jk}(t)\xi_j\xi_k \geq C_0|\xi|^2 \text{ for } \xi\in\mathbb{R}^d \text{ and } t\in(0,T) \tag{3.1}$$

with some positive constant $C_0>0$. Set

$$P_2(t,\partial_t,\xi)=\partial_t^2+\sum_{j,k=1}^{d} a_{jk}(t)\xi_j\xi_k \tag{3.2}$$

where $\xi\in\mathbb{R}^d$. Then we have the following result.

Theorem 3.1. *Assume that $a_{jk}(t)\in Z_\gamma((0,T))$ $(j,k=1,\dots,d)$ with $\gamma\geq 0$. Let $\xi\in\mathbb{R}^d$. If $u(t)\in C^1([0,T])$ satisfies $P_2(t,\partial_t,\xi)u=f(t)$ on $(0,T)$ with $f(t)\in L^1([0,T])$, then we have*

$$\begin{aligned}&(|\partial_t u(t_2)|^2+|\xi|^2|u(t_2)|^2)^{1/2}\\ &\leq C_1 e^{C_2(\log(|\xi|+1)+1+\gamma)^\gamma}\big((|\partial_t u(t_1)|^2+|\xi|^2|u(t_1)|^2)^{1/2}+\int_{t_1}^{t_2}|f(t)|\,dt\big)\end{aligned} \tag{3.3}$$

for any $0\leq t_1\leq t_2\leq T$. Here constants C_1 and C_2 depend on C_0 of (3.1) *and Z_γ-norm of coefficients $a_{jk}(t)$ but not on the length of the interval $[0,T]$.*

Before presenting the proof of Theorem above, we remark the following well known result. Let $L=\partial_t^2+a^2(t)\rho^2$ where $a(t)$ is smooth and positive and $\rho>0$. Noting that $(\partial_t-ia(t)\rho-\frac{a'(t)}{2a(t)})(\partial_t+ia(t)\rho+\frac{a'(t)}{2a(t)})$ and $(\partial_t+ia(t)\rho-\frac{a'(t)}{2a(t)})(\partial_t-ia(t)\rho+\frac{a'(t)}{2a(t)})$ are equal to

$$L-(\frac{a'(t)}{2a(t)})^2+(\frac{a'(t)}{2a(t)})',$$

we consider the energy

$$E(u)=\frac{1}{a(t)}|\partial_t u+\frac{a'(t)}{2a(t)}u|^2+a(t)\rho^2|u|^2.$$

Then we have

$$\frac{d}{dt}E(u)=\frac{2}{a(t)}\mathrm{Re}\big(\overline{(\partial_t u+\frac{a'(t)}{2a(t)}u)}\,(Lu-Ru)\big) \tag{3.4}$$

where $R=(\frac{a'(t)}{2a(t)})^2-(\frac{a'(t)}{2a(t)})'$.

Proof of Theorem 3.1. If $\xi=0$, $P_2u=f(t)$ is equal to $\partial_t^2u=f(t)$. Then we have immediately (3.3). In the following, we assume $\xi\neq 0$. First we extend the coefficients $a_{jk}(t)$ on $\mathbb{R}$ so that $\|a_{jk}(t)\|_{L^\infty(\mathbb{R})}=\|a_{jk}(t)\|_{L^\infty((0,T))}$ and the estimate (3.1) still holds for $t\in\mathbb{R}$. Then we consider the regularization $a_{jk,\varepsilon}(t)$ of $a_{jk}(t)$ given by $\int_{\mathbb{R}}\phi((t-s)/\varepsilon)a_{jk}(s)\,ds/\varepsilon$ with $\varepsilon>0$ using a non-negative, even and smooth function $\phi(s)$ as described in the section 2. Then we see that (3.1) with $a_{jk,\varepsilon}(t)$ in the place of $a_{jk}(t)$ holds. Then we define $a(t,\xi,\varepsilon)$ by

$$a(t,\xi,\varepsilon) = |\xi|^{-1}\Big(\sum_{j,k=1}^{d} a_{jk,\varepsilon}(t)\xi_j\xi_k\Big)^{1/2} \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

We have

$$C_1 \geq a(t,\xi,\varepsilon) \geq \sqrt{C_0} \tag{3.5}$$

with constants C_0 appearing in (3.1) and C_1 depending only on $\|a_{jk}(\cdot)\|_{L^\infty((0,T))}$. We see from (2.7), that

$$\int_0^T (|\partial_t a(t,\xi,\varepsilon)|^2 + |\partial_t^2 a(t,\xi,\varepsilon)|)\,dt \leq C_1\varepsilon^{-1}(|\log(\varepsilon^{-1}+1)| + 1 + \gamma)^\gamma \tag{3.6}$$

for any $\varepsilon > 0$. Furthermore Lemma 2.1 implies that

$$\int_0^T |a(t,\xi,\varepsilon)^2|\xi|^2 - \sum_{j,k=1}^{d} a_{jk}(t)\xi_j\xi_k|\,dt \leq C_2\varepsilon(|\log(\varepsilon^{-1}+1)| + 1 + \gamma)^\gamma|\xi|^2. \tag{3.7}$$

Here the constants above C_1 and C_2 may depend on Z_γ-norm of $a_{jk}(t)$ and the constant C_0 of (3.1) but not on the length of interval $[0,T]$.

Assume that $u(t) \in C^1([0,T])$ satisfies $\partial_t^2 u + \sum_{j,k=1}^d a_{jk}(t)\xi_j\xi_k u = f(t)$ on $(0,T)$ with $\xi \in \mathbb{R}^d \setminus \{0\}$ and $f(t) \in L^1([0,T])$. Let

$$E_\varepsilon(t) = \frac{1}{a(t,\xi,\varepsilon)}|\partial_t u + \frac{\partial_t a(t,\xi,\varepsilon)}{2a(t,\xi,\varepsilon)}u|^2 + a(t,\xi,\varepsilon)|\xi|^2|u|^2. \tag{3.8}$$

Then it follows from (3.4) that

$$\frac{d}{dt}E_\varepsilon(t) = \frac{2}{a(t,\xi,\varepsilon)}\mathrm{Re}\Big(\overline{(\partial_t u + \frac{\partial_t a(t,\xi,\varepsilon)}{2a(t,\xi,\varepsilon)}u)}\ (L_\varepsilon u - R_\varepsilon u)\Big) \tag{3.9}$$

where $L_\varepsilon u = \partial_t^2 - a(t,\xi,\varepsilon)^2|\xi|^2 u$ and $R_\varepsilon = (\frac{\partial_t a(t,\xi,\varepsilon)}{2a(t,\xi,\varepsilon)})^2 - \partial_t(\frac{\partial_t a(t,\xi,\varepsilon)}{2a(t,\xi,\varepsilon)})$. Note that

$$|L_\varepsilon u| \leq |a(t,\xi,\varepsilon)^2|\xi|^2 - \sum_{j,k=1}^{d} a_{jk}(t)\xi_j\xi_k||u| + |f(t)|$$

and

$$|R_\varepsilon u| \leq C(|\partial_t a(t,\xi,\varepsilon)|^2 + |\partial_t^2 a(t,\xi,\varepsilon)|).$$

Since

$$|(\partial_t u + \frac{\partial_t a(t,\xi,\varepsilon)}{2a(t,\xi,\varepsilon)}u)||u| \leq \frac{1}{2|\xi|}E_\varepsilon(t),$$

we see that

$$\Big|\frac{d}{dt}E_\varepsilon(t)\Big| \leq 2C(t,\xi,\varepsilon)E_\varepsilon(t) + E_\varepsilon(t)^{1/2}2C_0^{-1/4}|f(t)| \tag{3.10}$$

where

$$\begin{aligned} C(t,\xi,\varepsilon) = \frac{1}{2}C_0^{-1/2}\Big(&|\big(a(t,\xi,\varepsilon)^2|\xi|^2 - \sum_{j,k=1}^{d} a_{jk}(t)\xi_j\xi_k\big)| \\ &+ C|\partial_t a(t,\xi,\varepsilon)|^2 + |\partial_t^2 a(t,\xi,\varepsilon)|\Big)|\xi|^{-1}. \end{aligned}$$

Hence for any positive constant $\delta > 0$, we have

$$\Big|\frac{d}{dt}\big(E_\varepsilon(t)+\delta\big)\Big| \le 2C(t,\xi,\varepsilon)\big(E_\varepsilon(t)+\delta\big) + \big(E_\varepsilon(t)+\delta\big)^{1/2} 2C_0^{-1/4}|f(t)|,$$

from which we obtain

$$\Big|\frac{d}{dt}\big(E_\varepsilon(t)+\delta\big)^{1/2}\Big| \le C(t,\xi,\varepsilon)\big(E_\varepsilon(t)+\delta\big)^{1/2} + C_0^{-1/4}|f(t)|.$$

Then we see that, for $0 \le t_1 \le t_2 \le T$,

$$\big(E_\varepsilon(t_2)+\delta\big)^{1/2} \le e^{\int_{t_1}^{t_2} C(t,\xi,\varepsilon)\,dt}\big(E_\varepsilon(t_1)+\delta\big)^{1/2} + \int_{t_1}^{t_2} e^{\int_t^{t_2} C(s,\xi,\varepsilon)\,ds} C_0^{-1/4}|f(t)|\,dt.$$

It follows from (3.6) and (3.7) that

$$\int_0^T C(t,\xi,\varepsilon)\,dt \le C(\varepsilon|\xi| + \frac{1}{\varepsilon|\xi|})(\log(\varepsilon^{-1}+1)+1+\gamma)^\gamma.$$

Now picking $\varepsilon = 1/|\xi|$, we obtain

$$\big(E_{1/|\xi|}(t_2)+\delta\big)^{1/2} \le e^{C(\log(|\xi|+1)+1+\gamma)^\gamma}\Big(\big(\big(E_{1/|\xi|}(t_1)+\delta\big)^{1/2} + \int_{t_1}^{t_2} C_0^{-1/4}|f(t)|\,dt\Big). \tag{3.11}$$

By taking $\delta \to 0$, we obtain

$$\big(E_{1/|\xi|}(t_2)\big)^{1/2} \le e^{C(\log(|\xi|+1)+1+\gamma)^\gamma}\Big(\big((E_{1/|\xi|}(t_2))^{1/2} + \int_{t_1}^{t_2} C_0^{-1/4}|f(t)|\,dt\Big).$$

Since $|\partial_t a_{jk,\varepsilon}(t)| \le C\varepsilon^{-1}\|a_{jk}(\cdot)\|_{L^\infty((0,T))}$ and $\varepsilon = 1/|\xi|$, we see from (3.5) that there exists a constant $C > 0$ such that

$$C(|\partial_t u(t)|^2 + |\xi|^2|u(t)|^2) \le E_{1/|\xi|}(t) \le C^{-1}(|\partial_t u(t)|^2 + |\xi|^2|u(t)|^2)$$

for any $t \in [0,T]$ and any $\xi \in \mathbb{R}^d \setminus \{0\}$. Then we obtain the desired estimate (3.3). $\square$

Since $u(t_2) = u(t_1) + i\int_{t_1}^{t_2} \partial_t u(t)\,dt$, from (3.3) we obtain

$$\begin{aligned} &(|\partial_t u(t_2)|^2 + (|\xi|^2+1)|u(t_2)|^2)^{1/2} \\ &\le C_T e^{C_2(\log(|\xi|+1)+1+\gamma)^\gamma}\big((|\partial_t u(t_1)|^2 + (|\xi|^2+1)|u(t_1)|^2)^{1/2} + \int_{t_1}^{t_2} |f(t)|\,dt\big) \end{aligned} \tag{3.12}$$

where the constant C_T may depend on the length of the interval $[0,T]$.

Now consider $u(t,x) \in C^2([0,T],H^\infty)$. Let

$$f(t,x) = \partial_t^2 u(t,x) - \sum_{j,k=1}^d a_{jk}(t)\partial_{x_j}\partial_{x_k} u(t,x).$$

Then we have $P_2\hat{u}(t,\xi) = \hat{f}(t,\xi)$ where $\hat{u}(t,\xi)$ and $\hat{f}(t,\xi)$ are the Fourier transform of $u(t,x)$ and $f(t,x)$ in variables x respectively. Then from (3.12), we obtain

$$\begin{aligned} &(|\partial_t \hat{u}(t_2,\xi)|^2 + (|\xi|^2+1)|\hat{u}(t_2,\xi)|^2)^{1/2} \\ &\le C_T e^{C_2(\log(|\xi|+1)+1+\gamma)^\gamma}\big((|\partial_t \hat{u}(t_1,\xi)|^2 + (|\xi|^2+1)|\hat{u}(t_1,\xi)|^2)^{1/2} \\ &\quad + \int_{t_1}^{t_2} |\hat{f}(t,\xi)|\,dt\big) \quad \text{for } 0 \le t_1 < t_2 \le T. \end{aligned} \tag{3.13}$$

Hence from the estimate $\| \int_{t_1}^{t_2} g(t,\xi)\,dt\|_{L^2(\mathbb{R}^d_\xi)} \le \int_{t_1}^{t_2} \|g(t,\xi)\|_{L^2(\mathbb{R}^d_\xi)}\,dt$, which follows from the convexity of norm, and the Plancherel Theorem, we obtain

$$
\begin{aligned}
&\|\partial_t u(t_2,\cdot)\| + \sum_{|\alpha|\le 1} \|\partial_x^\alpha u(t_2,\cdot)\| \\
&\le C\big(\|A_\gamma \partial_t u(t_1,\cdot)\| + \sum_{|\alpha|\le 1} \|A_\gamma \partial_x^\alpha u(t_1,\cdot)\| + \int_{t_1}^{t_2} \|A_\gamma f(t,\cdot)\|\,dt\big)
\end{aligned}
\tag{3.14}
$$

where A_γ is a Fourier multiplier given by

$$
A_\gamma v(x) = \int e^{i(x-y)\xi} e^{\frac{1}{2}C_2(\log(|\xi|+1)+1+\gamma)^\gamma} v(y)\,d\xi dy/(2\pi)^d.
$$

Similarly multiplying (3.13) by $(|\xi|^2+1)^{s/2}$ with $s \in \mathbb{R}$, we obtain

$$
\begin{aligned}
&\|\partial_t u(t_2,\cdot)\|_s + \|u(t_2,\cdot)\|_{s+1} \\
&\le C\big(\|A_\gamma \partial_t u(t_1,\cdot)\|_s + \|A_\gamma u(t_1,\cdot)\|_{s+1} + \int_{t_1}^{t_2} \|A_\gamma f(t,\cdot)\|_s\,dt\big).
\end{aligned}
\tag{3.15}
$$

If $\gamma = 0$, then $A_0 v(x) = Cv(x)$ with $C = e^{C_2/2}$. Hence

$$
\|A_0 v(\cdot)\| \le C\|v(\cdot)\| \tag{3.16}
$$

for any $v \in L^2$, while $e^{C_2(\log(|\xi|+1)+2))/2} = C(|\xi|+1)^m$ with $m = C_2/2$ implies that

$$
\|A_1 v(\cdot)\|_s \le C\|v(\cdot)\|_{s+m} \tag{3.17}
$$

with some $m \ge 0$ for any $s \in \mathbb{R}$ and any $v \in H^{s+m}$. Then we have the following theorem.

Theorem 3.2. *Let $a_{jk}(t)$ $(j,k = 1,\dots,d)$ be a real-valued bounded measurable function on $(0,T)$ with $T > 0$ satisfying $a_{kj}(t) = a_{jk}(t)$ and (3.1). Let L be a second order hyperbolic operator given by*

$$
L = \partial_t^2 - \sum_{j,k=1}^d a_{jk}(t)\partial_{x_j}\partial_{x_k}.
$$

If $a_{jk}(t) \in Z_0((0,T))$ $(j,k = 1,\dots,d)$, then we have the estimate

$$
\sum_{l+|\alpha|\le 1} \|\partial_t^l \partial_x^\alpha u(t_2,\cdot)\| \le C(\sum_{l+|\alpha|\le 1} \|\partial_t^l \partial_x^\alpha u(t_1\cdot)\| + \int_{t_1}^{t_2} \|Lu(s,\cdot)\|\,ds) \tag{3.18}
$$

for any $0 \le t_1 \le t_2 \le T$. Here $u(t,x) \in \bigcap_{j=0}^1 C^j([0,T],H^{1-j})$ satisfying $Lu \in L^1([0,T],L^2)$.

If $a_{jk}(t) \in Z_1((0,T))$ $(1 \le j,k \le d)$, then the Cauchy problem for L is C^∞ well posed. Namely, for any $u_0(x), u_1(x) \in C^\infty(\mathbb{R}^d)$ and $f(t,x) \in L^1([0,T],C^\infty(\mathbb{R}^d))$, we have a unique solution $u(t,x) \in C^1([0,T],C^\infty(\mathbb{R}^d))$ to the equation $Lu = f(t,x)$ on $(0,T)\times\mathbb{R}^d$ with the initial conditions $u(0,x) = u_0(x)$ and $\partial_t u(0,x) = u_1(x)$.

Proof. Assume that $a_{jk}(t) \in Z_0((0,T))$ $(j,k = 1,\dots,d)$. If $u(t,x)$ belongs to $\bigcap_{j=0}^2 C([0,T],H^{2-j})$, the estimate (3.18) follows from (3.14) with $\gamma = 0$ and (3.16). In the case where $u(t,x) \in \bigcap_{j=0}^1 C^j([0,T],H^{1-j})$ and $f(t,x) = Lu \in L^1([0,T],L^2)$, we regularize $u(t,x)$ with respect to x-variables by setting $u_\delta(t,x) = \int e^{i(x-y)\xi}(1+\delta|\xi|^2)^{-1}u(t,y)\,d\xi dy/(2\pi)^d$ with $\delta > 0$. We denote this by $(1-\delta\Delta)^{-1}u(t,x)$. Then we regularize u_δ with respect to t-variable by setting $u_\delta^\varepsilon(t,x) = \int_{\mathbb{R}} \psi_\varepsilon(t-s) u_\delta(s,x)\, ds$ with $\varepsilon > 0$ where $\psi_\varepsilon(s)$ is given by $\psi_\varepsilon(s) = \psi(s/\varepsilon)/\varepsilon$ with a smooth function $\psi(s)$ on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} \psi(s)\, ds = 1 \text{ and } \psi(s) = 0 \quad \text{for } s \geq 0 \text{ or } s \leq -1.$$

We denote this convolution by $\psi_\varepsilon * u_\delta(t,x)$. Then we see that $Lu_\delta^\varepsilon(t,x) = F_\delta^\varepsilon(t,x)$ for $t \in [0, T-\varepsilon]$ where $F_\delta^\varepsilon(t,x) = f_\delta^\varepsilon(t,x) + R_\delta^\varepsilon$ with $f_\delta^\varepsilon(t,x) = \psi_\varepsilon * (1-\delta\Delta)^{-1} f(t,x)$ and

$$R_\delta^\varepsilon = \sum_{j,k=1}^{d} [\psi_\varepsilon *, a_{jk}(t)] \partial_{x_j} \partial_{x_k} u_\delta(t,x).$$

Here $[\cdot,\cdot]$ denotes the commutator. Since $u_\delta^\varepsilon(t,x) \in \bigcap_{j=0}^{2} C^j([0,T-\varepsilon], H^{2-j})$, the estimate (3.18) is valid for $u_\delta^\varepsilon(t,x)$ when $0 \leq t_1 \leq t_2 \leq T-\varepsilon$. Since $f(t,x) \in L^1([0,T], L^2)$, we see that, for $0 \leq t_1 \leq t_2 < T$, $\int_{t_1}^{t_2} \|f_\delta^\varepsilon(t,\cdot)\|\, dt$ converges to $\int_{t_1}^{t_2} \|f_\delta(t,\cdot)\|\, dt$ as ε tends to zero. While $u(t,x) \in C^0([0,T], H^1)$ implies $u_\delta(t,x) \in C^0([0,T], H^2)$. Then we have $\partial_{x_j}\partial_{x_k} u_\delta(t,x) \in C^0([0,T], L^2)$, which implies that $\psi_\varepsilon * \partial_{x_j}\partial_{x_k} u_\delta$ and $\psi_\varepsilon * (a_{jk}(t)\partial_{x_j}\partial_{x_k} u_\delta)$ converge to $\partial_{x_j}\partial_{x_k} u_\delta$ and $a_{jk}(t)\partial_{x_j}\partial_{x_k} u_\delta$ in $L^1([0,T], L^2)$ respectively as ε tends to zero. Hence we see that $\int_{t_1}^{t_2} \|R_\delta^\varepsilon\|\, dt \to 0$ as ε tends to zero when $0 \leq t_1 \leq t_2 < T$, Then the estimate (3.18) is valid for $u_\delta(t,x)$ when $0 \leq t_1 \leq t_2 < T$. Finally we obtain the desired estimate for $u(t,x)$ by taking $\delta \to 0$.

Now consider the case where $a_{jk}(t) \in Z_1((0,T))$ $(j,k = 1,\dots,d)$. The estimates (3.15) with $\gamma = 1$ and (3.17) imply that for any $u_0(x) \in H^{s+1}$, $u_1(x) \in H^s$ and $f(t,x) \in L^1([0,T], H^s)$ with arbitrarily chosen $s \in \mathbb{R}$, there exist a solution $u(t,x) \in \bigcap_{j=0,1} C^j([0,T], H^{s+1-j-m})$ with some positive m independent of s to the equation $Lu = f$ satisfying the initial condition $u(0,x) = u_0(x)$ and $\partial_t u(0,x) = u_1(x)$. The uniqueness of solutions follows from the existence of solutions to the adjoint Cauchy problem. Then the Cauchy problem is H^∞ well posed. Since in the article [2] one has shown the existence of the finite propagation speed for L with the coefficients in more general function classes, we see that the Cauchy problem is C^∞ well posed. We see also the existence of finite propagation speed for L by considering the wave operator $L_\varepsilon = \partial_t^2 - \sum_{j,k=1}^{d} a_{jk,\varepsilon}(t)\partial_{x_j}\partial_{x_k}$ where the coefficients $a_{jk,\varepsilon}(t)$ $(j,k = 1\dots,d)$ are defined at the beginning of the proof of Theorem 3.1 as the regularization of $a_{jk}(t)$ $(j,k = 1\dots,d)$. First remark that we see from (3.5) that the propagation speed for L_ε is not larger than C_1. For any smooth and compactly supported initial data u_0, u_1 and $f(t)$, solutions u_ε $(0 < \varepsilon < 1)$ to the equation $L_\varepsilon u_\varepsilon = f$ with the initial condition $u_\varepsilon(0,x) = u_0(x)$ and $\partial_t u_\varepsilon(0,x) = u_1(x)$ have the uniform estimate (3.15) with $\gamma = 1$ and (3.17). Hence we see that the solution u to the equation $Lu = f$ with the same initial condition $u(0,x) = u_0(x)$ and $\partial_t u(0,x) = u_1(x)$ can be obtained as a limit of a suitable subsequence $\{u_{\varepsilon_n}(t,x)\}$ with $\varepsilon_n \to 0$. Then we see the existence of finite propagation speed for L. $\square$

Example 3.3. From example 2.3 of the previous section and Theorem 3.2, we see that the L^2 estimate (1.2) for $L = \partial_t^2 - (2 + w_0(t))\partial_x^2$ holds where $w_0(t)$ is a continuous and nowhere differentiable function given by $w_0(t) = \sum_{n\geq 1} 2^{-n} \cos 2^n t$.

Remark 3.4. We assume the boundedness of the coefficients in the theorems above. While Colombini, De Giorgi and Spagnolo [2] have shown that the condition (1.4) without the assumption of boundedness is sufficient for C^∞ well posed. But we see from the example 2.2 of the previous section that, even for bounded functions, the condition (2.1) with $\gamma = 1$ is still less restrictive than that of (1.4). For the related problem for wave equations with unbounded coefficients having some special type of singularity see, for example, [6] or [9].

Remark 3.5. As mentioned in Theorem 3.1, the constants C_1 and C_2 in (3.3) are independent of the length of interval. Then we obtain the following from (3.3) with $\gamma = 0$. If $a_{jk}(t)$ $(j, k = 1, \dots, d)$ belongs to $Z_0((0,\infty))$, that is, $a_{jk}(t)$ is bounded measurable on $(0,\infty)$ and satisfies

$$\int_\varepsilon^\infty |a_{jk}(t+\varepsilon) + a_{jk}(t-\varepsilon) - 2a_{jk}(t)|\, dt \le C\varepsilon \quad \text{for any } \varepsilon > 0,$$

then under the condition (3.1) with $T = \infty$ we have the following estimate for the homogeneous energy $E_0(u)(t) = \|\partial_t u(t,\cdot)\|^2 + \sum_{j=1}^d \|\partial_{x_j} u(t,\cdot)\|^2$:

$$E_0(u)(t_1) \le C E_0(u)(t_0) \quad \big(t_0, t_1 \in [0,\infty)\big)$$

for any $u(t,x) \in \bigcap_{j=0,1} C^j([0,\infty), H^{1-j})$ satisfying $Lu = 0$ on $(0,\infty) \times \mathbb{R}^d$.

4. Appendix

In this section we show (2.8). Let $h_\gamma(t) = \sin(|\log t|^{1+\gamma})$ with $\gamma > 0$. For any positive integer n, let $t_n, t_{n-}, t_{n+} \in (0,1)$ be given by

$$t_n = e^{-(2\pi n)^{1/(1+\gamma)}}, \quad t_{n-} = e^{-(2\pi n - \pi/4)^{1/(1+\gamma)}}, \quad t_{n+} = e^{-(2\pi n + \pi/4)^{1/(1+\gamma)}}.$$

We note $t_{n+} < t_n < t_{n-}$ and

$$|\log t_n|^{1+\gamma} = 2\pi n, \quad |\log t_{n-}|^{1+\gamma} = 2\pi n - \pi/4, \quad |\log t_{n+}|^{1+\gamma} = 2\pi n + \pi/4.$$

We obtain $t_{n-} - t_n > t_n - t_{n+}$ from $\frac{d}{ds} e^{-s^{1/(1+\gamma)}} < 0$ and $\frac{d^2}{ds^2} e^{-s^{1/(1+\gamma)}} > 0$.

Since $h_\gamma'(t) = -(1+\gamma)\cos(|\log t|^{1+\gamma})|\log t|^\gamma/t$ on $(0,1)$, we see that

$$|h_\gamma'(t)| \ge C|\log t|^\gamma/t \quad \text{for } t_{n+} \le t \le t_{n-}.$$

Since $Cn^{-\gamma/(1+\gamma)} \le |(2n \pm 1/4)^{1/(1+\gamma)} - (2n)^{1/(1+\gamma)}| \le C^{-1} n^{-\gamma/(1+\gamma)}$, we see that

$$t_n - t_{n+} \ge C e^{-(2\pi n)^{1/(1+\gamma)}} n^{-\gamma/(1+\gamma)}$$

and $1 \le t_{n-}/t_{n+} \le C$. Then when $0 < \varepsilon \le t_n - t_{n+}$, we have

$$|h_\gamma(t+\varepsilon) - h_\gamma(t)| \ge C\varepsilon(|\log t|^\gamma/t) \quad \text{for } t_{n+} \le t \le t_n,$$

from which we have

$$I_n = \int_{t_{n+}}^{t_n} |h_\gamma(t+\varepsilon) - h_\gamma(t)|\, dt \ge C\varepsilon(|\log t_{n+}|^{1+\gamma} - |\log t_n|^{1+\gamma}.)$$

Then we have $I_n \ge C\varepsilon$ with some constant $C > 0$ for any positive integer n and $\varepsilon \in [0, t_n - t_{n+}]$. We pick a large positive integer n_0 so that we have $t_{n-} \le 1/2$ for $n \ge n_0$. For any large positive integer $N > n_0$, pick $\varepsilon = t_N - t_{N+}$. Then we see that

$$\int_0^{1/2-\varepsilon} |h_\gamma(t+\varepsilon) - h_\gamma(t)|\, dt \ge \sum_{n=n_0}^N I_n \ge C(N - n_0 + 1)\varepsilon.$$

Since $t_N - t_{N+} \geq Ce^{-(2\pi N)^{1/(1+\gamma)}} N^{-\gamma/(1+\gamma)}$, we see that $N \geq C|\log\varepsilon|^{1+\gamma}$ for large N. Then we see that

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon|\log\varepsilon|^{1+\gamma}} \int_0^{1/2-\varepsilon} |h_\gamma(t+\varepsilon) - h_\gamma(t)|\,dt > 0.$$

By a similar argument, we see that $h_\gamma(t) \notin Z_\sigma((0,1/2))$ when $0 \leq \sigma < 2\gamma$. Indeed noting that $h''_\gamma(t) = -(\gamma+1)^2 \sin(|\log t|^{\gamma+1})|\log t|^{2\gamma}/t^2 + r(t)$ where $|r(t)| \leq C|\log t|^\gamma/t^2$, we choose s_n and $s_{n\pm}$ in $(0,1)$ so that $|\log s_n|^{\gamma+1} = 2\pi n + \pi/2$ and $|\log s_{n\pm}|^{\gamma+1} = 2\pi n + \pi(1/2 \pm 1/4)$. Then if an integer n is large and $\varepsilon \leq (s_n - s_{n+})/2$, we have

$$\int_{s_{n+}+\varepsilon}^{s_{n-}-\varepsilon} |h_\gamma(t+\varepsilon) + h_\gamma(t-\varepsilon) - 2h_\gamma(t)|\,dt \geq C\varepsilon^2(s_n - s_{n+})|\log s_{n-}|^{2\gamma}/s_{n-}^2,$$

from which and from the arguments similar to the above we see that

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon(\log(\varepsilon^{-1}+1)+1+\gamma)^\gamma} \int_\varepsilon^{1/2-\varepsilon} |h_\gamma(t+\varepsilon) + h_\gamma(t-\varepsilon) - 2h_\gamma(t)|\,dt > 0.$$

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