WELL-POSEDNESS FOR SOME PERTURBATIONS OF THE KDV EQUATION WITH LOW REGULARITY DATA

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ABSTRACT. We study some well-posedness issues of the initial value problem associated with the equation
$$u_t + u_{xxx} + \eta Lu + uu_x = 0, \quad x \in \mathbb{R}, \ t \geq 0,$$
where $\eta > 0$, $\widehat{Lu}(\xi) = -\Phi(\xi)\hat{u}(\xi)$ and $\Phi \in \mathbb{R}$ is bounded above. Using the theory developed by Bourgain and Kenig, Ponce and Vega, we prove that the initial value problem is locally well-posed for given data in Sobolev spaces $H^s(\mathbb{R})$ with regularity below L^2. Examples of this model are the Ostrovsky-Stepanyams-Tsimring equation for $\Phi(\xi) = |\xi| - |\xi|^3$, the derivative Korteweg-de Vries-Kuramoto-Sivashinsky equation for $\Phi(\xi) = \xi^2 - \xi^4$, and the Korteweg-de Vries-Burguers equation for $\Phi(\xi) = -\xi^2$.

1. INTRODUCTION

In this paper we consider the initial value problem (IVP)
$$
\begin{aligned}
u_t + u_{xxx} + \eta Lu + uu_x = 0, \quad x \in \mathbb{R}, \ t \geq 0,\\
u(x,0) = u_0(x),
\end{aligned}
\tag{1.1}
$$
where $\eta > 0$ is a constant and the linear operator L is defined via the Fourier transform by $\widehat{Lu}(\xi) = -\Phi(\xi)\hat{u}(\xi)$. The Fourier symbol
$$
\Phi(\xi) = \sum_{j=0}^{n} \sum_{i=0}^{2m} c_{i,j} \xi^i |\xi|^j, \quad c_{i,j} \in \mathbb{R}, \ c_{2m,n} = -1.
\tag{1.2}
$$
is a real valued function which is bounded above; i.e., there is a constant C such that $\Phi(\xi) < C$. Without loss of generality, we suppose that $\Phi(\xi) < 1$. For this, let us perform the following scale change
$$
v(x,t) = \frac{1}{\lambda^2} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^3}\right).
$$
Then v satisfies the equation
$$
\lambda^3 v_t + \lambda^3 v_{xxx} + \eta Tv + \lambda^3 vv_x = 0,
\tag{1.3}
$$

where
$$\widehat{Tv}(\xi) = \Phi(\lambda\xi)\hat{v}(\xi).$$
If we take $\lambda^3 = C$, where C is as earlier, then the Fourier symbol of the new operator T in (1.3) is bounded above by 1. Finally, inverting the scale change, we obtain well-posedness result for the original IVP (1.1) from that of (1.3). So, throughout this work we consider the IVP (1.1) with $\Phi(\xi)$ in (1.2) satisfying $\Phi(\xi) < 1$.

Our interest here is to obtain well-posedness results for (1.1) with given data u_0 in the Sobolev spaces $H^s(\mathbb{R})$ with regularity below L^2. The L^2-based Sobolev space $H^s(\mathbb{R})$ is defined by
$$H^s(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{H^s} < \infty\},$$
where
$$\|f\|_{H^s}^2 = \int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi,$$
and $\hat{f}(\xi)$ is the usual Fourier transform given by
$$\hat{f}(\xi) \equiv \mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx.$$
However, from here onwards, we will neglect the factor 2π in the definition of the Fourier transform because it does not alter our analysis.

Also, we consider the homogeneous Sobolev space $\dot{H}^s(\mathbb{R})$ defined via the norm
$$\|f\|_{\dot{H}^s}^2 = \int_{\mathbb{R}} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi.$$

Before stating the main results of this work, we give some examples that belong to the class considered in (1.1).

The first example of this type is the generalized Ostrovsky-Stepanyams-Tsimring (OST) equation.
$$u_t + u_{xxx} - \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) + u^k u_x = 0, \quad x \in \mathbb{R},\ t \ge 0,\ k \in \mathbb{Z}^+, \tag{1.4}$$
$$u(x,0) = u_0(x),$$

where $\mathcal{H}$ denotes the Hilbert transform:
$$\mathcal{H}g(x) = \ \text{P.V.} \frac{1}{\pi} \int \frac{g(x-\xi)}{\xi} d\xi,$$
$u = u(x,t)$ is a real-valued function and $\eta > 0$ is a constant.

Equation (1.4) with $k = 1$ was derived by Ostrovsky et al. in [17] to describe the radiational instability of long waves in a stratified shear flow. Recently, Carvajal and Scialom in [6] considered the IVP (1.4) and proved the local well-posedness results for given data in H^s, $s \ge 0$ when $k = 1, 2, 3$. They also obtained the global well-posedness result for data in L^2 when $k = 1$. The earlier well-posedness results for (1.4) with $k = 1$ can be found in [1], where for given data in $H^s(\mathbb{R})$, local result when $s > 1/2$ and global result when $s \ge 1$ have been obtained.

Another model that fits in the class (1.1) is the derivative Korteweg-de Vries-Kuramoto Sivashinsky equation
$$u_t + u_{xxx} + \eta(u_{xx} + u_{xxxx}) + uu_x = 0, \quad x \in \mathbb{R},\ t \ge 0, \tag{1.5}$$
$$u(x,0) = u_0(x),$$

where $u = u(x,t)$ is a real-valued function and $\eta > 0$ is a constant.

This equation arises as a model for long waves in a viscous fluid flowing down an inclined plane and also describes drift waves in a plasma (see [8, 18]). The equation (1.5) is a particular case of Benney-Lin equation [2, 18]; i.e.,

$$u_t + u_{xxx} + \eta(u_{xx} + u_{xxxx}) + \beta u_{xxxxx} + u u_x = 0, \quad x \in \mathbb{R}, \ t \geq 0,$$
$$u(x,0) = u_0(x), \tag{1.6}$$

when $\beta = 0$.

The IVP associated to (1.5) was studied by Biagioni, Bona, Iorio and Scialom in [3]. They also determined the limiting behavior of solutions as the dissipation tends to zero. Biagioni and Linares proved global well-posedness for the IVP (1.6) for initial data in L^2 in [4].

Another example is the Korteweg-de Vries-Burgers equation

$$u_t + u_{xxx} - \eta u_{xx} + u u_x = 0, \quad x \in \mathbb{R}, \ t \geq 0, \ \eta > 0,$$
$$u(x,0) = u_0(x), \tag{1.7}$$

Recently, Molinet and Ribaud considered the IVP (1.7) in [15] and proved that it is locally well-posed for given data in H^s, $s > -1$. The equation (1.7) is also known as the parabolic regularization of the KdV equation with $\eta > 0$. Some years ago, when the interest was to obtain local results for given data in larger Sobolev spaces, this regularization was used to obtain well-posedness results for $\eta > 0$ and then pass the limit $\eta \downarrow 0$. However, this limit is a delicate matter.

Now, we state the main results of this work. The first result deals with the local well-posedness for given data in the Sobolev spaces of negative index.

Theorem 1.1. *The IVP (1.1) with $\eta > 0$ and $\Phi(\xi)$ given by (1.2) is locally well-posed for any data $u_0 \in H^s(\mathbb{R})$, $s > -3/4$.*

To prove this theorem we follow the theory developed by Bourgain [5] and Kenig, Ponce and Vega [11]. The main ingredients in the proof are estimates in the integral equation associated to an extended IVP that is defined for all $t \in \mathbb{R}$ (see IVP (1.12) below). The proof we presented here does not use the Bourgain type space associated to the linear part of the IVP (1.1); instead it uses the usual Bourgain space associated to the KdV equation. To carry out this scheme, the Proposition 2.2 plays a fundamental role which permits us to use a bilinear estimate for $\partial_x(u^2)$ (see [11]), that is a central part of our arguments.

The result of the Theorem 1.1 improves the known local well-posedness results for the IVP (1.4) and (1.5) described above. Note that, the value $s > -3/4$, in the case of the Korteweg-de Vries (KdV) equation, is sharp in the sense that for $s < -3/4$, the IVP associated to the KdV equation is ill-posed. We should mention that, the lack of conserved quantities in the spaces with regularity below L^2, prevents us to get global solution using the usual technique.

The second result is concerned with the particular case of the IVP (1.1) for given data in the homogeneous Sobolev space when the Fourier symbol is of the form $\Phi(\xi) = |\xi|^k - |\xi|^{k+2}$, $k \in \mathbb{Z}^+$.

Theorem 1.2. *The IVP (1.1) with $\eta > 0$ and $\Phi(\xi) = |\xi|^k - |\xi|^{k+2}$, $k \in \mathbb{Z}^+$, is locally well-posed for any data $u_0 \in \dot{H}^s(\mathbb{R})$, $s > -1/2$.*

Although this theorem does not improve the result obtained in Theorem 1.1, it is interesting on its own because the proof we present here uses different tools, that are simpler than the ones used in the proof of Theorem 1.1. The main ingredients

in the proof are the refined local smoothing effect (see (3.4) in Corollary 3.3 below), and a Strichartz type estimate (see Proposition 4.1 below). Using these estimates we are able to apply fixed point argument to obtain a local well-posedness result in the homogeneous Sobolev spaces of negative order without the use of Bourgain type spaces.

Now we introduce function spaces that will be used for proving Theorem 1.1. We consider the following IVP associated to the Linear KdV equation

$$u_t + u_{xxx} = 0, \quad x, \, t \in \mathbb{R},$$
$$u(0) = u_0. \tag{1.8}$$

The solution to (1.8) is given by $u(x,t) = U(t)u_0(x)$, where the unitary group $U(t)$ is defined as

$$\widehat{U(t)u_0}(\xi) = e^{it\xi^3}\widehat{u_0}(\xi). \tag{1.9}$$

For $s, b \in \mathbb{R}$, we define the space $X_{s,b}$ as the completion of the Schwartz space $S(\mathbb{R}^2)$ with respect to the norm

$$\|u\|_{X_{s,b}} \equiv \|U(-t)u\|_{H_{s,b}} := \|\langle\tau\rangle^b\langle\xi\rangle^s\widehat{U(-t)}u(\xi,\tau)\|_{L^2_\tau L^2_\xi}$$
$$= \|\langle\tau - \xi^3\rangle^b\langle\xi\rangle^s\widehat{u}(\xi,\tau)\|_{L^2_\tau L^2_\xi}, \tag{1.10}$$

where $\widehat{u}(\xi,\tau)$ is the Fourier transform of u in both space and time variables. The space $X_{s,b}$ is the usual Bourgain space for the KdV equation (see [5]).

Note that (1.1) is defined only for $t \geq 0$. To use Bourgain's type space, we should be able to write the IVP (1.1) for all $t \in \mathbb{R}$. For this, we define

$$\eta(t) \equiv \eta\operatorname{sgn}(t) = \begin{cases} \eta & \text{if } t \geq 0, \\ -\eta & \text{if } t < 0 \end{cases} \tag{1.11}$$

and write (1.1) in the form

$$u_t + u_{xxx} + \eta(t)Lu + uu_x = 0, \quad x, \, t \in \mathbb{R},$$
$$u(0) = u_0. \tag{1.12}$$

Now we consider the IVP associated to the linear part of (1.12),

$$u_t + u_{xxx} + \eta(t)Lu = 0, \quad x, \, t \in \mathbb{R},$$
$$u(0) = u_0. \tag{1.13}$$

The solution to (1.13) is given by $u(x,t) = V(t)u_0(x)$ where the semigroup $V(t)$ is defined as

$$\widehat{V(t)u_0}(\xi) = e^{it\xi^3 + \eta|t|\Phi(\xi)}\widehat{u_0}(\xi). \tag{1.14}$$

Observe that, defining $\widetilde{U}(t)$ by $\widehat{\widetilde{U}(t)u_0}(\xi) = e^{\eta|t|\Phi(\xi)}\widehat{u_0}(\xi)$, the semigroup $V(t)$ can be written as $V(t) = U(t)\widetilde{U}(t)$ where $U(t)$ is the unitary group associated to the KdV equation (see (1.9)).

This paper is organized as follows: In Section 2, we prove Theorem 1.1. In Section 3, we present a refined local smoothing effect when $\Phi(\xi) = |\xi|^k - |\xi|^{k+2}$, $k \in \mathbb{Z}^+$, in (1.2). In Section 4, we to obtain some Strichartz type estimates. In Section 5, we prove Theorem 1.2.

2. Local Well-posedness in H^s for $s > -3/4$

This section is devoted to supply the proof of the Theorem 1.1. We start by proving some preliminary results.

2.1. Preliminary estimates.

Proposition 2.1. *Let $s > -3/4$. There exist $b' \in (-\frac{1}{2}, 0)$ and $\epsilon_s > 0$ such that for any $b \in (\frac{1}{2}, b' + 1]$ with $1 - b + b' \leq \epsilon_s$, and $u, v \in X_{s,b}$*

$$\|(uv)_x\|_{X_{s,b'}} \leq c\|u\|_{X_{s,b}}\|v\|_{X_{s,b}}.$$

The proof of the above proposition can be found in [11].

We consider a cut-off function $\psi \in C^\infty(\mathbb{R})$, such that $0 \leq \psi(t) \leq 1$,

$$\psi(t) = \begin{cases} 1 & \text{if } |t| \leq 1, \\ 0 & \text{if } |t| \geq 2. \end{cases} \tag{2.1}$$

Let us define $\psi_T(t) \equiv \psi(\frac{t}{T})$ and $\tilde{\psi}_T(t) = sgn(t)\psi_T(t)$.

The following Proposition plays a central role in the proof of our first main result, Theorem 1.1. This Proposition allows us to work in the usual $X_{s,b}$ space associated to the KdV equation instead of the Bourgain space associated to the IVP (1.12).

Proposition 2.2. *Let $-1/2 < b' \leq 0$, $T \in [0, 1]$. Then we have*

$$\|\psi(t)V(t)u_0\|_{X_{s,b}} \leq c\|u_0\|_s. \tag{2.2}$$

If $1/2 < b \leq b'/3 + 2/3$, $s \in \mathbb{R}$ then

$$\|\psi_T(t)\int_0^t V(t - t')(uu_x)(t')dt'\|_{X_{s,b}} \leq cT^{1+b'/2-3b/2}\|uu_x\|_{X_{s,b'}}, \tag{2.3}$$

where c is a constant.

Before proving this proposition, we record the following results.

Lemma 2.3. *Let $0 < T \leq 1$, $1/2 < b < 1$ and $a < 1$. Then we have*

$$\|\psi_T(t)\|_{H_t^b} \leq c(T^{1/2} + T^{1/2-b}), \tag{2.4}$$

$$\|\psi_T(t)\,e^{a|t|}\|_{H_t^b} \leq T^{1/2}\langle\frac{1}{T^b}\rangle. \tag{2.5}$$

$$|\mathcal{F}(|t|\,\psi_T(t)\,e^{a|t|})(\tau)| \leq \frac{cT^2}{1 + (\tau^2 + a^2)T^2}, \tag{2.6}$$

$$|\mathcal{F}(|t|\,\tilde{\psi}_T(t)\,e^{a|t|})(\tau)| \leq \frac{cT^2}{1 + (\tau^2 + a^2)T^2}, \tag{2.7}$$

where c is a constant independent of T and a.

Proof. Using the definition of the space H^b, we have

$$\|\psi_T(t)\|_{H_t^b} \leq c\|\psi_T\|_{L^2} + c\|D_t^b\psi_T\|_{L^2} = cT^{1/2}\|\psi\|_{L^2} + cT^{1/2-b}\|D_t^b\psi\|_{L^2},$$

where we used the fact that $\widehat{h(t/T)}(\tau) = T\hat{h}(T\tau)$. To prove (2.5), we exploit the compact support of ψ_T to get

$$\|\psi_T(t)\,e^{a|t|}\|_{H_t^b} \leq cT^{1/2}\|h\|_{L^2} + c\|D_t^b(\psi_T(t)\,e^{a|t|})\|_{L^2} \leq c(T^{1/2} + T^{1/2-b}\|D_t^bh\|_{L^2}),$$

where $h(t) = \psi(t)e^{aT|t|}$. Integrating by parts twice we obtain

$$|\hat{h}(\tau)| \leq \frac{c}{\langle\tau\rangle^2},$$

where, in the case when $a < 0$, we have used the fact that $|x|e^x \leq e$, $\forall x \leq 1$, in particular for $x = aT$. This proves inequality (2.5).

To prove inequality (2.6) we have

$$\mathcal{F}(|t|\,\psi_T(t)\,e^{a|t|})(\tau) = T^2\hat{p}(T\tau) \tag{2.8}$$

where $p(t) = |t|\psi(t)e^{aT|t|}$. Integrating by parts, we obtain

$$|\hat{p}(\tau)| \leq \frac{c}{|\tau - iaT|^k}, \quad k = 0, 1, 2.$$

Therefore,

$$|\hat{p}(\tau)| \leq \frac{c}{1 + \tau^2 + a^2T^2}. \tag{2.9}$$

Combining (2.8) and (2.9) yields the desired inequality (2.6). The proof of (2.7) is similar. $\square$

Remark. It's not possible to obtain similar inequalities as (2.4) and (2.5) for $\tilde{\psi}_T(t)$ because of the discontinuity.

In the following estimates, without loss of generality, we suppose that $\eta = 1$.

Lemma 2.4. *Let $-1/2 < b' \leq 0$, $1/2 < b \leq b'/3 + 2/3$, $T \in (0, 1]$, $a < 1$. Then*

$$\|\psi_T(t)\int_0^t e^{|t-t'|a}f(t')dt'\|_{H_t^b} \leq c\,T^{1+b'/2-3b/2}\|f\|_{H^{b'}}, \tag{2.10}$$

where c is a constant independent of a, f and T.

Proof. It is sufficient to prove Lemma 2.4 when $|a| \leq 1$. In fact, let us suppose that Lemma 2.4 has been established in the case $|a| \leq 1$. Then when $a < -1$, we use the change of variable $t'a \equiv t'$, to obtain

$$\psi_T(t)I_a(t) := \psi_T(t)\int_0^t e^{|t-t'|a}f(t')dt' = \frac{1}{a}\psi_{aT}(at)\int_0^{at} e^{|at-t'|}f_a(t')dt' = \frac{1}{a}J(at). \tag{2.11}$$

where $f_a(t') = f(t'/a)$ and $J(t) = \psi_{aT}(t)\int_0^t e^{|t-t'|}f_a(t')dt'$. Note that for $a < -1$,

$$\|J(t)\|_{H^b} \leq c|aT|^{1+b'/2-3b/2}\|f_a\|_{H^{b'}} \leq c|a|^{3/2-b'/2-3b/2}T^{1+b'/2-3b/2}\|f\|_{H^{b'}}. \tag{2.12}$$

Since $b' + b > 0$ and $|a| > 1$, from (2.11) and (2.12) we obtain

$$\|\psi_T(t)I_a(t)\|_{H^b} = \frac{1}{|a|}\|J(at)\|_{H^b}$$

$$\leq c\frac{\langle a\rangle^b}{|a|^{3/2}}\|J(t)\|_{H^b}$$

$$\leq \frac{c}{|a|^{(b'+b)/2}}T^{1+b'/2-3b/2}\|f\|_{H^{b'}}.$$

Hence, we arrived at (2.10) in this case too.

From here onwards, we consider $|a| \leq 1$. Now let $b > 1/2$, then we have

$$I_a(t) := \int_0^t e^{|t-t'|a} f(t') dt' = \int_0^t e^{(t-t')\, sgn(t) a} f(t') dt'$$

$$= e^{a|t|} \int_0^t e^{-sgn(t)at'} \int_{\mathbb{R}} e^{it'\tau} \hat{f}(\tau) d\tau dt'$$

$$= e^{a|t|} \int_{\mathbb{R}} \hat{f}(\tau) \int_0^t e^{(i\tau - sgn(t)a)t'} dt' d\tau$$

$$= e^{a|t|} \int_{\mathbb{R}} \hat{f}(\tau) \frac{e^{(i\tau - sgn(t)a)t} - 1}{i\tau - sgn(t)a} d\tau$$

$$= \int_{\mathbb{R}} \hat{f}(\tau) \frac{e^{i\tau t} - e^{a|t|}}{i\tau - sgn(t)a} d\tau.$$

We have

$$\frac{1}{sgn(t)a - i\tau} = sgn(t) \frac{a}{a^2 + \tau^2} + i \frac{\tau}{a^2 + \tau^2}.$$

If we define

$$p_a(t) = \frac{a}{a^2 + t^2}, \quad q_a(t) = \frac{t}{a^2 + t^2}$$

and replace τ by t' we obtain

$$I_a(t) = sgn(t) \int_{\mathbb{R}} p_a(t') [e^{a|t|} - e^{it't}] \widehat{f}(t') dt' + ic \int_{\mathbb{R}} q_a(t') [e^{a|t|} - e^{it't}] \widehat{f}(t') dt'$$

$$:= I_{a,1}(t) + I_{a,2}(t). \tag{2.13}$$

Estimate for $I_{a,1}$. We consider two cases.

Case 1: $|t'| > 1/T$. Let $\widehat{f}(t') \equiv \widehat{f}(t')\chi_{\{|t'|>1/T\}}$. From the definition of $I_{a,1}$ we have

$$\psi_T(t) I_{a,1}(t) = a\, sgn(t) \psi_T(t) \int_{\mathbb{R}} \frac{\widehat{f}(t')}{a^2 + t'^2} [e^{a|t|} - e^{itt'}] dt' = ah\Big(\frac{t}{T}\Big), \tag{2.14}$$

where $h(t) = sgn(t)\psi(t) \int_{\mathbb{R}} \{\widehat{f}(t')/(a^2 + t'^2)\} [e^{aT|t|} - e^{iTtt'}] dt'$. We have

$$\widehat{h(t)}(\tau) = \int_{\mathbb{R}} \frac{\widehat{f}(t')}{a^2 + t'^2} K(a, T, \tau, t') dt', \tag{2.15}$$

where

$$K(a, T, \tau, t') = \int_{\mathbb{R}} sgn(t)\psi(t) [e^{aT|t|} - e^{iTtt'}] e^{-it\tau} dt.$$

Integrating by parts,

$$|K(a, T, \tau, t')| \leq c\frac{\langle t'\rangle}{\langle \tau\rangle}, \quad \text{and} \quad |K(a, T, \tau, t')| \leq c\frac{\langle t'\rangle}{\langle \tau\rangle^2} + c\frac{\langle t'\rangle^2}{\langle \tau\rangle^2} \leq c\frac{\langle t'\rangle^2}{\langle \tau\rangle^2}.$$

Hence

$$|K(a, T, \tau, t')| \leq c\frac{\langle t'\rangle^{2b}}{\langle \tau\rangle^{2b}}.$$

Therefore, from (2.15) we obtain

$$|\widehat{h(t)}|(\tau) \le \frac{c}{\langle\tau\rangle^{2b}} \int_{|t'|>1/T} \frac{|\widehat{f}(t')|}{a^2+t'^2} \langle t'\rangle^{2b} dt' \le c\frac{T^{3/2+b'-2b}}{\langle\tau\rangle^{2b}} \|f\|_{H^{b'}}.$$

Now, using (2.14) we have

$$\|\psi_T(t)I_a(t)\|_{H^b} = |a| \|h\Big(\frac{t}{T}\Big)\|_{H^b} \le |a|T^{1/2-b}\|h(t)\|_{H^b} \le cT^{3/2+b'-2b}T^{1/2-b}\|f\|_{H^{b'}}.$$

Hence

$$\|\psi_T(t)I_a(t)\|_{H^b} \le cT^{2+b'-3b}\|f\|_{H^{b'}} \le cT^{1+b'/2-3b/2}\|f\|_{H^{b'}}.$$

Case 2: $|t'| \le 1/T.$ Let $\widehat{f}(t') \equiv \widehat{f}(t')\chi_{\{|t'|\le 1/T\}}$ and as earlier $\tilde{\psi}_T(t) = sgn(t)\psi_T(t).$ We have

$$\begin{aligned}
\mathcal{F}(\psi_T(t)I_{a,1}(t))(\tau) &= \int_{\mathbb{R}} e^{-it\tau}\tilde{\psi}_T(t) \int_{\mathbb{R}} p_a(t')\big[e^{a|t|} - e^{it't}\big]\widehat{f}(t')dt'dt \\
&= \int_{\mathbb{R}} p_a(t')\widehat{f}(t') \int_{\mathbb{R}} \tilde{\psi}_T(t)e^{-it\tau}\big[e^{a|t|} - e^{it't}\big]dtdt' \\
&= \int_{\mathbb{R}} p_a(t')\widehat{f}(t')\{\mathcal{F}\big(\tilde{\psi}_T(t)\,e^{a|t|}\big)(\tau) - \mathcal{F}\big(\tilde{\psi}_T(t)\big)(\tau-t')\}dt' \\
&= \int_{\mathbb{R}} p_a(t')\widehat{f}(t')\{\mathcal{F}\big(\tilde{\psi}_T(t)\,e^{a|t|}\big)(\tau) - \mathcal{F}\big(\tilde{\psi}_T(t)\,e^{a|t|}\big)(\tau-t')\}dt' \\
&\quad + \int_{\mathbb{R}} p_a(t')\widehat{f}(t')\{\mathcal{F}\big(\tilde{\psi}_T(t)\,e^{a|t|}\big)(\tau-t') - \mathcal{F}\big(\tilde{\psi}_T(t)\big)(\tau-t')\}dt' \\
&:= I_{a,11}(\tau) + I_{a,12}(\tau).
\end{aligned}$$

Since $|p_a(t')| \le 1/|t'|$, we can estimate the term $I_{a,11}(\tau)$ as in [9]. Therefore we will estimate only the term $I_{a,12}(\tau)$.

Let us define $h(t',\tau) := \mathcal{F}\big(\tilde{\psi}_T(t)[e^{a|t|} - 1]\big)(\tau-t')$, then we have

$$h(t',\tau) = \int_0^a \mathcal{F}\big(|t|\,\tilde{\psi}_T(t)\,e^{s|t|}\big)(\tau-t')ds. \tag{2.16}$$

From (2.7) we have that

$$|\mathcal{F}\big(|t|\,\tilde{\psi}_T(t)\,e^{s|t|}\big)(\tau-t')| \le \frac{cT^2}{(1+(|\tau-t'|+|s|)T)^2}, \tag{2.17}$$

where c is independent of s, τ, t' and T.

Observe that $0 \le s \le a$ if $a \ge 0$ and $a \le s \le 0$ if $a \le 0$. Thus we obtain

$$\begin{aligned}
|h(t',\tau)| &\le cT^2 \int_0^{|a|} \frac{1}{(1+(|\tau-t'|+|s|)T)^2}ds \\
&= cT^2 \frac{|a|}{(1+|\tau-t'|T)(1+|\tau-t'|T+|a|T)}.
\end{aligned}$$

As $|Tt'| \le 1$, we have

$$\frac{1}{1+|\tau-t'|T} \le \frac{2}{1+|\tau|T}.$$

Hence

$$|h(t',\tau)| \le cT^2 \frac{|a|}{(1+|\tau|T)^2}.$$

Using the Hölder's inequality we obtain

$$
\begin{aligned}
|I_{a,12}(\tau)| &= |\int_{\mathbb{R}} p_a(t')\hat{f}(t')h(t',\tau)dt'| \\
&\leq \|f\|_{H^{b'}} \Big(\int_{|t'|\leq 1/T} \frac{\langle t'\rangle^{-2b'}|h(t',\tau)|^2}{(a^2+t'^2)^2}dt' \Big)^{1/2} \\
&\leq \frac{cT^2}{(1+|\tau|T)^2} \|f\|_{H^{b'}} \Big(\int_{|t'|\leq 1/T} (1+|t'|^{-2b'})dt' \Big)^{1/2} \\
&\leq \frac{cT^{3/2+b'}}{(1+|\tau|T)^2} \|f\|_{H^{b'}}.
\end{aligned}
\tag{2.18}
$$

Finally, we arrive at

$$
\begin{aligned}
\Big(\int_{\mathbb{R}} (1+|\tau|)^{2b}|I_{a,12}(\tau)|^2\,d\tau \Big)^{1/2} &\leq cT^{3/2+b'} \|f\|_{H^{b'}} \Big(\int_{\mathbb{R}} \frac{1+|\tau|^{2b}}{(1+|\tau|T)^4}d\tau \Big)^{1/2} \\
&\leq cT^{3/2+b'} \|f\|_{H^{b'}} \Big(\frac{1}{T^{1/2}} + \frac{1}{T^{b+1/2}} \Big) \\
&\leq cT^{1-b+b'} \|f\|_{H^{b'}}.
\end{aligned}
$$

Therefore, in this case we have

$$
\|\psi_T I_{a,1}\|_{H^b} \leq cT^{1-b+b'} \|f\|_{H^{b'}}.
$$

Estimate for $I_{a,2}$. The estimate for $I_{a,2}$ is similar to that of $I_{a,1}$, exchanging p_a by q_a and $\tilde{\psi}_T(t)$ by $\psi_T(t)$. So, we omit its calculation. $\qquad\square$

In the following remark we present improvement of the estimate obtained in the Lemma 2.4 in some particular cases. Although, this improvement does not help to improve our main result, it will be of interest on its own.

Remark. (1) The proof in the case $|t'| \leq 1/T$ is valid for all $a < 1$.
(2) We know that

$$
\widehat{\mathcal{H}g}(\eta) = -i\,\mathrm{sgn}(\eta)\,\widehat{g}(\eta) \quad \text{and} \quad \widehat{q_a}(t) = -i\,\mathrm{sgn}(t)\,e^{a|t|}.
$$

Thus

$$
\begin{aligned}
I_{a,2}(t) &= -\tilde{\psi}_T(t)\widehat{q_a}(t) \int_{\mathbb{R}} q_a(t')\widehat{f}(t')dt' + \sqrt{2\pi}\tilde{\psi}_T(t)\mathcal{F}^{-1}\mathcal{H}(q_a\widehat{f})(t) \\
&= \tilde{\psi}_T(t)\mathcal{F}^{-1}(q_a)(t) \int_{\mathbb{R}} q_a(t')\widehat{f}(t')dt' + \sqrt{2\pi}\tilde{\psi}_T(t)\mathcal{F}^{-1}\mathcal{H}(q_a\widehat{f})(t),
\end{aligned}
$$

where $\tilde{\psi}_T(t) = sgn(t)\psi_T(t)$. Consequently,

$$
\begin{aligned}
\widehat{I_{a,2}}(\tau) &= \widehat{\tilde{\psi}_T} \star q_a(\tau) \int_{\mathbb{R}} q_a(t')\widehat{f}(t')dt' + \sqrt{2\pi}\widehat{\tilde{\psi}_T} \star \mathcal{H}(q_a\widehat{f})(\tau) \\
&= \widehat{\tilde{\psi}_T} \star q_a(\tau) \int_{\mathbb{R}} q_a(t')\widehat{f}(t')dt' + \sqrt{2\pi}\mathcal{H}(\widehat{\tilde{\psi}_T}) \star (q_a\widehat{f})(\tau).
\end{aligned}
$$

Similarly,

$$
\widehat{I_{a,1}}(\tau) = \frac{1}{i}\widehat{\psi_T} \star q_a(\tau) \int_{\mathbb{R}} q_a(t')\widehat{f}(t')dt' + \frac{\sqrt{2\pi}}{i}\mathcal{H}(\widehat{\psi_T}) \star (q_a\widehat{f})(\tau).
$$

(3) If $1/2 < b < b'/3 + 2/3$, then $1-b+b' > 3/4 + b'/2 - b > 1 + b'/2 - 3b/2 > 0$.

(4) If $|a| > 1$, then $|q_a(t')| \le c/\langle t' \rangle$, hence

$$\int_{\mathbb{R}} |q_a(t')\widehat{f}(t')|dt' \le c\int_{\mathbb{R}} \frac{|\widehat{f}(t')|\,\langle t' \rangle^{1-b}}{\langle t' \rangle \,\langle t' \rangle^{1-b}}dt' \le c\|f\|_{H^{b-1}}\left\|\frac{1}{\langle t' \rangle^b}\right\|_{L^2},$$

and therefore we obtain a more refined estimate than (2.19).

(5) In the case $|t'| > 1/T$ we can to obtain a better estimate for $I_{a,2}$ because ψ_T is regular (using the inequalities (2.4) and (2.5)). In fact, let $\widehat{f}(t') \equiv \widehat{f}(t')\chi_{\{|t'|>1/T\}}$. We have that

$$\|\psi_T(t)I_{a,2}(t)\|_{H_t^b} \le \|\psi_T(t)e^{|t|a}\|_{H_t^b}\Big|\int_{\mathbb{R}} q_a(t')\widehat{f}(t')dt'\Big| + \|\psi_T(t)\|_{H_t^b}\|\mathcal{F}^{-1}(q_a\widehat{f})(t)\|_{H_t^b}.$$

Since $|t'| > 1/T$ implies $|t'| \simeq \langle t' \rangle$, using the Cauchy-Schwartz inequality we obtain

$$
\begin{aligned}
\int_{\mathbb{R}} |q_a(t')\widehat{f}(t')|dt' &\le \int_{|t'|>1/T} \frac{|\widehat{f}(t')|}{|t'|}dt' \\
&\lesssim \int_{|t'|>1/T} \frac{|\widehat{f}(t')|\,|t'|^{-b'}}{|t'|\,|t'|^{-b'}}dt' \\
&\le \|f\|_{H^{b'}}\Big(\int_{|t'|>1/T} \frac{dt'}{|t'|^{2(1+b')}}\Big)^{1/2} \lesssim T^{1/2+b'}\,\|f\|_{H^{b'}}.
\end{aligned}
\tag{2.19}
$$

Similarly,

$$
\begin{aligned}
\int_{\mathbb{R}} |q_a(t')\widehat{f}(t')|dt' &\le c\|f\|_{H^{b'}}\Big(\int_{|t'|>1/T} \frac{dt'}{|t'|^{2b'}(a^2+t'^2)}\Big)^{1/2} \\
&\le c\frac{1}{|a|^{1/2+b'}}\Big(\int_{\mathbb{R}} \frac{dt'}{|t'|^{2b'}(1+t'^2)}\Big)^{1/2}\|f\|_{H^{b'}} \\
&\le c\frac{1}{|a|^{1/2+b'}}\,\|f\|_{H^{b'}}.
\end{aligned}
\tag{2.20}
$$

Hence from (2.19) and (2.20) we obtain

$$\int_{\mathbb{R}} |q_a(t')\widehat{f}(t')|dt' \le c\Big(\frac{T}{|a|}\Big)^{1/4+b'/2}\|f\|_{H^{b'}}, \tag{2.21}$$

and

$$
\begin{aligned}
\|\mathcal{F}^{-1}(q_a\widehat{f})(t)\|_{H_t^b}^2 &\lesssim \int_{|t'|>1/T} \frac{|\widehat{f}(t')|^2}{\langle t' \rangle^{2(1-b)}}dt' \\
&\lesssim \int_{|t'|>1/T} \frac{|\widehat{f}(t')|^2}{\langle t' \rangle^{-2b'}|t'|^{2(1-b+b')}}dt' \\
&\le T^{2(1-b+b')}\|f\|_{H^{b'}}^2.
\end{aligned}
$$

So from inequality (2.4) we have

$$\|\psi_T(t)\|_{H_t^b}\|\mathcal{F}^{-1}(q_a\widehat{f})(t)\|_{H_t^b} \le cT^{1/2-b}\,T^{(1-b+b')}\|f\|_{H^{b'}} \le cT^{3/2-2b+b'}\|f\|_{H^{b'}}.$$

On the other hand, if $a < -1$ from inequalities (2.5) and (2.21) we obtain

$$
\|\psi_T(t)\, e^{a|t|}\|_{H_t^b} \int_{\mathbb{R}} |q_a(t')\widehat{f}(t')|dt'
$$

$$
\leq c\Big(\frac{T}{|a|}\Big)^{1/2}\Big(1+\frac{1}{T^b}\Big)(1+|a|^b)\Big(\frac{T}{|a|}\Big)^{1/4+b'/2}\|f\|_{H^{b'}}
$$

$$
\leq c\Big(\frac{T}{|a|}\Big)^{3/4+b'/2}\Big(1+\frac{1}{T^b}\Big)(1+|a|^b)\|f\|_{H^{b'}}
$$

$$
\leq cT^{3/4+b'/2-b}\Big(1+\frac{1}{|a|^{3/4+b'/2-b}}\Big)\|f\|_{H^{b'}}
$$

$$
\leq cT^{3/4+b'/2-b}\|f\|_{H^{b'}}.
$$

Now if $|a| < 1$ using the inequality (2.5) we obtain:

$$
\|\psi_T(t)\, e^{a|t|}\|_{H_t^b} \int_{\mathbb{R}} |q_a(t')\widehat{f}(t')|dt' \leq cT^{1/2+b'}\|f\|_{H^{b'}} T^{1/2}\Big(1+\frac{1}{|T|^b}\Big)
$$

$$
\leq cT^{1/2+b'}\, T^{1/2-b}\|f\|_{H^{b'}}
$$

$$
\leq cT^{1-b+b'}\|f\|_{H^{b'}}
$$

$$
\leq cT^{3/4+b'/2-b}\|f\|_{H^{b'}}.
$$

Therefore, in this case

$$
\|\psi_T(t)I_a(t)\|_{H_t^b} \leq cT^{3/4+b'/2-b}\|f\|_{H^{b'}},
$$

where c is a constant independent of a, f and T.

Now we prove Proposition 2.2 which plays a crucial role in the proof of the first main result of this work.

Proof of Proposition 2.2. To prove (2.2), we use the estimate (2.5) with $T = 1$ and $a = \Phi(\xi) < 1$, to obtain

$$
\|\psi(t)\, e^{\eta\Phi(\xi)|t|}\|_{H_t^b}^2 \leq c(\eta),
$$

where $c(\eta)$ is a constant. Therefore,

$$
\|\psi_T(t)V(t)u_0\|_{X_{s,b}} \leq c(\eta)\Big(\int_{\mathbb{R}}\langle\xi\rangle^{2s}|\widehat{u_0}(\xi)|^2 d\xi\Big)^{1/2} \leq c(\eta)\|u_0\|_{H^s}.
$$

Now, we move to prove (2.3). From definition (1.10) of the $X_{s,b}$ norm, we have

$$
\|\psi_T(t)\int_0^t V(t-t')(uu_x)(t')dt'\|_{X_{s,b}}
$$

$$
= \|U(-t)\psi_T(t)\int_0^t V(t-t')(uu_x)(t')dt'\|_{H_{s,b}}
$$

$$
= \|\langle\tau\rangle^b\langle\xi\rangle^s\mathcal{F}_{\xi\tau}\big[\psi_T(t)\int_0^t U(-t')\tilde{U}(t-t')(uu_x)(t')dt'\big]\|_{L_\tau^2 L_\xi^2}
$$

$$
= \|\langle\xi\rangle^s\|\psi_T(t)\int_0^t e^{-it'\xi^3}e^{|t-t'|\Phi(\xi)}\widehat{uu_x}(t',\xi)dt'\|_{H_t^b}\|_{L_\xi^2}.
$$

(2.22)

If we fix the variable ξ and suppose $f_\xi(t') = e^{-it'\xi^3}\widehat{uu_x}(t',\xi)$ the estimate (2.3) follows from (2.22) using (2.10). □

2.2. **Proof of the Theorem 1.1.**

Proof. As discussed in the introduction, we will use Bourgain's space associated to the KdV group to prove well-posedness of (1.1); therefore we need to consider that (1.12) is defined for all t. Now consider (1.12) in its equivalent integral form

$$u(t) = V(t)u_0 - \int_0^t V(t-t')(uu_x)(t')dt', \qquad (2.23)$$

where $V(t)$ is the semigroup associated with the linear part given by (1.14). Note that, if for all $t \in \mathbb{R}$, $u(t)$ satisfies

$$u(t) = \psi(t)V(t)u_0 - \psi_T(t)\int_0^t V(t-t')(uu_x)(t')dt',$$

then $u(t)$ satisfies (2.23) in $[-T, T]$. We define an application

$$\Psi(u)(t) = \psi(t)\,V(t)u_0 - \psi_T(t)\int_0^t V(t-t')(uu_x)(t')dt'.$$

Let $s > -3/4$, and $u_0 \in H^s$. Let b and b' be two numbers given by Proposition 2.1, such that $\theta \equiv \min\{1 + b'/2 - 3b/2,\ 3/4 + s/3 - b\} > 0$. For $M > 0$, let us define a ball

$$X_{s,b}^M = \{u \in X_{s,b} : \|u\|_{X_{s,b}} \le M\}.$$

We will prove that there exists M such that the application Ψ maps $X_{s,b}^M$ into $X_{s,b}^M$ and is a contraction. Let $u \in X_{s,b}^M$. Then using Propositions 2.1, 2.2 and the definition of $X_{s,b}^M$ we obtain

$$\|\Psi(u)\|_{X_{s,b}} \le c\|u_0\|_s + cT^\theta\|(uu_x)\|_{X_{s,b'}} \le \frac{M}{4} + cT^\theta M^2 \le \frac{M}{2},$$

where we have chosen $M = 4c\|u_0\|_{H^s}$ and $0 < T < 1$ such that $cT^\theta M = 1/4$. Therefore, Ψ maps $X_{s,b}^M$ into itself. With a similar argument we can prove that Ψ is a contraction. Hence Ψ has a unique fixed point u which is a solution to (1.1) such that $u \in C([-T, T], H^s)$.

The rest of the proof follows in an analogous way to [11], so we omit the details. $\qquad\square$

3. A REFINED LOCAL SMOOTHING EFFECT

In this section we prove the following local smoothing effect for the semigroup $V_k(t)$ defined by (1.14) with $\Phi(\xi) = |\xi|^k - |\xi|^{k+2}$. Similar results can also be obtained for more general Φ as in (1.2). Our proof follows the ideas of [6].

Theorem 3.1. *Let $T > 0$, $u_0 \in L^q$, $0 \le s < (k+3-p)/p + 1/p_1$ and $p \ge 2$, $p_1 \ge 2$, then*

$$\|D_x^s V_k(t)u_0\|_{L_T^p L_x^{p_1}} \le \frac{c(\eta)}{(k+3 - p(s+1) + p/p_1)^{1/p}}\,(T^{1/p}e^{2\eta T} + T^\epsilon)\,\|u_0\|_{L^q}, \quad (3.1)$$

where $\epsilon = \epsilon(p,k,s,p_1) = (k+3-p(s+1))/(k+2) + p/((k+1)p_1)$ and $1/p + 1/q = 1$.

Corollary 3.2. *Let $u_0 \in L^q$, $T > 0$, $2 \le p < k+3$, and $0 \le s < (k+3-p)/p$, then*

$$\|D_x^s V_k(t)u_0\|_{L_T^p L_x^\infty} \le \frac{c(\eta)}{(k+3 - p(s+1))^{1/p}}\,(T^{1/p}e^{2\eta T} + T^{\epsilon_0})\,\|u_0\|_{L^q}, \qquad (3.2)$$

where $\epsilon_0 = \epsilon(p, k, s) = (k + 3 - p(s + 1))/(k + 2)$ and $1/p + 1/q = 1$.

In particular, the case when $p = 2$ is interesting, which is stated as follows.

Corollary 3.3. (1) If $u_0 \in L^2$, $p_1 \geq 2$, $0 \leq s < 1 + (k - 1)/2 + 1/p_1$, $0 < T < 1$ and $\gamma = \min\{1/2, \epsilon(2, k, s, p_1)\}$ then

$$\|D_x^s V_k(t)u_0\|_{L_T^2 L_x^{p_1}} \leq \frac{c(\eta)\, T^\gamma}{(1 + (k - 1)/2 + 1/p_1 - s)^{1/2}}\|u_0\|_{L^2}. \tag{3.3}$$

(2) If $u_0 \in \dot{H}^s$, $-k/2 < s \leq 0$, $0 < T < 1$ and $\gamma = \min\{1/2, \epsilon(2, k, 1 - s, p_1)\}$ then

$$\|D_x V_k(t)u_0\|_{L_T^2 L_x^{p_1}} \leq \frac{c(\eta)\, T^\gamma}{((k - 1)/2 + 1/p_1 + s)^{1/2}}\|D^s u_0\|_{L^2}, \tag{3.4}$$

in the following cases:
(i) when $-(k - 1)/2 \leq s \leq 0$ and $2 \leq p_1$.
(ii) when $-k/2 < s < -(k - 1)/2$ and $2 \leq p_1 \leq (-s - (k - 1)/2)^{-1}$.

In the proof of Theorem 3.1 we will use the following result.

Proposition 3.4. Let $p \geq 2$, and $1/p + 1/q = 1$, then

$$\|\hat{u}\|_{L^p} \leq c\|u\|_{L^q}, \tag{3.5}$$

The proof of the above corollary can be found in [13, Corollary 1.43].

Proof of Theorem 3.1. We can assume that $u_0 \in S(\mathbb{R})$. We consider a cut-off function $\varphi \in C(\mathbb{R} \setminus \{0\})$, $0 \leq \varphi \leq 1$ defined by

$$\varphi(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t < 0 \text{ or } t \geq 2. \end{cases} \tag{3.6}$$

Let us define $\varphi_T(t) \equiv \varphi(\frac{t}{T})$, then

$$\|D_x^s V_k(t)u_0(x)\|_{L_T^p L_x^{p_1}} \leq \|\varphi_T(t)D_x^s V_k(t)u_0(x)\|_{L_t^p L_x^{p_1}}.$$

Let $1/p_1 + 1/q_1 = 1$, using duality it is enough to prove for u_0 in L^q and g in $L_t^q L_x^{q_1}$

$$J \equiv \Big| \int_{\mathbb{R}^2} \varphi_T(t)D_x^s V_k(t)u_0(x)g(x,t)dx\, dt \Big| \leq c\|u_0\|_{L^q}\|g\|_{L_t^q L_x^{q_1}}.$$

Using (1.14), we have

$$D_x^s V_k(t)u_0(x) = i \int_{\mathbb{R}} |\xi|^s\, e^{it\xi^3 + \eta t\Phi(\xi) + ix\xi}\hat{u}_0(\xi)d\xi.$$

Therefore, by Fubini's theorem, Proposition 3.4 and Hölder's inequality we obtain

$$J \leq c\|u_0\|_{L^q}\|Lg\|_{L^q},$$

where $Lg(\xi)$ is defined by

$$Lg(\xi) \equiv |\xi|^s\Big| \int_{\mathbb{R}^2} \varphi_T(t)g(x,t)e^{it\xi^3 + \eta t\Phi(\xi) + ix\xi}dx\, dt\Big|$$

and

$$|Lg(\xi)| \leq |\xi|^s \int_{\mathbb{R}} \varphi_T(t)e^{\eta t\Phi(\xi)}|\mathcal{F}^{-1}(g(\cdot, t))(\xi)|dt. \tag{3.7}$$

We have

$$\|Lg(\xi)\|_{L^q(\mathbb{R})} \leq \|Lg(\xi)\|_{L^q(|\xi| \leq 2)} + \|Lg(\xi)\|_{L^q(|\xi| > 2)} \equiv J_1 + J_2. \tag{3.8}$$

In J_1 by (3.7), Minkowski and Hölder's inequalities and Proposition (3.4) we obtain

$$
\begin{aligned}
J_1 &\leq c \int_{\mathbb{R}} \varphi_T(t) \|e^{\eta t \Phi(\xi)}\|_{L^{r_1}(|\xi|\leq 2)} \|\mathcal{F}^{-1}(g(\cdot,t))(\xi)\|_{L^{p_1}(|\xi|\leq 2)} dt \\
&\leq c e^{2\eta T} \|\varphi_T\|_{L^p} \|g\|_{L^q_t L^{q_1}_x} \leq c e^{2\eta T} T^{1/p} \|g\|_{L^q_t L^{q_1}_x},
\end{aligned}
\tag{3.9}
$$

where c is a constant, $1/q = 1/r_1 + 1/p_1$ and $1/p_1 + 1/q_1 = 1$. Similarly for J_2 we have

$$
J_2 \leq \int_{\mathbb{R}} \varphi_T(t) \| \xi^s e^{\eta t \Phi(\xi)}\|_{L^{r_1}(|\xi|>2)} \|g(\cdot,t)\|_{L^{q_1}_x} dt.
$$

For $t > 0$, we have

$$
\| |\xi|^s e^{-\eta t |\xi|^{k+2}/2}\|_{L^{r_1}(|\xi|>2)} \leq \frac{c(\eta)}{t^{s/(k+2)+1/((k+2)r_1)}}.
$$

Therefore, for $0 \leq s < (k+3-p)/p + 1/p_1$ we obtain

$$
J_2 \leq c(\eta) \Big\| \frac{\varphi_T(t)}{t^{s/(k+2)+1/((k+2)r_1)}} \Big\|_{L^p} \|g\|_{L^q_t L^{q_1}_x} \leq c(\eta) T^\epsilon \|g\|_{L^q_t L^{q_1}_x},
\tag{3.10}
$$

where $\epsilon = (k+3-p(s+1))/(k+2) + p/((k+1)p_1)$. From (3.8), (3.9) and (3.10) we obtain

$$
\|Lg\|_{L^q} \leq \frac{c(\eta)}{(k+3-p(s+1)+p/p_1)^{1/p}} (T^{1/p} e^{2\eta T} + T^\epsilon) \|g\|_{L^q_t L^{q_1}_x}.
$$

$\square$

4. SOME STRICHARTZ TYPE ESTIMATES

Proposition 4.1. *Let* $2 \leq p$, $k \geq 1$, $c_{p,k} = \frac{p-2}{2p(k+2)}$, $0 < T < 1$, $s \leq 0$ *and*

$$
\frac{1}{r} + \frac{s}{(k+2)} - c_{p,k} > 0.
$$

Then

$$
\|V_k(t)u_0\|_{L^r_T L^p_x} \leq c(\eta, sqr_0) T^{\{\frac{1}{r} + \frac{s}{(k+2)} - c_{p,k}\}} \|u_0\|_{H^s},
$$

where $1/q + 1/p = 1$, $r_0 = 2/(2-q)$.

Proof. Let $\Phi(\xi) = |\xi|^k - |\xi|^{k+2}$. By (3.5) we have

$$
\begin{aligned}
\|V_k(t)u_0\|_{L^r_T L^p_x} &\leq c \|\widehat{V_k(t)u_0}\|_{L^r_T L^q_\xi} \\
&\leq c \|e^{\eta t \Phi(\xi)} \widehat{u_0}\|_{L^r_T L^q_\xi(|\xi|\leq 2)} + c \|e^{\eta t \Phi(\xi)} \widehat{u_0}\|_{L^r_T L^q_\xi(|\xi|>2)} \\
&\equiv J_1 + J_2.
\end{aligned}
$$

In J_1, using Hölder's inequality we have

$$
J_1 \leq c 3^{-s} e^{\eta T} T^{1/r}.
$$

To estimate J_2, by Hölder's inequality we obtain

$$\int_{|\xi|>2} e^{q\eta t\Phi(\xi)}|\widehat{u_0}(\xi)|^q d\xi$$

$$\leq \int_{|\xi|>2} e^{-q\eta t|\xi|^{k+2}/2}(1+|\xi|)^{-sq}(1+|\xi|)^{sq}|\widehat{u_0}(\xi)|^q d\xi$$

$$\leq \|e^{-q\eta t|\xi|^{(k+2)}/2}(1+|\xi|)^{-sq}\|_{L^{r_0}(|\xi|>2)}\|u_0\|_{H^s}^q$$

$$\leq c(\eta)\Big[\frac{1}{t^{1/((k+2)r_0)}} + \frac{1}{t^{1/((k+2)r_0)-sq/(k+2)}}\Big(\int_{\mathbb{R}} e^{-|y|^{(k+2)}}|y|^{-sqr_0}dy\Big)^{1/r_0}\Big]\|u_0\|_{H^s}^q,$$

where $r_0 = 2/(2-q)$. Therefore

$$J_2 \leq c(\eta, sqr_0)\Big\|\frac{1}{t^{1/((k+2)r_0q)-s/(k+2)}}\Big\|_{L_T^r}\|u_0\|_{H^s} \leq c(\eta, sqr_0)\, T^{\frac{1}{r}+\frac{s}{(k+2)}-c_{p,k}}\|u_0\|_{H^s},$$

where $c_{p,k} = \frac{p-2}{2p(k+2)}$. $\qquad\square$

Corollary 4.2. *Let $0 < T < 1$, $s \leq 0$, $1 < r < 2(k+2)/(1-2s)$. Then*

$$\|V_k(t)u_0\|_{L_T^r L_x^\infty} \leq c(\eta, s)T^{1/r-(1-2s)/(2(k+2))}\|u_0\|_{H^s}.$$

Proposition 4.3. *Let $u_0 \in L^2$. Then*

$$\|V_k(t)u_0\|_{L_T^\infty L_x^2} \leq e^{\eta T}\|u_0\|_{L^2}.$$

Proof. Using Plancherel identity

$$\|V_k(t)u_0\|_{L_x^2} = \|e^{\eta t(|\xi|^k-|\xi|^{k+2})}\widehat{u_0}(\xi)\|_{L_\xi^2} \leq e^{\eta T}\|u_0\|_{L^2},$$

where we used the estimate $e^{\eta t(|\xi|^k-|\xi|^{k+2})} \leq e^{\eta T}$. $\qquad\square$

In the following section we give an application of the above results.

5. Proof of Theorem 1.2

This section is devoted to give proof of the local well-posedness result for given data in homogeneous Sobolev space with regularity below L^2. We consider the IVP

$$\partial_t u + \partial_x^3 u + \eta L_k(u) + u\partial_x u = 0, \quad x \in \mathbb{R},\ t \geq 0,$$
$$u(x,0) = u_0(x), \tag{5.1}$$

which is a special case of (1.1) with Fourier symbol $\Phi(\xi) = |\xi|^k - |\xi|^{k+2}$.

To prove Theorem 1.2, we need the following proposition.

Proposition 5.1. *Let $0 \leq -s < 1/2$. If $u \in L^{2/(1-2s)}$, then*

$$\|D_x^s(u)\|_{L^2} \leq c\|u\|_{L^{2/(1-2s)}}. \tag{5.2}$$

If $u \in L^1 \cap L^2$, then

$$\|D_x^s(u)\|_{L^2} \leq c(\|u\|_{L^1} + \|u\|_{L^2}). \tag{5.3}$$

Proof. The inequality (5.2) follows using the Hardy, Sobolev, Littlewood inequality and

$$\Big(\widehat{\frac{1}{|\xi|^{1+s}}}\Big)(\eta) = \frac{1}{|\eta|^{-s}}.$$

The inequality (5.3) follows from

$$\|D_x^s(u)\|_{L^2}^2 = \int_{|\eta|<1} \frac{|\widehat{u}(\eta)|^2}{|\eta|^{-2s}} d\eta + \int_{|\eta|\geq 1} \frac{|\widehat{u}(\eta)|^2}{|\eta|^{-2s}} d\eta.$$

$\square$

Now we are in position to supply proof of our second main result.

Proof of Theorem 1.2. Now let $u_0 \in \dot{H}^s$, with $0 \leq -s < 1/2$. For $0 < T < 1$, define a ball

$$Z_{a,T} = \{w \in C([0,T], \dot{H}^s); \quad |||w|||_T \leq a\},$$

where

$$|||w|||_T = \|w\|_{\dot{H}^s} + \|w_x\|_{L_T^2 L_x^{p_1}} + T^{\left\{\frac{q_1-2}{2q_1(k+2)} - \frac{1}{2} - \frac{s}{k+2}\right\}} \|w\|_{L_T^2 L_x^{q_1}},$$

$-1/s \leq p_1 < \infty$, $q_1 \geq 2$, $1/p_1 + 1/q_1 = (1-2s)/2$ and p_1 is chosen as in inequality (3.4) of Corollary 3.3. Using Corollary 3.3 and Proposition 4.1 we obtain

$$|||V_k(t)u_0|||_T \leq c\|D_x^s(u_0)\|_{L^2}. \tag{5.4}$$

Also, using the inequality (5.2) we obtain

$$\begin{aligned}
\int_0^T \|D_x^s(vv_x)\|_{L_x^2} &\leq c \int_0^T \|vv_x\|_{L_x^{2/(1-2s)}} \\
&\leq c\|v_x\|_{L_T^2 L_x^{p_1}} \|v\|_{L_T^2 L_x^{q_1}} \\
&\leq cT^{\left\{-\frac{q_1-2}{2q_1(k+2)} + \frac{1}{2} + \frac{s}{k+2}\right\}} a^2.
\end{aligned} \tag{5.5}$$

Now, define an application

$$\Psi(v)(t) = V_k(t)u_0 - \int_0^t V_k(t-\tau)vv_x(\tau)d\tau,$$

where $V_k(t)$ is the evolution operator defined in (1.14) with $\Phi(\xi) = |\xi|^k - |\xi|^{k+2}$. With the help of the inequalities (5.4) and (5.5), it can be shown that the application Ψ maps $Z_{a,T}$ into $Z_{a,T}$ and is a contraction considering $a = 2c\|D_x^s(u_0)\|_{L^2}$, and $cT^{\left\{-\frac{q_1-2}{2q_1(k+2)} + \frac{1}{2} + \frac{s}{k+2}\right\}} a < 1/2$. The rest of the proof follows from a standard argument. $\square$

Remark. If we use the inequality (5.3) we can also take the following space in the proof of the Theorem 1.2

$$Z_{a,T} = \{w \in C([0,T], \dot{H}^s); \quad |||w|||_T \leq a\},$$

where

$$\begin{aligned}
|||w|||_T = \|w\|_{\dot{H}^s} &+ \|w_x\|_{L_T^2 L_x^{p_1}} + \|w_x\|_{L_T^2 L_x^2} + T^{\left\{\frac{q_1-2}{2q_1(k+2)} - \frac{1}{2} - \frac{s}{k+2}\right\}} \|w\|_{L_T^2 L_x^{q_1}} \\
&+ T^{\left\{-\frac{1}{2} - \frac{s}{k+2}\right\}} \|w\|_{L_T^2 L_x^2},
\end{aligned}$$

$-1/s < p_1 < \infty$, $q_1 \geq 2$, $1/p_1 + 1/q_1 = 1/2$ and p_1 is chosen as in Corollary 3.3 inequality (3.4). By Corollary 3.3 and Proposition 4.1 we obtain

$$|||V_k(t)u_0|||_T \leq c\|D_x^s(u_0)\|_{L^2}.$$

Then using (5.3),

$$
\begin{aligned}
\int_0^T \|D_x^s(vv_x)\|_{L_x^2} &\le c\int_0^T (\|vv_x\|_{L_x^1} + \|vv_x\|_{L_x^2}) \\
&\le c\|v_x\|_{L_T^2 L_x^2}\,\|v\|_{L_T^2 L_x^2} + c\|v_x\|_{L_T^2 L_x^{p_1}}\,\|v\|_{L_T^2 L_x^{q_1}} \\
&\le cT^{\left\{\frac{1}{2} + \frac{s}{k+2}\right\}} a^2 + cT^{\left\{-\frac{q_1-2}{2q_1(k+2)} + \frac{1}{2} + \frac{s}{k+2}\right\}} a^2 \\
&\le cT^{\left\{-\frac{q_1-2}{2q_1(k+2)} + \frac{1}{2} + \frac{s}{k+2}\right\}} a^2.
\end{aligned}
$$

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