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MULTIPLE SOLUTIONS TO A SINGULAR LANE-EMDEN-FOWLER EQUATION WITH CONVECTION TERM

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ABSTRACT. This article concerns the existence of multiple solutions for the problem

$$\begin{gathered}-\Delta u = K(x)u^{-\alpha} + s(\mathcal{A}u^{\beta} + \mathcal{B}|\nabla u|^{\zeta}) + f(x) \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,,\end{gathered}$$

where Ω is a smooth, bounded domain in $\mathbb{R}^n$ with $n \geq 2$, α, β, ζ, $\mathcal{A}$, $\mathcal{B}$ and s are real positive numbers, and $f(x)$ is a positive real valued and measurable function. We start with the case $s = 0$ and $f = 0$ by studying the structure of the range of $-u^{\alpha}\Delta u$. Our method to build K's which give at least two solutions is based on positive and negative principal eigenvalues with weight. For s small positive and for values of the parameters in finite intervals, we find multiplicity via estimates on the bifurcation set.

1. Introduction

Singular bifurcation problems of the form

$$\begin{gathered}-\Delta u = K(x)u^{-\alpha} + s\mathcal{G}(x, u, \nabla u) + f(x) \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\end{gathered} \tag{1.1}$$

where α is a positive number, $K(x)$ is a bounded measurable function, $\mathcal{G}(x, \cdot, \cdot)$ a non-negative Carathéodory function, $f(x)$ a non-negative bounded measurable function and Ω a bounded domain in $\mathbb{R}^n$, are used in several applications. As examples, we mention: Modelling heat generation in electrical circuits [17], fluid dynamics [7, 8, 27], magnetic fields [25], diffusion in contained plasma [26], quantum fluids [18], chemical catalysis [2, 28], boundary layer theory of viscous fluids [37], super-diffusivity for long range Van der Waal interactions in thin films spreading on solid surfaces [19], laser beam propagation in gas vapors [31, 32] and plasmas [33], exothermic reactions [6, 36], cellular automata and interacting particles systems with self-organized criticality [9], etc.

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Our main concern in this paper is on the existence of multiple solutions for the problem

$$
\begin{gathered}
-\Delta u = K(x)u^{-\alpha} + s(\mathcal{A}u^{\beta} + \mathcal{B}|\nabla u|^{\zeta}) + f(x) \quad \text{in } \Omega \\
u > 0 \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega\,,
\end{gathered}
\tag{1.2}
$$

where Ω is a smooth, bounded domain in $\mathbb{R}^n$ with $n \geq 2$, α, β, ζ, $\mathcal{A}$, $\mathcal{B}$ and s are real positive numbers and $f(x)$ is a non-negative measurable function.

We start with the case $s = 0$ and $f \equiv 0$. The situation with positive K has been widely studied by several authors. For example in [4, 14, 17, 22, 24, 29], under different hypothesis on K, they prove the existence and unicity of solutions for equation (1.2). In Theorem 2.4, we build a family of K's, such that problem (1.2), with $s = 0$, $f \equiv 0$ and α positive small enough has at least two solutions. We apply the classical Lyapunov-Schmidt method to the map $F : \mathcal{C}^+ \to \mathcal{D}$,

$$
F(u) = -u^{\alpha}\Delta u \tag{1.3}
$$

where $\mathcal{C}^+$ is defined in (3.4, 3.5) and $\mathcal{D}$ is defined in (3.6) to search a bifurcation point for $F(u)$. This point will be an eigenfunction corresponding to a negative principal eigenvalue of a linear weighted eigenvalue problem. To prove it, we give a Lemma concerning the localization of the maximum value of such an eigenfunction (see Lemma 2.1). We also use a Harnack inequality to establish a necessary estimate (see Lemma 2.3). A final technical matter is differentiability of $F(u)$ (Lemma 3.1). To our knowledge there are no previous similar results for (1.2) with $s = 0$ and $f \equiv 0$.

Concerning the existence of at least one solution to (1.1) or (1.2) we may recall:

For $K(x) \equiv 1$, $\mathcal{A} = 1$, $\mathcal{B} = 0$, $f \equiv 0$, $\alpha > 0$ and $\beta > 0$ in (1.2), Coclite-G. Palmieri [13] have shown that there exists $0 < s^* \leq \infty$ such that this problem (1.2) has at least one solution for all $s \in (0, s^*)$.

Similar results for problem (1.2) can be found in Zhang and Yu [35] under the conditions $K(x) \equiv 1$, $\alpha > 0$, $\mathcal{A} \equiv 0$, $\mathcal{B} \equiv 1$, $0 < \zeta \leq 2$ and $f(x)$ equivalent to a non-negative constant.

In a recent work about (1.1), Ghergu and Rădulescu [20] prove existence and nonexistence results for a more general singular equation. They study

$$
\begin{gathered}
-\Delta u = g(u) + \lambda|\nabla u|^{\zeta} + \mu f(x, u) \quad \text{in } \Omega \\
u > 0 \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega\,,
\end{gathered}
\tag{1.4}
$$

where $g : (0, \infty) \to (0, \infty)$ is a Hölder continuous function which is non-increasing and $\lim_{s \searrow 0} g(s) = \infty$. They prove in [20, Theorem 1.4]) that for $\zeta = 2$, $f \equiv 1$ and fixed μ, (1.4) has a unique solution. Under the assumption $\limsup_{s \searrow 0} s^{\alpha} g(s) < +\infty$, they also prove existence of a bifurcation at infinity for some $\lambda^* < \infty$. In this article we also obtain bifurcations from infinity at $s = 0$ (see Theorems 2.7 and 2.8).

Concerning existence of multiple solutions for problem (1.2), Haitao [23], using a variational method, proves existence of two classical solutions under the assumptions $K(x) \equiv 1$, $0 < \alpha < 1 < \beta \leq \frac{N+2}{N-2}$, $\mathcal{A} = 1$ $s \in (0, s^*)$ for some $s^* > 0$, $\mathcal{B} \equiv 0$ and $f \equiv 0$. We remark that our problem (1.2) has not a variational structure because of the convection term $\mathcal{B}|\nabla u|^{\zeta}$.

Aranda and Godoy [5] proved the existence of two weak solutions for the problem, involving the p-laplacian,

$$\begin{gathered}-\Delta_p u = g(u) + s\mathcal{G}(u) \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,,\end{gathered} \tag{1.5}$$

where $s > 0$ is small enough. This is done under the assumptions

- (i) $g : (0,\infty) \to (0,\infty)$ is a locally Lipschitz and non-increasing function such that $\lim_{s\searrow 0} g(s) = \infty$.
- (ii) $1 < p \le 2$, $\mathcal{G}$ is a locally Lipschitz on $[0,\infty)$, $\inf_{s>0} \mathcal{G}(s)/s^{p-1} > 0$ and $\lim_{s\to\infty} \mathcal{G}(s)/s^q < \infty$ for some $q \in \big(p-1, n(p-1)/(n-p)\big]$.
- (iii) Ω is a bounded convex domain.

We remark that for $p = 2$ and using the change of variable $v = e^u - 1$ (see [20]), we can immediately obtain existence of two classical solutions of the singular problem with a particular convection term

$$\begin{gathered}-\Delta u = \frac{g(e^u-1)}{e^u} + s\frac{\mathcal{G}(e^u-1)}{e^u} + |\nabla u|^2 \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,,\end{gathered}$$

for s is small enough. In comparison with this result, Theorems 2.8 and 2.9 give results on the existence of two classical solutions for $\zeta \ne 2$. This indicates a complex relation between the convection term, the function $f(x)$ and the domain Ω.

For dimension $n = 1$ results on multiplicity can be found, for example, in Agarwal and O'Reagan [1].

To prove Theorems 2.7, 2.8 and 2.9, we apply an "inverse function" strategy. We use that problem $-\Delta u = u^{-\alpha} + f(x)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ on Ω (see Theorem 3.1 in [4]) has a unique solution for $f(x) \ge 0$. Moreover the solution operator defined by $H(f) := u$ is a continuous and compact map from P into P, where P is the positive cone in $C^1(\overline{\Omega})$ (see Lemma 3.2 and Lemma 3.3). Therefore, we may write the problem (1.1) as $u = H\big(s\mathcal{G}(x,u,\nabla u) + f(x)\big)$.

Properties of H and a classical theorem on nonlinear eigenvalue problems stated in [3], give existence of an unbounded connected set of solution pairs (s,u), in an appropriate norm, to problem (1.1). Estimates on this solution set, combined with nonexistence results, give a bifurcation from infinity at $s = 0$. We use similar ideas to establish Theorems 2.8 and 2.9.

2. Statement of the main results

Let us consider the weighted eigenvalue problem

$$\begin{gathered}-\Delta u = \lambda m(x) u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,,\end{gathered} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$. Suppose $m = m^+ - m^-$ in $L^\infty(\Omega)$, where $m^+ = \max(m,0)$, $m^- = -\min(m,0)$. Denote

$$\Omega_+ = \{x \in \Omega : m(x) > 0\}, \quad \Omega_- = \{x \in \Omega : m(x) < 0\}$$

and $|\Omega_+|$, $|\Omega_-|$ its Lebesgue measures. It is well known (see [16] for a nice survey) that if $|\Omega_+| > 0$ and $|\Omega_-| > 0$, then (2.1) has a double sequence of eigenvalues

$$\dots \le \lambda_{-2} < \lambda_{-1} < 0 < \lambda_1 < \lambda_2 \le \dots,$$

where λ_1 and λ_{-1} are simple and the associated eigenfunctions $\varphi_1 \in C(\overline{\Omega})$, $\varphi_{-1} \in C(\overline{\Omega})$ can be taken $\varphi_1 > 0$ on Ω, $\varphi_{-1} > 0$ on Ω. Where λ_1 and λ_{-1} are the principal eigenvalues of (2.1) φ_1 and φ_{-1} are the associated principal eigenfunctions. Our first result is as follows.

Lemma 2.1. *Suppose $m = m^+ - m^-$ in $L^\infty(\Omega)$ such that $|\Omega^+| > 0$, $|\Omega^-| > 0$. Then the principal eigenfunctions $\varphi_1 > 0$, $\varphi_{-1} > 0$ of (2.1) satisfy*

$$\begin{gathered} \|\varphi_1\|_{L^\infty(\Omega)} = \|\varphi_1\|_{L^\infty(rmsupp\, m^+,\, m^+dx)} \\ \|\varphi_{-1}\|_{L^\infty(\Omega)} = \|\varphi_{-1}\|_{L^\infty(rmsupp\, m^-,\, m^-dx)} \end{gathered} \tag{2.2}$$

where $\|\varphi_1\|_{L^\infty(rmsupp\, m^+,\, m^+dx)}$ (respectively $\|\varphi_{-1}\|_{L^\infty(rmsupp\, m^-,\, m^-dx)}$) is the essential supremum on $rmsupp\, m^+$ with respect to the measure m^+dx (respectively on $rmsupp\, m^-$ w. r. t. m^-dx).

Here $rmsupp\, m^+$ is the support of the distribution m^+ in Ω. We take $s = 0$ in (1.1) or (1.2) and look for multiple solutions of

$$\begin{gathered} -u^\alpha \Delta u = K(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{2.3}$$

We fix $p > n$ and consider $K \in L^p(\Omega)$. It is shown in [4] that for $\alpha > 0$, $0 < K \in L^p(\Omega)$, (2.3) has a unique solution $u \in W^{2,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$. On the other hand, for $\alpha > 0$ and $K < 0$, we deduce from the Maximum Principle that (2.3) has no solution. Thus, if we want multiple solutions, K should change sign.

We give now two auxiliary results which will provide a family of α and K's giving multiple solutions to (2.3) Let $\lambda_{\pm j}((m))$ denote the eigenvalues of the problem $-\Delta u = \lambda m(x) u$ in Ω, $u = 0$ on $\partial\Omega$.

Lemma 2.2. *The function*

$$\alpha(t) := -\frac{\lambda_1((m^+ - tm^-))}{\lambda_{-1}((m^+ - tm^-))}$$

is continuous on $(0, \infty)$ and satisfies $\lim_{t\to 0^+} \alpha(t) = 0$ and $\lim_{t\to\infty} \alpha(t) = \infty$.

Our next lemma says that a weight m with "a positive and a negative bump" gives a bifurcation point to $F(u)$ for the proof of Theorem 2.4.

Lemma 2.3. *Let y_+, y_- be fixed points of Ω, let $\delta > 0$ be such that the ball $B_{20\delta}\big(\frac{y_+ + y_-}{2}\big)$ with radius 20δ centered at $\frac{y_+ + y_-}{2}$ is contained in Ω, in such a way that the distance between y_+ and y_- is 8δ. If φ_{-1} is the principal positive eigenfunction associated to the principal negative eigenvalue λ_{-1} and φ_1 is the principal positive eigenfunction associated to the principal positive eigenvalue λ_1 of the problem*

$$\begin{gathered} -\Delta u = \lambda(m^+(x) - tm^-(x))u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{2.4}$$

where $m(x) = m^+(x) - m^-(x) \in C(\overline{\Omega})$, is such that $rmsupp\, m^+ = \overline{B_\delta(y_+)}$, $rmsupp\, m^- = \overline{B_\delta(y_-)}$ and $m^-(x) > 0$ in $B_\delta(y_-)$. Then there exists a positive *constant* $\epsilon(m^+, m^-) > 0$ *depending on* m^+, m^- *such that for all* $t \in (0, \epsilon(m^+, m^-))$

$$\int_\Omega (m^+ - tm^-)\varphi_{-1}^{-1}\varphi_1^3 dx \neq 0\,. \tag{2.5}$$

We give now a family of α and K providing multiple solutions to (2.3).

Theorem 2.4. *Suppose* $m = m^+ - m^-$ *as in Lemma 2.3. For* $t > 0$, *denote* $m_t = m^+ - tm^-$. *Let* $\lambda_1(m_t) > 0$ *in* $\mathbb{R}$, $\varphi_1(t) > 0$ *in* $C(\overline{\Omega})$, $\lambda_{-1}(m_t) < 0$ *in* $\mathbb{R}$, $\varphi_{-1}(t) > 0$ *in* $C(\overline{\Omega})$, *be the principal eigenvalues and eigenfunctions of*

$$\begin{gathered} -\Delta u = \lambda m_t(x) u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered}$$

Define

$$\alpha(t) = -\frac{\lambda_1(m_t)}{\lambda_{-1}(m_t)}, \quad t > 0\,.$$

If $\alpha = \alpha(t)$ *in* (2.3) *and*

$$K = K(t, \rho) = \lambda_{-1}(m_t) m_t \varphi_{-1}(t)^{\alpha(t)+1} + \rho\varphi_{-1}(t)$$

Then (2.3) *has at least two solutions for* $t > 0$ *and* $\rho > 0$ *small enough.*

Remark 2.5. The first term in K is a negative function on Ω^+, the second a positive one.

Remark 2.6. For $\rho = 0$, $(\alpha(t), \varphi_{-1}(t)) \in \mathbb{R}^+ \times C(\overline{\Omega})^+$ could be a bifurcation pair for (2.3) since $u = \varphi_{-1}$ is a solution for $\alpha = \alpha(t)$ and $K = K(t, 0)$.

Now we consider $K(x) \equiv 1$. Hence for $s = 0$, (1.1) has a unique solution. Our next theorem is related to the topological nature of this nonlinear eigenvalue problem (1.1). Let P be the positive cone in $C^1(\overline{\Omega})$ with its usual norm.

Theorem 2.7. *Suppose* $0 < \alpha < 1/n$, $K(x) \equiv 1$, $\mathcal{G}$ *is nonnegative continuous and let* $f(x)$ *be a non-negative bounded measurable function. Then, the set of pairs* (s, u) *of solutions of* (1.1) *is unbounded in* $\mathbb{R}^+ \times P$. *Moreover, if* $\mathcal{G}(x, \eta, \xi) \geq g_0 + |\xi|^2$ *where* $g_0 > 0$ *in* $\mathbb{R}$. *Then, we have* $s \leq 2n/\sqrt{g_0}\, r(\Omega)$, *where* $r(\Omega)$ *is the inner radius of* Ω. *As a consequence, there is bifurcation at infinity for some* $s_* < \infty$.

Recall that the inner radius of Ω is given by $\sup\{r : B_r(x) \subset \Omega\}$.

Finally, we obtain two results dealing with multiplicity for our singular elliptic problem (1.2) with a convection term, as in our title.

Theorem 2.8. *Suppose that*

(i) $0 < \alpha < \frac{1}{n}$, $1 < \beta < \frac{n+1}{n-1}$ *and* $0 < \zeta < \frac{2}{n}$.
(ii) $f \in L^\infty(\Omega)$, $f > 0$.
(iii) $K(x) \equiv 1$.
(iv) $\mathcal{A} = 1$ *and*

$$0 \leq \mathcal{B} < C\Big\{\frac{\int_\Omega f\varphi_1 dx \int_\Omega \varphi_1^2 dx}{\int_\Omega \varphi_1 dx}\Big\}^{\beta-1}$$

where φ_1, λ_1 *are the principal eigenfunction an principal eigenvalue of the operator* $-\Delta$ $(-\Delta\varphi_1 = \lambda_1\varphi_1)$ *with Dirichlet boundary conditions and* C *is a constant depending only in* Ω, β, λ_1.

Then there exist $0 < s^{**} \le s^* < \infty$ *such that for all* $s \in (0, s^{**})$ *problem* (1.2) *admits at least two solutions and no solutions for* $s > s^*$. *Furthermore there is bifurcation at infinity at* $s = 0$.

For a particular form of f and for K with indefinite sign but in a more restricted class we have the following result.

Theorem 2.9. *Suppose that*

(i) $0 < \alpha < \frac{1}{n}$, $1 < \beta < \frac{n+1}{n-1}$, *and* $\zeta < \frac{2}{n}$.
(ii) $f = t\varphi_1$, $t \ge B^{\frac{1}{1+\alpha}}\big[\lambda_1(\frac{\alpha}{\lambda_1})^{\frac{1}{1+\alpha}} + (\frac{\lambda_1}{\alpha})^{\frac{\alpha}{1+\alpha}}\big]$.
(iii) $|K(x)| \le B\varphi_1^{1+\alpha}(x)$.
(iv) $\mathcal{A} = 1$ *and* $0 \le \mathcal{B} < C$ *where* C *is a constant depending only in* λ_1, β, B.

Then there exists $0 < s^{**} \le s^* < \infty$ *such that for all* $s \in (0, s^{**})$ *problem* (1.2) *has at least two solutions and no solutions for* $s > s^*$. *Furthermore there is bifurcation at infinity for s=0.*

We remark that estimate (ii) is needed at the end of the following section.

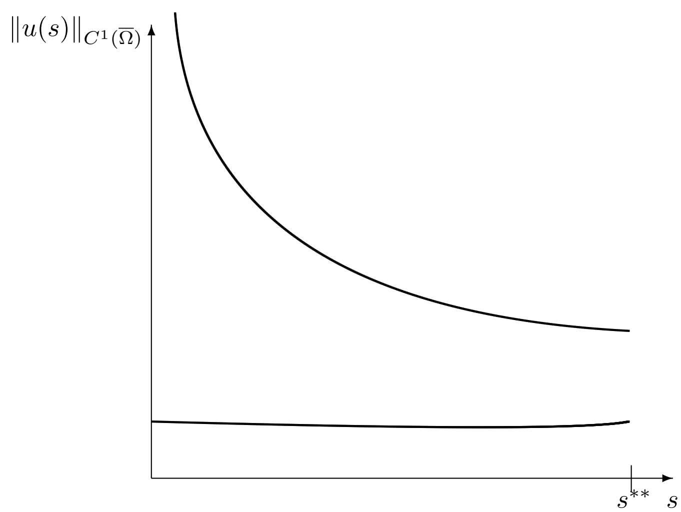

FIGURE 1. Behaviour of the two branches near $s = 0$ in Theorem 2.9

3. Auxiliary Results

It is our purpose in this section to prove some preliminary results.

Proof of Lemma 2.1. We set $\gamma > 2$. Then from the identity

$$-\Delta\varphi_{-1}^{\gamma} = \gamma\lambda_{-1}(m^+ - m^-)\varphi_{-1}^{\gamma} - \gamma(\gamma-1)\varphi_{-1}^{\gamma-2}|\nabla\varphi_{-1}|^2$$

and using that

$$\int_\Omega \Delta\varphi_{-1}^{\gamma} dx = \int_\Omega \operatorname{div} \nabla\varphi_{-1}^{\gamma} dx = \int_{\partial\Omega} \langle \nabla\varphi_{-1}^{\gamma}, n\rangle dx = \int_{\partial\Omega} \gamma\varphi_{-1}^{\gamma-1}\langle \nabla\varphi_{-1}^{\gamma}, n\rangle dx = 0,$$

where the last equality holds because $\varphi_{-1}^{\gamma-1}=0$ on $\partial\Omega$. So

$$\begin{aligned}-\gamma\lambda_{-1}\int_\Omega m^-\varphi_{-1}^\gamma dx &= -\gamma\lambda_{-1}\int_\Omega m^+\varphi_{-1}^\gamma dx+\gamma(\gamma-1)\int_\Omega \varphi_{-1}^{\gamma-2}|\nabla\varphi_{-1}|^2dx\\ &\geq \gamma(\gamma-1)\int_\Omega \varphi_{-1}^{\gamma-2}|\nabla\varphi_{-1}|^2dx,\end{aligned}$$

and consequently

$$\gamma^{1/\gamma}(-\lambda_{-1})^{1/\gamma}\Big(\int_\Omega m^-\varphi_{-1}^\gamma dx\Big)^{1/\gamma}\geq \gamma^{1/\gamma}(\gamma-1)^{1/\gamma}\Big(\int_\Omega \varphi_{-1}^{\gamma-2}|\nabla\varphi_{-1}|^2dx\Big)^{1/\gamma}.$$

Letting $\gamma\to\infty$, we find

$$\|\varphi_{-1}\|_{L^\infty(rmsupp\, m^-,m^-dx)}\geq \|\varphi_{-1}\|_{L^\infty(\Omega,|\nabla\varphi_{-1}|^2dx)}$$

where $\|\varphi_{-1}\|_{L^\infty(\Omega,|\nabla\varphi_{-1}|^2dx)}=\operatorname{ess\,sup}_\Omega|\varphi_{-1}|$ is taken with respect the measure $|\nabla\varphi_{-1}|^2dx$. We observe that $-\Delta\varphi_{-1}=0$ in $\Omega-\{rmsupp\, m^-\cup\operatorname{supp}\, m^+\}$ to conclude that the Lebesgue's measure of thee set $\{x\in\Omega-\{rmsupp\, m^-\cup rmsupp\, m^+\}:\nabla\varphi_{-1}(x)=0\}$ is zero.

From $-\Delta\varphi_{-1}<0$ in $rmsupp\, m^+$, we infer that

$$\sup_{rmsupp\, m^+}\varphi_{-1}\leq \sup_{\partial\, rmsupp\, m^+}\varphi_{-1}$$

and find that

$$\begin{aligned}\|\varphi_{-1}\|_{L^\infty(\Omega,|\nabla\varphi_{-1}|^2dx)} &\geq \|\varphi_{-1}\|_{L^\infty(\Omega-\{rmsupp\, m^+\cup rmsupp\, m^-\},|\nabla\varphi_{-1}|^2dx)}\\ &= \|\varphi_{-1}\|_{L^\infty(\Omega-\{rmsupp\, m^+\cup rmsupp\, m^-\})}\\ &= \|\varphi_{-1}\|_{L^\infty(\Omega-\{rmsupp\, m^-\})};\end{aligned}$$

hence

$$\|\varphi_{-1}\|_{L^\infty(rmsupp\, m^-,\ m^-dx)}\geq \|\varphi_{-1}\|_{L^\infty(\Omega-\{rmsupp\, m^-\})}$$

With the aid of this last expression, we arrive to the desired conclusion. □

Proof of Lemma 2.2. Continuity follows from well known results ([16]). Since $m^+-tm^-<m^+$ for all $t>0$, we conclude that $\lambda_1((m^+-tm^-))>\lambda_1((m^+))$ ([16]). Clearly

$$\lim_{t\to\infty}\lambda_{-1}((m^+-tm^-))=\lim_{t\to\infty}\frac{1}{t}\lambda_{-1}((\frac{m^+}{t}-m^-))=0.$$

Then $\lim_{t\to\infty}\alpha(t)=\infty$. Using $m^+-tm^->-tm^-$, we deduce that $\lambda_{-1}((m^+-tm^-))<\lambda_{-1}((-tm^-))=\frac{1}{t}\lambda_{-1}((-m^-))$ and therefore

$$\lim_{t\to 0^+}\lambda_{-1}((m^+-tm^-))=-\infty\,.$$

Finally, from $\lim_{t\to0^+}\lambda_1((m^+-tm^-))=\lambda_1((m^+))$, we find $\lim_{t\to0^+}\alpha(t)=0$. □

Proof of Lemma 2.3. To prove this lemma, we bound $t|\lambda_{-1}((m^+-tm^-))|$. From $m^+-tm^->-tm^-$, we deduce $\lambda_{-1}((m^+-tm^-))<\lambda_{-1}((-tm))$ ([16]) and therefore

$$-t\lambda_{-1}((m^+-tm^-))>-\lambda_{-1}((-m^-))>0\,.$$

From the equation

$$\begin{gathered}-\Delta\varphi_{-1}=\lambda_{-1}(m^+-tm^-)\varphi_{-1}\quad\text{in }\Omega\\ \varphi_{-1}=0\quad\text{on }\partial\Omega\,,\end{gathered}$$

we see that

$$-\Delta\varphi_{-1} = -\lambda_{-1}(tm^- - m^+)\varphi_{-1} \quad \text{in } \Omega$$
$$\varphi_{-1} = 0 \quad \text{on } \partial\Omega\,.$$

We conclude that

$$-\lambda_{-1}((m^+ - tm^-;\Omega)) = \lambda_1((tm^- - m^+;\Omega))\,.$$

Using $rmsupp\, m^- \subset \Omega$, it follows that

$$\lambda_1((tm^- - m^+;\Omega)) \le \lambda_1((tm^- - m^+; rmsupp\, m^-)) = \lambda_1((tm^-; rmsupp\, m^-))$$

Thus, we have

$$0 < -\lambda_{-1}((-m^-)) < t|\lambda_{-1}((m^+ - tm^-;\Omega))| < \lambda_1((m^-; rmsupp\, m^-)) \tag{3.1}$$

Our next tool is Harnack inequality. It asserts that if $u \in W^{1,2}(\Omega)$ satisfies

$$-\Delta u + mu = 0 \quad \text{in } \Omega$$
$$u \ge 0 \quad \text{on } \Omega,$$

then for any ball $B_{4R}(y) \subset \Omega$, we have

$$\sup_{B_R(y)} u \le C(N)^{1+R\sqrt{\|m\|_{L^\infty(\Omega)}}} \inf_{B_R(y)} u$$

(see Theorem 8.20 [21]).

Now we are ready to deal with (2.5). We may suppose $\|\varphi_{-1}\|_{L^\infty(\Omega)} = 1$. From Harnack inequality and Lemma 2.1, we find

$$1 \le C(N)^{1+R\sqrt{t|\lambda_{-1}|}} \inf_{rmsupp\, m^-} \varphi_{-1}\,.$$

Then

$$t\int_\Omega m^-\varphi_{-1}^{-1}\varphi_1^3 dx \le tC(N)^{1+R\sqrt{t|\lambda_{-1}|}} \int_\Omega m^-\varphi_1^3 dx\,. \tag{3.2}$$

Assume the claim in this Lemma false, i. e.,

$$\int_\Omega (m^+ - tm^-)\varphi_{-1}^{-1}\varphi_1^3 dx = 0\,.$$

Then

$$\begin{aligned}\int_\Omega m^+\varphi_1^3 dx &\le \int_\Omega m^+\varphi_{-1}^{-1}\varphi_1^3 dx\\ &= t\int_\Omega m^-\varphi_{-1}^{-1}\varphi_1^3 dx\\ &\le tC(N)^{1+R\sqrt{t|\lambda_{-1}|}} \int_\Omega m^-\varphi_1^3 dx\,.\end{aligned}$$

Thus

$$\begin{aligned}\big(\inf_{rmsupp\, m^+} \varphi_1\big)^3 \int_{rmsupp\, m^+} m^+ dx &\le tC(N)^{1+R\sqrt{t|\lambda_{-1}|}} \int_\Omega m^-\varphi_1^3 dx\\ &\le tC(N)^{1+R\sqrt{t|\lambda_{-1}|}} \big(\sup{}_{rmsupp\, m^-}\varphi_1\big)^3 \int_{rmsupp\, m^-} m^- dx\,.\end{aligned}$$

Consequently,

$$\Big(\inf_{B_{5R}(\frac{1}{2}(y_++y_-))}\varphi_1\Big)^3 \le tC(N)^{1+R\sqrt{t|\lambda_{-1}|}}\big(\sup_{B_{5R}(\frac{1}{2}(y_++y_-))}\varphi_1\big)^3\frac{\int_{rm supp\, m^-} m^-dx}{\int_{\text{supp }m^+} m^+dx}$$

Hence

$$\frac{1}{C(N)^{(1+R\sqrt{t|\lambda_{-1}|})+3+15R\sqrt{\max(\lambda_1,t\lambda_1)}}}\frac{\int_{rm supp\, m^+} m^+dx}{\int_{rm supp\, m^-} m^-dx}\le t\,. \tag{3.3}$$

For small t, using (3.1), we deduce that (3.3) is a contradiction. □

Recall that the vector space

$$C(\bar\Omega)_e=\{u\in C(\bar\Omega); -se\le u\le se \text{ for some } s>0 \text{ in } \mathbb{R}\},$$

where e is the solution of $-\Delta e=1$ in Ω, $e=0$ on $\partial\Omega$, endowed with the norm

$$||u||_e=\inf\{s>0; -se\le u\le se\}$$

is a Banach space [3]. We will use the Banach space

$$\mathcal{C}=W^{2,p}(\Omega)\cap C(\overline\Omega)_e \tag{3.4}$$

for the norm $\|\cdot\|_{\mathcal{C}}=\|\cdot\|_{W^{2,p}(\Omega)}+\|\cdot\|_e$. Hence, the cone of positive functions

$$\mathcal{C}^+=W^{2,p}(\Omega)\cap C(\overline\Omega)_e^+ \tag{3.5}$$

has non empty interior $\mathring{\mathcal{C}}^+$. We also need

$$\mathcal{D}=\{f: fe^{-\alpha}\in L^p(\Omega)\} \tag{3.6}$$

which is a Banach space for the norm

$$\|f\|_{\mathcal{D}}=\Big(\int_\Omega |f|^pe^{-p\alpha}dx\Big)^{1/p}$$

Note that all principal eigenfunctions are in $\mathring{\mathcal{C}}^+$.

Lemma 3.1. *The map $F:\mathring{\mathcal{C}}^+\to\mathcal{D}$,*

$$F(u)=-u^\alpha\Delta u,$$

is regular and has first and second derivatives

$$dF(u)v=-\alpha u^{\alpha-1}v\Delta u-u^\alpha\Delta v,$$
$$d^2F(u)[v,h]=-\alpha(\alpha-1)u^{\alpha-2}vh\Delta u-\alpha u^{\alpha-1}v\Delta h-\alpha u^{\alpha-1}h\Delta v$$

Proof. Consider

$$\omega(t)=\frac{F(u+tv)-F(u)}{t}+\alpha u^{\alpha-1}v\Delta u+u^\alpha\Delta v \tag{3.7}$$

To prove Gateaux differentiability, we need to establish

$$\lim_{t\to0}\|\omega(t)\|_{\mathcal{C}}=0 \tag{3.8}$$

From the Mean-Value Theorem one has (at almost every $x\in\Omega$)

$$\begin{aligned}F(u+tv)-F(u)&=-\int_0^1\frac{d}{d\xi}\left\{(u+\xi tv)^\alpha\Delta(u+\xi tv)\right\}d\xi\\&=-t\int_0^1\Big\{\alpha(u+\xi tv)^{\alpha-1}v\Delta(u+\xi tv)+(u+\xi tv)^\alpha\Delta v\Big\}d\xi\,.\end{aligned}$$

Thus

$$\begin{aligned} \|\omega(t)\|_{\mathcal{D}} \le \| \int_0^1 \alpha v \left\{ u^{\alpha-1}\Delta u - (u+\xi tv)^{\alpha-1}\Delta(u+\xi tv)\right\} d\xi \|_{\mathcal{D}} \\ + \| \int_0^1 \Delta v \left\{ u^{\alpha} - (u+\xi tv)^{\alpha} \right\} d\xi \|_{\mathcal{D}} \,. \end{aligned} \tag{3.9}$$

Using the definition of $\|\cdot\|_{\mathcal{D}}$, Jensen inequality and Fubini Theorem, we obtain

$$\begin{aligned} \| \int_0^1 \Delta v\{u^{\alpha} - (u+\xi tv)^{\alpha}\} d\xi\|_{\mathcal{D}}^p &= \int_\Omega | \int_0^1 \Delta v\{u^{\alpha} - (u+\xi tv)^{\alpha}\} d\xi|^p \ e^{-p\alpha} dx \\ &\le \int_0^1 d\xi \int_\Omega |\Delta v\{u^{\alpha} - (u+\xi tv)^{\alpha}\}|^p e^{-p\alpha} dx \,. \end{aligned}$$

A similar estimate is valid for the second term in (3.9) and consequently, the Lebesgue Dominated-Convergence Theorem implies (3.8). Next we prove continuity of the map

$$d_G F : \mathring{\mathcal{C}}^+ \to L(\mathcal{C}, \mathcal{D})$$

where $L(\mathcal{C}, \mathcal{D})$ is provided with the operator norm. Recall that

$$\|d_G F(u_j) - d_G F(u)\|_{L(\mathcal{C},\mathcal{D})} = \sup_{v\in\mathcal{C}, \|v\|_{\mathcal{C}}\le 1} \|d_G F(u_j) v - d_G F(u) v\|_{\mathcal{D}} \,.$$

Furthermore,

$$\begin{aligned} \|d_G F(u_j)v - d_G F(u)v\|_{\mathcal{D}} &= \| - \alpha u_j^{\alpha-1} v\Delta u_j - u_j^{\alpha}\Delta v + \alpha u^{\alpha-1} v \Delta u + u^{\alpha}\Delta v\|_{\mathcal{D}} \\ &\le \|\alpha v (u^{\alpha-1}\Delta u - u_j^{\alpha-1}\Delta u_j)\|_{\mathcal{D}} + \|(u^{\alpha} - u_j^{\alpha})\Delta v\|_{\mathcal{D}} \\ &\le \|\alpha v \Delta u (u^{\alpha-1} - u_j^{\alpha-1})\|_{\mathcal{D}} + \|\alpha v u_j^{\alpha-1}(\Delta u - \Delta u_j)\|_{\mathcal{D}} \\ &\quad + \|(u^{\alpha} - u_j^{\alpha})\Delta v\|_{\mathcal{D}} \,. \end{aligned}$$

If $\|u - u_j\|_{\mathcal{C}}$, that is $|u - u_j| \le \frac{1}{j}\ e$ in Ω, we prove now that each of these last three terms tends to zero. From

$$\begin{aligned} |u(x)^{\alpha-1} - u_j(x)^{\alpha-1}| &= |(\alpha-1)\int_0^1 (\xi u_j(x) + (1-\xi)u(x))^{\alpha-2} d\xi (u(x) - u_j(x))| \\ &\le \frac{|1-\alpha|}{j} C\ e(x)^{\alpha-1} \end{aligned}$$

and using $|v| \le \varphi_{-1}$, we get

$$\|\alpha v \Delta u (u^{\alpha-1} - u_j^{\alpha-1})\|_{\mathcal{D}} \le C\frac{\alpha|1-\alpha|}{j} \|\ e^{\alpha}\Delta u\|_{\mathcal{D}} = C\frac{\alpha|1-\alpha|}{j}\|\Delta u\|_{L^p(\Omega)} \,.$$

Similarly,

$$\begin{gathered} \|\alpha v u_j^{\alpha-1}(\Delta u - \Delta u_j)\|_{\mathcal{D}} \le C\|\Delta u - \Delta u_j\|_{L^p(\Omega)}, \\ \|(u^{\alpha} - u_j^{\alpha})\Delta v\|_{\mathcal{D}} \le C\frac{\alpha}{j} \,. \end{gathered}$$

This proves continuity of the Gateaux derivative and hence F is Fréchet differentiable. For the second derivative we proceed similarly. □

In [4, Theorem 3.1] it is stated that

$$\begin{gathered} -\Delta u = u^{-\alpha} + f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.10}$$

with non-negative $f \in L^p(\Omega)$ ($p > n$), has a unique solution $u \in W^{2,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$.

Lemma 3.2. *Suppose* $0 < \alpha < \frac{1}{n}$. *Then the solution map of problem* (3.10) $f \to u$, *denoted* H *is well defined from* $\{f \in C(\overline{\Omega}) : f(x) \geq 0,\ x \in \Omega\}$ *into* $\{u \in C^1(\overline{\Omega}) : u(x) \geq 0,\ x \in \Omega,\ u(x) = 0$ *and* $\frac{\partial u}{\partial n}(x) < 0,\ x \in \partial\Omega\}$. *Moreover* H *is a continuous and compact map.*

Proof. $0 < \alpha < \frac{1}{n}$ allow us to fix $p > n$ such that $\alpha p < 1$. In the proof of this Lemma we will use this p. From the proof in [4, Theorem 1], we know that $u_j = Hf_j \geq w$, where w satisfies

$$\begin{gathered} -\Delta w = u_1^{-\alpha} \quad \text{in } \Omega \\ w = 0 \quad \text{on } \partial\Omega \end{gathered}$$

and $u_1 \in W^{2,p}(\Omega)$ is the unique solution of the problem

$$\begin{gathered} -\Delta u_1 = u_1^{-\alpha} + f_j \quad \text{in } \Omega \\ u_1 = 1 \quad \text{on } \partial\Omega\,. \end{gathered}$$

Using the Maximum Principle, we have $u_1^{-\alpha} \leq w_1^{-\alpha}$, where w_1 is the solution of the problem

$$\begin{gathered} -\Delta w_1 = f_j \quad \text{in } \Omega \\ w_1 = 1 \quad \text{on } \partial\Omega\,. \end{gathered}$$

Using again the Maximum Principle we see that $u_1^{-\alpha} \leq 1$ on $x \in \overline{\Omega}$. We recall a Uniform Hopf Principle as it is formulated in Diaz-Morel-Oswald [15]. It asserts that there exists a constant C, depending only on Ω, such that for all $f \geq 0$, $f \in L^1(\Omega)$, each weak solution u of

$$\begin{gathered} -\Delta u = f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.11}$$

satisfies

$$u \geq C\Big(\int_\Omega fe\Big)e\,. \tag{3.12}$$

Applying this Uniform Hopf Principle, we get

$$w(x) \geq C(\Omega)\Big(\int_\Omega u_1^{-\alpha} e dx\Big)e(x)\,.$$

Jensen inequality implies

$$\Big(\int_\Omega u_1^{-\alpha} e dx\Big)^{-\alpha} \leq \Big(\int_\Omega e\, dx\Big)^{\alpha-1}\Big(\int_\Omega u_1^{\alpha^2} e dx\Big)\,.$$

As before, we have $u_1 \leq w_j$ where w_j is the unique solution of

$$\begin{gathered} -\Delta w_j = 1 + f_j \quad \text{in } \Omega \\ w_j = 1 \quad \text{on } \partial\Omega\,. \end{gathered}$$

Thus

$$u_j(x)^{-\alpha} \leq C(\Omega)^{-\alpha}\Big(\int_\Omega e dx\Big)^{\alpha-1}\Big(\int_\Omega w_j^{\alpha^2} e\, dx\Big)e^{-\alpha}\,. \tag{3.13}$$

If $f_j \to f$ in $C(\overline{\Omega})$, then there exist a constant C, independent of j, such that

$$\|u_j^{-\alpha}\|_{L^p(\Omega)} < C\,.$$

Then $\|u_j\|_{W^{2,p}(\Omega)} < C$, so Rellich-Kondrachov Theorem implies $u_j \to u$ strongly in $C^1(\overline{\Omega})$. Using (3.13) we conclude that $u_j^{-\alpha} \to u^{-\alpha}$ strongly in $L^p(\Omega)$, and therefore u is a solution of the problem

$$\begin{gathered}-\Delta u = u^{-\alpha} + f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,.\end{gathered}$$

Compactness is deduced from (3.13). □

Lemma 3.3. *Suppose $\mathcal{L} = \Delta + c(x)$ satisfies the maximum principle and suppose*

$$|K(x)| \le B\varphi_1^{1+\alpha}(x) \quad \text{for some } B > 0 \text{ in } \mathbb{R}, \tag{3.14}$$

where φ_1 is the principal eigenfunction corresponding to the principal positive eigenvalue of the problem $-\mathcal{L}u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$. If $f \in L^p(\Omega)$, $p > n$, satisfies

$$f \ge t_0\varphi_1 \quad p.\ p.$$

where $t_0 = B^{\frac{1}{1+\alpha}}\big[\lambda_1(\frac{\alpha}{\lambda_1})^{\frac{1}{1+\alpha}} + (\frac{\lambda_1}{\alpha})^{\frac{\alpha}{1+\alpha}}\big]$. Then

$$\begin{gathered}-\mathcal{L}u + K(x)u^{-\alpha} = f(x) \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \Omega\end{gathered} \tag{3.15}$$

has a strong solution $u \in W^{2,p}(\Omega)$. Moreover, if $f > t_0\varphi_1$ then $u > (\frac{\alpha B}{\lambda_1})^{\frac{1}{1+\alpha}}\varphi_1$ and it is unique within the set $\{v > (\frac{\alpha B}{\lambda_1})^{\frac{1}{1+\alpha}}\varphi_1\}$. If instead of f we consider $f_1 > f_2 \ge t\varphi_1$ in $C(\overline{\Omega})$ with $t > t_0$, then corresponding solutions u_1, u_2 in $\{u \in C(\overline{\Omega}) : u \ge C(t)\varphi_1\}$ satisfy $u_1 > u_2$.

Proof. Let us consider, for $g \in L^\infty(\Omega)$, the solution operator $h = (-\mathcal{L})^{-1}g$ defined by $-\mathcal{L}h = g$ in Ω, $h = 0$ on $\partial\Omega$. Then h lies in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ for all $1 < p < \infty$. We define

$$G_C = \{u \in C(\overline{\Omega}) : u \ge C\varphi_1\}$$

If $t \ge t_0$, then there exists a unique $C(t) \ge (\frac{\alpha B}{\lambda_1})^{\frac{1}{1+\alpha}}$ satisfying $t = \lambda_1 C(t) + \frac{B}{C(t)^\alpha}$. We prove now that for $f \in G_t$, $u \in G_{C(t)}$ the operator

$$F(u) = (-\mathcal{L})^{-1}(f - Ku^{-\alpha})$$

is well defined from $G_{C(t)}$ into $G_{C(t)}$. Moreover, it is continuous for the usual topology on $C(\overline{\Omega})$. Indeed, if $u \in G_{C(t)}$ then $-Ku^{-\alpha} \ge -C(t)^{-\alpha}B\varphi_1$ and consequently $f - Ku^{-\alpha} \ge \lambda_1 C(t)\varphi_1$. Now positivity of $\mathcal{L}^{-1}$ implies $(-\mathcal{L})^{-1}(f - Ku^{-\alpha}) \ge C(t)\varphi_1$.

To see that F is a continuous map, let $(u_n) \in G_{C(t)}$ be a sequence such that $u_n \to u$ in $C(\overline{\Omega})$, then $K(x)u_n(x)^{-\alpha} \to K(x)u(x)^{-\alpha}$, pointwise on Ω. Since $|K(x)u_n^{-\alpha}(x)| \le C(t)^{-\alpha}B\varphi_1(x)$, Lebesgue's Dominated Convergence Theorem gives $f - Ku_n^{-\alpha} \to f - Ku^{-\alpha}$ in $L^p(\Omega)$, $1 < p < \infty$. Then the classical L^p theory for elliptic operators implies

$$(-\mathcal{L})^{-1}(f - Ku_n^{-\alpha}) \to (-\mathcal{L})^{-1}(f - Ku^{-\alpha})$$

in $W^{2,p}(\Omega)$ for all $1 < p < \infty$ and then $F(u_n) \to F(u)$ in $C(\overline{\Omega})$. Moreover $\overline{F(G_{C(t)})}$ is a compact set in $C(\overline{\Omega})$. In fact, we have

$$\|(-\mathcal{L})^{-1}(f - Ku^{-\alpha})\|_{W^{2,p}(\Omega)} \le C_0\|f - Ku^{-\alpha}\|_{L^p(\Omega)} \le C,$$

for all $u \in G_{C(t)}$, $1 < p < \infty$, then it is clear that $\overline{F(G_C)}$ is compact in $C(\overline{\Omega})$. Since $G_{C(t)}$ is a convex closed set, Schauder Fixed Point Theorem provides a fixed point for F in $G_{C(t)}$, so a solution to (3.15).

Suppose now that for $f \in G_t$ there exist two different solutions, u and v of (3.15), then

$$\begin{aligned} -\mathcal{L}(u-v) &= -K(u^{-\alpha} - v^{-\alpha}) \\ &= \alpha K\Big(\int_0^1 (ru + (1-r)v)^{-\alpha-1}\,dr\Big)(u-v). \end{aligned}$$

We define $m = K\int_0^1 (ru+(1-r)v)^{-\alpha-1}\,dr$. Thus, we can write, recalling that $\mathcal{L} = \Delta + c(x)$,

$$\begin{gathered} \Delta(u-v) + (c+\alpha m)(u-v) = 0 \quad \text{in } \Omega \\ u - v = 0 \quad \text{on } \partial\Omega\,. \end{gathered}$$

Since $u \not\equiv v$ we may suppose $u - v$ is positive somewhere in Ω. Now, [10, Corollary 1.1] implies that the principal eigenvalue $\lambda_1((\Delta + c + \alpha m))$ of the problem

$$\begin{gathered} \Delta h + (c+\alpha m)h = \lambda h \quad \text{in } \Omega \\ h = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

is a nonpositive number. We recall Lipschitz continuity of this eigenvalue with respect to L^∞-norm of the coefficient function $c + \alpha m$ (see for example [10, Proposition 2.1]) and the estimate $|m| \le BC(t)^{-1-\alpha}$ to infer that

$$|\lambda_1((\Delta + c + \alpha m)) - \lambda_1((\Delta + c))| \le \|c + \alpha m - c\|_{L^\infty(\Omega)} \le \frac{\alpha B}{C(t)^{1+\alpha}}$$

Considering the choice of $C(t)$, we find

$$0 < \lambda_1 - \frac{\alpha B}{C(t)^{1+\alpha}} \le \lambda_1((\Delta + c + \alpha m)),$$

and this is a contradiction.

If $u_1 \not> u_2$ in our last assertion, then there exists $x_0 \in \Omega$ such that $u_2(x_0) \ge u_1(x_0)$, and $u_2 - u_1$ is a nontrivial solution of

$$\begin{gathered} \mathcal{L}(u_2 - u_1) + \alpha\tilde{m}(u_2 - u_1) \ge 0 \quad \text{in } \Omega \\ u_2 - u_1 = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $\tilde{m}$ is similar to m. From [10, Corollary 1.1] we obtain $\lambda_1((\Delta + c + \alpha\tilde{m})) \le 0$ and this is a contradiction, because $0 \le \tilde{m} \le BC(t)^{-1-\alpha}$ and as before, we have $\lambda_1((\Delta + c + \alpha\tilde{m})) > 0$. □

Remark 3.4. When $\mathcal{L} = \Delta$, t_0 is sharp under condition (3.14) for $K = B\varphi_1^{1+\alpha}$ and $f \in \{t\varphi_1 : t > 0\}$. Indeed

$$\begin{gathered} -\Delta u + B\varphi_1^{1+\alpha} u^{-\alpha} = t\varphi_1 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

implies

$$t_0 \int_\Omega \varphi_1^2 dx \le \int_\Omega \Big(\lambda_1 \frac{u}{\varphi_1} + B\big(\frac{u}{\varphi_1}\big)^{-\alpha}\Big)\varphi_1^2 dx = t\int_\Omega \varphi_1^2 dx.$$

4. Proofs

Proof of Theorem 2.4. Consider the map $F : \mathring{\mathcal{C}}^{+} \to \mathcal{D}$ given by $F(u) = -u^{\alpha}\Delta u$. According to Lemma 3.1, $dF(u)v = 0$ if and only if v satisfies

$$\begin{gathered} -\Delta v = \alpha \frac{\Delta u}{u} v \quad \text{in } \Omega \\ v = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{4.1}$$

Suppose m is as in Lemma 2.1 and consider the eigenvalue problem

$$\begin{gathered} -\Delta u = \lambda m u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered}$$

At $u = \varphi_{-1}$ and for $\alpha = -\frac{\lambda_1}{\lambda_{-1}}$ in (4.1), $dF(\varphi_{-1})v = 0$ is equivalent to

$$\begin{gathered} -\Delta v = \lambda_1 m v \quad \text{in } \Omega \\ v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{4.2}$$

which implies $\ker dF(\varphi_{-1}) = \langle \varphi_1 \rangle$. The equation $dF(\varphi_{-1})v = f$ is equivalent to

$$\begin{gathered} -\Delta v = \lambda_1 m v + \varphi_{-1}^{-\alpha} f \quad \text{in } \Omega \\ v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{4.3}$$

By hypothesis $f\varphi_{-1}^{-\alpha} \in L^p(\Omega)$ with $p > n$, hence the Fredholm alternative yields that (4.3) has a solution $v \in H_0^{1,2}(\Omega)$ if and only if $\int_\Omega \varphi_{-1}^{-\alpha} f \varphi_1 dx = 0$. If we have a solution v since $m \in L^\infty(\Omega)$ a Brezis-Kato result (see for example Struwe appendix B [14]) implies that $v \in \mathcal{C}$.

We want to solve the equation

$$F(\varphi_{-1} + \widehat{v}) = F(\varphi_{-1}) + \rho\varphi_{-1} \tag{4.4}$$

Inserting Taylor formula in (4.4),

$$F(\varphi_{-1} + \widehat{v}) = F(\varphi_{-1}) + dF(\varphi_{-1})\widehat{v} + \Psi(\widehat{v})$$

we find

$$dF(\varphi_{-1})\widehat{v} + \Psi(\widehat{v}) = \rho\varphi_{-1} \tag{4.5}$$

We use now the well known Lyapunov-Schmidt method. First we denote

$$\begin{gathered} \langle \varphi_{-1}^{-\alpha}\varphi_1 \rangle_{\mathcal{C}}^{\perp} = \{ w \in \mathcal{C} : \int_\Omega w \varphi_{-1}^{-\alpha}\varphi_1 dx = 0 \}, \\ \langle \varphi_{-1}^{-\alpha}\varphi_1 \rangle_{\mathcal{D}}^{\perp} = \{ w \in \mathcal{D} : \int_\Omega w \varphi_{-1}^{-\alpha}\varphi_1 dx = 0 \}\,. \end{gathered}$$

Observe that $\int_\Omega \varphi_{-1}\varphi_{-1}^{-\alpha}\varphi_1 dx \neq 0$, thus we have the decompositions as direct sums

$$\mathcal{C} = \langle \varphi_{-1} \rangle \oplus \langle \varphi_{-1}^{-\alpha}\varphi_1 \rangle_{\mathcal{C}}^{\perp}, \quad \mathcal{D} = \langle \varphi_{-1} \rangle \oplus \langle \varphi_{-1}^{-\alpha}\varphi_1 \rangle_{\mathcal{D}}^{\perp}$$

and consequently if $\widehat{v} \in \mathcal{D}$, we get the unique decomposition

$$\widehat{v} = \widehat{s}\varphi_{-1} + w$$

with $w \in \langle \varphi_{-1}^{-\alpha}\varphi_1 \rangle_{\mathcal{D}}^{\perp}$. Let us denote

$$P : \mathcal{D} \to \langle \varphi_{-1} \rangle, \quad Q : \mathcal{D} \to \langle \varphi_{-1}^{-\alpha}\varphi_1 \rangle_{\mathcal{D}}^{\perp}$$

linear operators such that $P\widehat{v} = \widehat{s}\varphi_{-1}$ and $Q\widehat{v} = w$. We can replace (4.5) by the equivalent system

$$QdF(\varphi_{-1})\widehat{v} + Q\Psi(\widehat{v}) = 0, \tag{4.6}$$

$$P\Psi(\widehat{v}) = \rho\varphi_{-1}\,. \tag{4.7}$$

To solve (4.6), we define the function

$$\Gamma : \mathbb{R} \times \langle\varphi_{-1}^{-\alpha}\varphi_1\rangle_{\mathcal{C}}^{\perp} \to \langle\varphi_{-1}^{-\alpha}\varphi_1\rangle_{\mathcal{D}}^{\perp},$$
$$\Gamma(\widehat{s}, w) = QdF(\varphi_{-1})(\widehat{s}\varphi_{-1} + w) + Q\Psi(\widehat{s}\varphi_{-1} + w)\,.$$

This function satisfies

$$\Gamma(0, 0) = 0, \tag{4.8}$$

$$d_w\Gamma(0, 0)w_0 = QdF(\varphi_{-1})w_0, \tag{4.9}$$

$$d_{\widehat{s}}\Gamma(0, 0) = QdF(\varphi_{-1})\varphi_{-1}\,. \tag{4.10}$$

The operator $d_w\Gamma(0, 0)$ has inverse from $\langle\varphi_{-1}^{-\alpha}\varphi_1\rangle_{\mathcal{C}}^{\perp}$ to $\langle\varphi_{-1}^{-\alpha}\varphi_1\rangle_{\mathcal{D}}^{\perp}$. The Implicit Function Theorem applies to Γ: there exist an interval $(-s^*, s^*)$ and a function

$$W : (-s^*, s^*) \to \langle\varphi_{-1}^{-\alpha}\varphi_1\rangle_{\mathcal{C}}^{\perp}$$

such that $\widehat{v} = s\varphi_{-1} + W(s)$ solves (4.6), with

$$W(0) = 0 \quad \text{and} \quad W'(0) = -[QdF(\varphi_{-1})]^{-1}QdF(\varphi_{-1})\varphi_{-1}\,.$$

Using $\operatorname{Im} dF(\varphi_{-1}) = \langle\varphi_{-1}^{-\alpha}\varphi_1\rangle_{\mathcal{D}}^{\perp}$ and $W'(0) \in \langle\varphi_{-1}^{-\alpha}\varphi_1\rangle_{\mathcal{C}}^{\perp}$, we conclude

$$dF(\varphi_{-1})W'(0) = -dF(\varphi_{-1})\varphi_{-1}\,.$$

Hence $W'(0) + \varphi_{-1} \in \operatorname{Ker} dF(\varphi_{-1}) = \langle\varphi_1\rangle$. Thus

$$W'(0) = r\varphi_1 - \varphi_{-1} \tag{4.11}$$

with $r \neq 0$ because $\varphi_{-1} \notin \langle\varphi_{-1}^{\alpha}\varphi_1\rangle^{\perp}$. From (4.7), we find

$$\rho = \int_\Omega \varphi_{-1}P\Psi(s\varphi_{-1} + W(s))dx = \langle\varphi_{-1}, P\Psi(s\varphi_{-1} + W(s))\rangle\,.$$

The function

$$\chi(s) = \langle\varphi_{-1}, P\Psi(s\varphi_{-1} + W(s))\rangle$$

is regular and has first and second derivatives given by

$$\chi'(s) = \langle\varphi_{-1}, Pd\Psi(s\varphi_{-1} + W(s))[\varphi_{-1} + W'(s)]\rangle\,,$$

$$\chi''(s) = \langle\varphi_{-1}, Pd^2\Psi(s\varphi_{-1} + W(s))[\varphi_{-1} + W'(s), \varphi_{-1} + W'(s)]\rangle$$
$$+ \langle\varphi_{-1}, Pd\Psi(s\varphi_{-1} + W(s))[W''(s)]\rangle\,.$$

From $d\Psi(0) = 0$ and $d^2\Psi(0) = d^2F(\varphi_{-1})$, we obtain

$$\chi'(0) = 0,$$
$$\chi''(0) = \langle\varphi_{-1}, Pd^2F(\varphi_{-1})[r\varphi_1, r\varphi_1]\rangle\,.$$

Direct calculations show that

$$d^2F(\varphi_{-1})[\varphi_1, \varphi_1] = \lambda_1(1 - \frac{\lambda_1}{\lambda_{-1}})\varphi_{-1}^{\alpha-1}\varphi_1^2 m\,.$$

Using the decomposition $d^2F(\varphi_{-1})[r\varphi, r\varphi] = s\varphi_{-1} + w$ with $w \in \langle \varphi_{-1}^{-\alpha}\varphi_1\rangle_{\mathcal{D}}^{\perp}$, we find

$$s = r^2\lambda_1(1 - \frac{\lambda_1}{\lambda_{-1}})\frac{\int_\Omega m\varphi_{-1}^{-1}\varphi_1^3 dx}{\int_\Omega \varphi_{-1}^{1-\alpha}\varphi_1 dx}\,.$$

Then $\chi''(0) \neq 0$ is equivalent to

$$\int_\Omega m\varphi_{-1}^{-1}\varphi_1^3 dx \neq 0\,. \tag{4.12}$$

If (4.12) is true, then there exist an nonempty open interval such that the equation (4.7) has at least two solutions. Lemma 2.3 states the existence of a class m's satisfying (4.12). □

Proof of Theorem 2.7. From Lemma 3.2 the operator

$$F(s, u) := H(s\mathcal{G}(x, u, \nabla u) + f)$$

is well defined and is continuous, compact from $\mathbb{R}_{\geq 0} \times P^+$ to P where P is the cone of positive functions in $C^1(\overline{\Omega})$ with the usual norm. Furthermore a solution v of the equation

$$F(s, v + u_*) - u_* = v \tag{4.13}$$

where u_* is the unique solution of the problem

$$\begin{gathered} -\Delta u_* = u_*^{-\alpha} + f \quad \text{in } \Omega \\ u_* = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{4.14}$$

satisfies the equation

$$\begin{gathered} -\Delta(v + u_*) = (v + u_*)^{-\alpha} + s\mathcal{G}(x, v + u_*, \nabla(v + u_*)) + f \quad \text{in } \Omega \\ v + u_* > 0 \quad \text{in } \Omega \\ v + u_* = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{4.15}$$

The operator $T(s, v) := F(s, v + u_*) - u_*$ is well defined from $\mathbb{R}_{\geq 0} \times P$ to P and is a continuous compact operator, moreover $T(0, 0) = 0$ and since $T(0, v) = 0$ for all $v \in P \cup \{0\}$, $v = 0$ is the unique fixed point of $T(0, \cdot)$. For each $\sigma \geq 1$ and $\rho > 0$, we have also that $T(0, v) \neq \sigma v$ for $v \in P \cap \rho\partial B$ where B denotes the open unit ball centered at 0 in $C^1(\overline{\Omega})$. Using Theorem 17.1 in Amman's article [3] there exist a nonempty set Σ of pairs (s, v) in $\mathbb{R}_{\geq 0} \times P$ that solves the equation (4.16). Moreover Σ is a closed, connected and unbounded subset of $\mathbb{R}_{\geq 0} \times P$ containing $(0, 0)$. The nonexistence Corollary 1.1 in [34] implies the last affirmation. □

Proof of Theorem 2.8. We start as in the proof of Theorem 2.7. Hence, from Lemma 3.2, the operator

$$F(s, u) := H(s(\mathcal{A}u^\beta + \mathcal{B}|\nabla u|^\zeta) + f)$$

is well defined, continuous and compact from $\mathbb{R}_{\geq 0} \times P^+$ to P where P is the cone of positive functions in $C^1(\overline{\Omega})$ with the usual norm. We study the fixed point equation

$$F(s, v + u_*) - u_* = v \tag{4.16}$$

where u_* is the unique solution of

$$\begin{gathered} -\Delta u_* = u_*^{-\alpha} + f \quad \text{in } \Omega \\ u_* = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{4.17}$$

Moreover if v is a solution of (4.16), $v + u_*$ is a solution of problem (1.2). Using Amman's article [3, Theorem 17.1], we obtain the existence of a nonempty, closed, connected and unbounded set Σ of pairs (s, v) in $\mathbb{R}_{\geq 0} \times P$ that solves (4.16).

To prove existence of two solutions we obtain a constant C_1 and a estimate $C(\delta) > 0$ for $\delta > 0$ such that:

(a) If (s, u) solves equation (1.2) then $s \leq C_1$.
(b) If (s, u) solves (1.2) then $\|u\|_{L^\infty(\Omega)} \leq C(\delta)$ for all $s \geq \delta$.

Using that Σ is unbounded, the conclusion of Theorem 2.8 follows.

First we prove (a). The function $Q(u) = \lambda_1 \beta u - su^\beta$ where and $1 < \beta < \infty$, has a global maximum on the set of positive real numbers at $u = (\frac{\lambda_1}{s})^{\frac{1}{\beta-1}}$, furthermore

$$Q\big((\frac{\lambda_1}{s})^{\frac{1}{\beta-1}}\big) = C(\beta, \lambda_1) s^{-\frac{1}{\beta-1}}$$

where $C(\beta, \lambda_1)$ is a strictly positive constant depending only on β and λ_1. From the inequality

$$\lambda_1 \beta u - su^\beta \leq C(\beta, \lambda_1) s^{-\frac{1}{\beta-1}}\,.$$

Using equation (1.2), we deduce

$$-\Delta u \geq \lambda_1 \beta u - C(\beta, \lambda_1) s^{-\frac{1}{\beta-1}}$$

and therefore

$$\lambda_1 \int_\Omega u\varphi_1 dx \geq \lambda_1 \beta \int_\Omega u\varphi_1 dx - C(\beta, \lambda_1) s^{-\frac{1}{\beta-1}} \int_\Omega \varphi_1 dx\,.$$

Finally

$$\int_\Omega u\varphi_1 dx \leq \frac{C(\beta, \lambda_1) s^{-\frac{1}{\beta-1}}}{\lambda_1(\beta - 1)} \int_\Omega \varphi_1 dx\,. \tag{4.18}$$

From (1.2), we have $-\Delta u \geq f$. Using the Uniform Hopf Principle (3.11), (3.12) and (4.18), it follows that

$$s \leq \Big\{\frac{C(\beta, \lambda_1) \int_\Omega \varphi_1 dx}{\lambda_1(\beta - 1) C(\Omega) \int_\Omega f\varphi_1 dx \int_\Omega \varphi_1^2 dx}\Big\}^{\beta-1} \tag{4.19}$$

This is the constant C_1 and (a) is proved.

Now we prove (b). We establish a priori bounds for solutions of problem (1.2) using a Brezis-Turner technique (see [12]). Multiplying (1.2) by φ_1 and integrating, we find

$$\lambda_1 \int_\Omega u\varphi_1 dx = s \int_\Omega u^\beta \varphi_1 dx + s\mathcal{B} \int_\Omega |\nabla u|^\zeta \varphi_1 dx + \int_\Omega u^{-\alpha} \varphi_1 dx + \int_\Omega f\varphi_1 dx\,.$$

From (4.18) it follows that

$$s \int_\Omega u^\beta \varphi_1 dx \leq \frac{\lambda_1 C(\beta, \lambda_1) s^{-\frac{1}{\beta-1}}}{\lambda_1(\beta - 1)} \int_\Omega \varphi_1 dx\,. \tag{4.20}$$

Using the hypothesis $\zeta < \frac{2}{n}$ and Young inequality, we obtain a $q \geq 1$ such that $0 < \zeta q \leq 2$, $\frac{1}{q} + \frac{1}{\vartheta+1} = 1$, $0 \leq \vartheta < \frac{n+1}{n-1}$ and

$$|\nabla u|^\zeta u \leq \frac{|\nabla u|^{\zeta q}}{q} + \frac{u^{\vartheta+1}}{\vartheta + 1} \leq |\nabla u|^2 + 1 + u^\vartheta u\,. \tag{4.21}$$

Using the assumption

$$\mathcal{B} < \Big\{\frac{\lambda_1(\beta-1)C(\Omega)\int_\Omega f\varphi_1 dx \int_\Omega \varphi_1^2 dx}{C(\beta,\lambda_1)\int_\Omega \varphi_1 dx}\Big\}^{\beta-1},$$

inequalities (4.19), (4.21), and multiplying (1.2) by u and then integrating, we find

$$C_1\int_\Omega |\nabla u|^2 dx \le s\int_\Omega u^\beta u\,dx + sC_2\int_\Omega u^\vartheta u\,dx + C_3\|u\|_{H_0^1(\Omega)} + C_4\,, \tag{4.22}$$

where C_i for $i=1,\dots 4$ are positive constants independent of s. Using Hölder inequality, (4.20) and the fact that if $1<\beta<\frac{n+1}{n-1}$ then for all $\epsilon>0$ there exist a positive constant C_ϵ such that for all $s>0$ holds $s^\beta \le \epsilon s^{\frac{n+1}{n-1}} + C_\epsilon$, we deduce

$$\begin{aligned}
\int_\Omega u^\beta u\,dx &= \int_\Omega u^{\gamma\beta}\varphi_1^\gamma u^{(1-\gamma)\beta}\varphi_1^{-\gamma} u\,dx \\
&\le \Big(\int_\Omega u^\beta \varphi_1 dx\Big)^\gamma \Big(\int_\Omega u^\beta \varphi_1^{\frac{-\gamma}{1-\gamma}} u^{\frac{1}{1-\gamma}} dx\Big)^{1-\gamma} \\
&\le \big(Cs^{-1-\frac{1}{\beta-1}}\big)^\gamma \Big(\int_\Omega u^\beta (\frac{u}{\varphi_1^\gamma})^{\frac{1}{1-\gamma}} dx\Big)^{1-\gamma} \\
&\le Cs^{-\gamma-\frac{\gamma}{\beta-1}}\Big\{\epsilon^{1-\gamma}\Big(\int_\Omega \frac{u^{\frac{n+1}{n-1}+\frac{1}{1-\gamma}}}{\varphi_1^{\frac{\gamma}{1-\gamma}}} dx\Big)^{1-\gamma} \\
&\quad + C_\epsilon^{1-\gamma}\Big(\int_\Omega (\frac{u}{\varphi_1^\gamma})^{\frac{1}{1-\gamma}} dx\Big)^{1-\gamma}\Big\}.
\end{aligned}$$

For $\gamma = 2/(n+1)$, we find

$$\begin{aligned}
\int_\Omega u^\beta u\,dx &\le Cs^{-\gamma-\frac{\gamma}{\beta-1}}\epsilon^{1-\gamma}\Big(\int_\Omega \big(\frac{u}{\varphi_1^{1/(n+1)}}\big)^{2\frac{n+1}{n-1}} dx\Big)^{\frac{n-1}{2(n+1)}2} \\
&\quad + Cs^{-\gamma-\frac{\gamma}{\beta-1}} C_\epsilon^{1-\gamma}\Big(\int_\Omega \big(\frac{u}{\varphi_1^{2/(n+1)}}\big)^{\frac{n+1}{n-1}} dx\Big)^{\frac{n-1}{n+1}}.
\end{aligned}$$

Since

$$\frac{1}{2\frac{n+1}{n-1}} = \frac{1}{2} - \frac{1}{n} + \frac{\frac{1}{n+1}}{n}, \quad \frac{1}{q} = \frac{1}{2} - \frac{1}{n} + \frac{\frac{2}{n+1}}{n},$$

with $q > \frac{n+1}{n-1}$, we apply Hardy-Sobolev inequality in [12, Lemma 2.2],

$$\|\frac{v}{\varphi_1^\tau}\|_{L^q(\Omega)} \le C\|v\|_{H_0^1(\Omega)} \quad \text{for all } v \text{ in } H_0^1(\Omega)$$

where C is a non-negative constant, $0\le\tau\le 1$, $\frac{1}{q} = \frac{1}{2} - \frac{1}{n} + \frac{\tau}{n}$, φ_1 is the principal eigenfunction of the operator $-\Delta$ ($-\Delta\varphi_1 = \lambda_1\varphi_1$) with Dirichlet boundary condition, and the Hölder inequality to obtain

$$\int_\Omega u^\beta u\,dx \le Cs^{-\gamma-\frac{\gamma}{\beta-1}}\big\{\epsilon^{1-\gamma}\|\nabla u\|_{L^2(\Omega)}^2 + C_\epsilon^{1-\gamma}\|\nabla u\|_{L^2(\Omega)}\big\}.$$

From (4.22), we conclude that

$$\begin{aligned}
C_1\|\nabla u\|_{L^2(\Omega)}^2 \le\;& Cs^{1-\gamma-\frac{\gamma}{\beta-1}}\Big\{\epsilon^{1-\gamma}\|\nabla u\|_{L^2(\Omega)}^2 + C_\epsilon^{1-\gamma}\|\nabla u\|_{L^2(\Omega)}\Big\} \\
& + C\|\nabla u\|_{L^2(\Omega)} + C(\delta)\,,
\end{aligned} \tag{4.23}$$

where C is a non-negative constant independent of s. The condition $\beta < \frac{n+1}{n-1}$ implies

$$1-\gamma-\frac{\gamma}{\beta-1}=\frac{n-1}{n+1}-\frac{2}{(n+1)(\beta-1)}<0\,.$$

Therefore if $s \geq \delta$, we can choose $\epsilon > 0$ such that

$$Cs^{1-\gamma-\frac{\gamma}{\beta-1}}\epsilon^{1-\gamma}\leq\frac{C_1}{2}\,.$$

It now follows from (4.23) that

$$\frac{C_1}{2}\|\nabla u\|^2_{L^2(\Omega)}\leq C\{1+C_\epsilon^{1-\gamma}s^{1-\gamma-\frac{\gamma}{\beta-1}}\}\|\nabla u\|_{L^2(\Omega)}+C(\delta)\,. \tag{4.24}$$

Finally if u is a solution of the problem (1.2) with $s > \delta > 0$, there exists a constant $C(\delta) > 0$ such that $\|u\|_{H_0^{1,2}(\Omega)} < C(\delta)$ and using classical Hölder estimates for weak solutions (see [21]) and Sobolev imbedding theorem we conclude the proof of (b). The proof is complete. □

Proof of Theorem 2.9. From Lemma 3.3, the problem

$$\begin{gathered}-\Delta u=K(x)u^{-\alpha}+f\quad\text{in }\Omega\\ u=0\quad\text{on }\partial\Omega\end{gathered}$$

under the conditions $|K(x)| \leq B\varphi_1^{1+\alpha}(x)$ for some $B > 0$ in $\mathbb{R}$, $f > t_0\varphi_1$ where $t_0 = B^{\frac{1}{1+\alpha}}\big[\lambda_1(\frac{\alpha}{\lambda_1})^{\frac{1}{1+\alpha}} + (\frac{\lambda_1}{\alpha})^{\frac{\alpha}{1+\alpha}}\big]$, has a unique strong solution $u \in W^{2,p}(\Omega)$ within the set $\{v > (\frac{\alpha B}{\lambda_1})^{\frac{1}{1+\alpha}}\varphi_1\}$. Furthermore if we denote H the solution map $f \to u$, it is a continuous and compact map from the set $\{f \in C^1(\overline{\Omega}) : f > t_0\varphi_1\}$ to $\{u \in C^1(\overline{\Omega}) : u > (\frac{\alpha B}{\lambda_1})^{\frac{1}{1+\alpha}}\varphi_1\}$ (see Lemma 3.3). Hence the map

$$F(s,u)=H(s(u^\beta+|\nabla u|^\zeta)+t\varphi_1).$$

with $t \geq t_0$ is well from $\mathbb{R}_{\geq 0} \times P$ to P, where P is the cone of positive functions in $C^1(\overline{\Omega})$. Like in the proof of previous theorems, we study the fixed point equation

$$F(s,u+u_*)-u_*=u\,, \tag{4.25}$$

where u_* is the unique solution in in the set $\{v > (\frac{\alpha B}{\lambda_1})\varphi_1\}$ (see Lemma 3.3)

$$\begin{gathered}-\Delta u_*=Ku_*^{-\alpha}+t\varphi_1\quad\text{in }\Omega\\ u_*=0\quad\text{on }\partial\Omega\,.\end{gathered}$$

If (s, u) solves (4.25) then $(s, u + u_*)$ solves equation (1.2). Now using again the Corollary 17.2 in [3], we find a connected, closed unbounded in $\mathbb{R} \times P$ and emanating from $(0, 0)$ set Σ of pairs (s, u) satisfying the equation (4.25). Since the obtained solution u of problem (1.2) satisfies $u \geq (\frac{\alpha B}{\lambda_1})^{\frac{1}{1+\alpha}}\varphi_1$, we deduce

$$|K|u^{-\alpha}\leq B^{\frac{1}{1+\alpha}}\big(\frac{\lambda_1}{\alpha}\big)^{\frac{\alpha}{1+\alpha}}\varphi_1$$

and from (1.2), we have

$$-\Delta u\geq su^\beta\geq\lambda_1\beta u-C(\beta,\lambda_1)s^{-\frac{1}{\beta-1}}\,.$$

Multiplying by φ_1 and integrating, we find

$$\lambda_1\int_\Omega u\varphi_1dx\geq\lambda_1\beta\int_\Omega u\varphi_1dx-C(\beta,\lambda_1)s^{-\frac{1}{\beta-1}}\int_\Omega\varphi_1dx\,.$$

Thus

$$(\frac{\alpha B}{\lambda_1})^{\frac{1}{1+\alpha}}\int_\Omega \varphi_1^2 dx \le \int_\Omega u\varphi_1 dx \le \frac{C(\beta,\lambda_1)s^{-\frac{1}{\beta-1}}}{\lambda_1(\beta-1)}\int_\Omega \varphi_1 dx\,.$$

Consequently,

$$s \le \{\frac{C(\beta,\lambda_1)}{\lambda_1(\beta-1)}(\frac{\lambda_1}{\alpha B})^{\frac{1}{1+\alpha}}\frac{\int_\Omega \varphi_1 dx}{\int_\Omega \varphi_1^2 dx}\}^{\beta-1}\,.$$

Recalling that

$$\lambda_1\int_\Omega u\varphi_1 dx = s\int_\Omega u^\beta\varphi_1 dx + t\int_\Omega \varphi_1^2 dx - \int_\Omega K(x)u^{-\alpha}\varphi_1 dx\,,$$

we see that

$$s\int_\Omega u^\beta\varphi_1 dx \le \frac{C(\beta,\lambda_1)s^{-\frac{1}{\beta-1}}}{\beta-1}\int_\Omega \varphi_1 dx\,.$$

The rest of the proof is similar to that one of Theorem 2.8. □

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