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PERIODIC SOLUTIONS AND EXPONENTIAL STABILITY FOR SHUNTING INHIBITORY CELLULAR NEURAL NETWORKS WITH CONTINUOUSLY DISTRIBUTED DELAYS

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Abstract. In this paper, we consider a class of shunting inhibitory cellular neural networks with continuously distributed delays (SICNNs). The delays are unbounded and the activation function is not assumed to be bounded. Using the continuation theorems of coincidence degree theory, Lyapunov functional method, we obtain new sufficient conditions for the existence and local exponential stability of periodic solutions of (SICNNs). Numerical examples illustrated our results are given.

1. Introduction

Cellular neural networks, which was introduced in [5, 6], have received much attention in the past years due to their extensive applications in signal processing, moving image processing, vision, pattern recognition, optimization and many other area [6, 2] and references therein.

Now, it has been shown that such applications of neural networks rely on the dynamical behaviors of the networks. Therefore, the existence of periodic, almost periodic solutions, stability analysis for neural networks have been wildly investigated [3, 17, 16, 9] and references therein.

Shunting inhibitory cellular neural networks (SICNNs), which was first proposed by Bouzerdoum and Pinter [1], has been found applications in may areas, such as psychophysics, speech, perception, robotics, adaptive pattern recognition, vision and image processing. So its dynamic behavior research has an important significance for theory and applications.

In this paper, we study a class of shunting inhibitory cellular neural networks with distributed time delay. The dynamics of a cell C_{ij} are described by the following equation

$$\begin{aligned}\dot{x}_{ij}(t) = -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f\Big(\int_0^\infty K_{ij}(u)x_{kl}(t-u)du\Big)x_{ij}(t) \\ + L_{ij}(t), \quad i = 1,2,\dots,m;\ j = 1,2,\dots,n,\end{aligned} \tag{1.1}$$

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where C_{ij} denote the cell at the (i, j) position of the lattice, the r-neighborhood $N_r(i, j)$ of C_{ij} is determined by

$$N_r(i,j) = \{C_{kl} : \max(|k-i|, |l-j|) \le r,\ 1 \le k \le m,\ 1 \le 1 \le n\}.$$

Here x_{ij} is the activity of the cell C_{ij}, L_{ij} is the external input to $C_{ij}, a_{ij}(t) > 0$ represent a passive decay rate of the cell activity, $C_{ij}^{kl} \ge 0$ is the connection of coupling strength of postsynaptic activity of the cell transmitted to the cell C_{ij}, the activity function f is a positive continuous function, representing the output or firing rate of cell C_{kl}.

Using Poincaré mapping, the authors in [4] proved the existence and global exponential stability of periodic solutions. However, the activation function $f(.)$ was bounded, Lipschitzian and the time delay was finite.

In recent paper [17], with assumptions the activation function $f(.)$ was Lipschitzian and $f(0) = 0$, the authors give conditions for the existence and stability of almost periodic solutions of (1.1). However, we see that the conditions for the stability depend on each solution of (1.1). So, it's difficult for the stability test.

In this paper, by using coincidence degree theory, we prove the existence and exponential stability of periodic solution of (1.1) without assumptions on boundedness and $f(0) = 0$ of activation function.

Denote by BC the Banach space of bounded continuous functions $\phi : (-\infty, 0] \to \mathbb{R}^{mn}$ with the norm $\|\phi\| = (\sum_{i,j} \sup_{-\infty<s\le 0} |\phi_{ij}(s)|^2)^{1/2}$.

The initial conditions associated with (1.1) are of the form

$$x(\theta) = \phi(\theta), \quad \theta \in (-\infty, 0],\ \phi \in BC. \tag{1.2}$$

For system (1.1) we consider the following hypotheses

- (H1) The delay kernels $K_{ij} : [0,\infty) \to \mathbb{R}$ are piecewise continuous and $P_{ij}(\varepsilon) = \int_0^\infty K_{ij}(u)e^{\varepsilon u}du$ is continuous on $[0, \delta), P_{ij}(0) = 1$ for some $\delta > 0$.
- (H2) The functions $a_{ij}(t), L_{ij}(t)$ are ω-periodic, $f(.)$ is positive, continuous and not assumed to be bounded on $\mathbb{R}$.

Let $L_{ij} = \sup_{t\in\mathbb{R}} |L_{ij}(t)|, a_{ij} = \inf_{t\in\mathbb{R}} a_{ij}(t) > 0$.

This paper is organized as follows. Section 2 presents notations, mathematical definitions and some results from coincidence degree theory that needed to use in the proof of main results in section 3. In section 3, we give new sufficient conditions for the existence of periodic solutions of (SICNNs). Based on Lyapunov functional method, the local exponential stability of the periodic solution of (SICNNs) is established. An example illustrates our main results is given in section 4. The paper ends with conclusion and cited references.

2. Preliminaries

In this section, we recall some notations and results in coincidence degree theory that to be used in the proof of our main results.

Let $\mathbb{X}, \mathbb{Y}$ be normed vector spaces, $L : \operatorname{Dom} L \subset \mathbb{X} \to \mathbb{Y}$ be a linear operator and $N : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L will be called a Fredholm mapping if

- (a) $\operatorname{Ker} L$ is a finite dimensional subspace of $\mathbb{X}$;
- (b) $\operatorname{Im} L$ is closed;
- (c) $\operatorname{Im} L$ has a finite co-dimension.

When L is Fredholm mapping, its index is a integer defined by

$$\operatorname{Ind} L := \dim \operatorname{Ker} L - \operatorname{codim} \operatorname{Im} L.$$

Suppose that L is Fredholm mapping of index zero, then there exist continuous projectors $P : \mathbb{X} \to \mathbb{Y}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ such that

$$\operatorname{Im} P = \operatorname{Ker} L, \quad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q).$$

It follows that mapping $L\big|_{\operatorname{Dom} L \cap \operatorname{Ker} P} : \operatorname{Dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible. Denoted by K_P the inversion of $L\big|_{\operatorname{Dom} L \cap \operatorname{Ker} P}$.

Let Ω be an open and bounded subset of $\mathbb{X}$, the mapping N is called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \Omega \to \mathbb{X}$ is compact mapping. Since ImQ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : ImQ \to \operatorname{Ker} L$.

Next, we give the following lemma, which known as Mawhin's continuation theorem [8], that to be used in next section.

Lemma 2.1 ([8]). *Let $\Omega \subset \mathbb{X}$ be an open bounded set, L be a Fredholm mapping of index zero and $N : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping which is $L-$ compact on Ω. Assume that*

(a) *For $\lambda \in (0, 1)$, $Lx \neq \lambda Nx$ for all $x \in \partial\Omega \cap \operatorname{Dom} L$;*
(b) *$QNx \neq 0$ for every $x \in \partial\Omega \cap \operatorname{Ker} L$;*
(c) $\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

Then equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

3. Main results

In this section, by applying coincidence degree theory we give sufficient conditions for the existence of periodic solutions of the system (1.1). Next, we prove the local exponential stability of periodic solutions of (SICNNs) (1.1).

3.1. Existence of Periodic solutions.

Theorem 3.1. *Let hypotheses* (H1), (H2) *hold. Then* (1.1) *has at least one ω-periodic solution.*

Proof. We denote

$$\mathbb{X} = \{u \in C(\mathbb{R}, \mathbb{R}^{mn}) : u(t + \omega) = u(t),\ \forall t \in \mathbb{R}\}$$

with norm

$$\|u\| = \Big(\sum_{i,j} \max_{t\in[0,\omega]} |u_{ij}(t)|^2\Big)^{1/2}.$$

It's easy to verify that $(\mathbb{X}, \|.\|)$ is a Banach space. Denote $\operatorname{Dom} L = \mathbb{X} \cap C^1(\mathbb{R}, \mathbb{R}^{mn})$. Consider the linear operator

$$L : \operatorname{Dom} L \to \mathbb{X}, \quad Lu = \dot{u}(t). \tag{3.1}$$

Then $\operatorname{Ker} L = \mathbb{R}^{mn}$ and

$$\operatorname{Im} L = \big\{x \in \mathbb{X} : \int_0^\omega x_{ij}(t)dt = 0,\ i = 1, \dots, m;\ j = 1, \dots n\big\}.$$

Clearly $\operatorname{Im} L$ is closed in $\mathbb{X}$ and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = mn$. Hence, L is a Fredholm mapping of index zero.

For convenience, we denote

$$y(t) = y_{ij}(t) = -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f\Big(\int_0^\infty K_{ij}(u)x_{kl}(t-u)du\Big)x_{ij}(t) + L_{ij}(t).$$

Consider the mapping $N : \mathbb{X} \to \mathbb{X}$, $Nx(t) = y(t)$. Define two projectors $P, Q : \mathbb{X} \to \mathbb{X}$ as

$$Pu = Qu = \frac{1}{\omega}\int_0^\omega u(t)dt. \tag{3.2}$$

Let Ω be an open bounded set in $\mathbb{X}$. Using the Arzela-Ascoli theorem [13], it is easy to show that N is L-compact on $\overline{\Omega}$. For $\lambda \in (0,1)$, corresponding to operator equation $Lx = \lambda Nx$, we have

$$\dot{x}_{ij}(t) = \lambda\Big[- a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f\Big(\int_0^\infty K_{ij}(u)x_{kl}(t-u)du\Big)x_{ij}(t) + L_{ij}(t)\Big] \tag{3.3}$$

Suppose that $x \in \mathbb{X}$ is a solution of (3.3) for some $\lambda \in (0,1), x(t) = (x_{ij}(t))$. Let $\overline{\eta_{ij}} \in [0,\omega]$ such that $x_{ij}(\overline{\eta_{ij}}) = \max_{t\in[0,\omega]} x_{ij}(t)$, then

$$a_{ij}(\overline{\eta_{ij}})x_{ij}(\overline{\eta_{ij}}) + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f\Big(\int_0^\infty K_{ij}(u)x_{kl}(\overline{\eta_{ij}} - u)du\Big)x_{ij}(\overline{\eta_{ij}}) = L_{ij}(\overline{\eta_{ij}}) \tag{3.4}$$

Therefore, for all i, j,

$$x_{ij}(\overline{\eta_{ij}}) \le \frac{L_{ij}}{a_{ij}}\,.$$

By the same argument, let $\underline{\eta_{ij}} \in [0,\omega]$ such that $x_{ij}(\underline{\eta_{ij}}) = \min_{t\in[0,\omega]} x_{ij}(t)$, we also have

$$x_{ij}(\underline{\eta_{ij}}) \ge -\frac{L_{ij}}{a_{ij}}$$

for all i, j. Denote

$$C = \Big(\sum_{ij} L_{ij}^2 \frac{1}{a^2} + T\Big)^{1/2},$$

where $T > 0$, $a = \min_{i,j} a_{ij}$. Then C independent of λ. We will show that the conditions (a), (b), (c) in Lemma 2.1 are satisfied.

We take $\Omega = \{u \in \mathbb{X} : \|u\| < C\}$. Then Ω satisfies condition (a) in Lemma 2.1. For $u \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^{mn}$, u is a constant vector in $\mathbb{R}^{mn}$ with $\|u\| = C$, we have

$$\begin{aligned} u^T QNu &\le \sum_{i,j} \Big[- a_{ij}u_{ij}^2 - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(\int_0^\infty K_{ij}(s)u_{kl}ds)u_{ij}^2 + L_{ij}|u_{ij}|\Big] \\ &\le \sum_{i,j}[-a_{ij}u_{ij}^2 + L_{ij}|u_{ij}|] \\ &\le -a\|u\|^2 + \sum_{i,j} L_{ij}|u_{ij}| < 0. \end{aligned}$$

So for any $u \in \partial\Omega \cap \operatorname{Ker} L$, then $QNu \ne 0$. It follows that condition (b) is satisfied.

Furthermore, from $\operatorname{Im} Q = \operatorname{Ker} L$, we choose $J = Id$. Let

$$\Phi(\gamma; u) = -\gamma u + (1-\gamma) QNu, \quad \gamma \in [0,1], u \in \mathbb{R}.$$

Then for any $x \in \partial\Omega \cap \operatorname{Ker} L, x^T \Phi(\gamma; x) < 0$ implies $0 \notin \Phi([0,1] \times \partial\Omega \cap \operatorname{Ker} L)$. According to the homotopy invariance property of mapping degree [8], we get

$$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} = \deg\{-Id, \Omega \cap \operatorname{Ker} L, 0\} \neq 0.$$

Hence, condition (c) of Lemma 2.1. is satisfied.

Thus, by Lemma 2.1 we conclude that $Lx = Nx$ has at least one solution in the ball $B(0, C) = \{x \in \mathbb{X} : \|x\| < C\}$, which concludes the proof. □

3.2. **Stability of periodic solutions.** In this subsection, first we prove the boundedness of solutions of the system (1.1) and then we deal with the stability of the periodic solutions of (1.1).

Definition 3.2. The periodic solution $x^*(t, \varphi^*)$ of the system (1.1) is said to be locally exponentially stable, if there are constants $\varepsilon > 0, \beta > 0$ and $M \geq 1$ such that for any solution $x(t, \varphi)$ of the system (1.1) which satisfies $\|\varphi - \varphi^*\| < \beta$, one has

$$|x_{ij}(t) - x^*_{ij}(t)| \leq M\|\varphi - \varphi^*\| e^{-\varepsilon t}, \quad \forall t \in \mathbb{R}^+,\ i = 1, \dots, m;\ j = 1, \dots, n.$$

If $\beta = \infty$ then (1.1) is said to be globally exponentially stable.

Lemma 3.3. *Assume that the hypotheses H_1, H_2 hold. Then every solution $x(t, \varphi)$ of the system* (1.1) *is bounded. Moreover, we have*

$$|x_{ij}(t)| \leq N_{ij} := \max\Big\{\frac{L_{ij}}{a_{ij}}, \sup_{\theta \in (-\infty, 0]} |\varphi_{ij}(\theta)|\Big\}, \quad \forall t \in \mathbb{R}.$$

Proof. Suppose that the conclusion in Lemma 3.3 is not true. Then there exist a solution $x(t, \varphi)$ and $t > 0$ such that $|x_{ij}(t)| > N_{ij}$.

If $x_{ij}(t) > N_{ij}$ then there exists a $t_{ij} > 0$ such that $x_{ij}(t_{ij}) > N_{ij}$ and $D^+ x_{ij}(t_{ij}) \geq 0$. On the other hand, we have

$$\begin{aligned}
\dot{x}_{ij}(t_{ij}) &= -a_{ij}(t_{ij}) x_{ij}(t_{ij}) \\
&\quad - \sum_{C_{kl} \in N_r(i,j)} C^{kl}_{ij} f\Big(\int_0^\infty K_{ij}(u) x_{kl}(t_{ij} - u) du\Big) x_{ij}(t_{ij}) + L_{ij}(t_{ij}) \\
&\leq -a_{ij}(t_{ij}) x_{ij}(t_{ij}) + L_{ij}(t_{ij}) \\
&< -a_{ij} N_{ij} + L_{ij} \leq 0.
\end{aligned}$$

Hence, $D^+ x_{ij}(t_{ij}) < 0$. This is a contradiction.

If $x_{ij}(t) < -N_{ij}$ then also there exists $t_{ij} > 0$ such that $x_{ij}(t_{ij}) < -N_{ij}$ and $D^+ x_{ij}(t_{ij}) \leq 0$. But

$$\begin{aligned}
\dot{x}_{ij}(t_{ij}) &= -a_{ij}(t_{ij}) x_{ij}(t_{ij}) \\
&\quad - \sum_{C_{kl} \in N_r(i,j)} C^{kl}_{ij} f\Big(\int_0^\infty K_{ij}(u) x_{kl}(t_{ij} - u) du\Big) x_{ij}(t_{ij}) + L_{ij}(t_{ij}) \\
&\geq -a_{ij}(t_{ij}) x_{ij}(t_{ij}) + L_{ij}(t_{ij}) \\
&> a_{ij} N_{ij} - L_{ij} \geq 0.
\end{aligned}$$

Then we also find that it is a contradiction. Finally, we obtain $|x_{ij}(t)| \leq N_{ij}$ for all $t \in \mathbb{R}$, which concludes the proof. □

In what follows, we consider the assumption

(H3) There exists $\mu > 0$ such that $|f(x) - f(y)| \le \mu|x - y|$ for all $x, y \in \mathbb{R}$. Also, there are constants $\xi_{ij} > 0$, $\beta > 2\widetilde{C}$ such that
$$-a_{ij}\xi_{ij} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} M_f \xi_{ij} + \sum_{C_{kl}\in N_r(i,j)} \widetilde{C} C_{ij}^{kl} \mu \xi_{kl} < 0, \tag{3.5}$$
for $i = 1, \dots, m$; $j = 1, \dots, n$, where
$$\widetilde{C} = \Big(\sum_{ij} L_{ij}^2 \frac{1}{a^2}\Big)^{1/2}, \quad M_f = \sup\big\{f(u) : \quad |u| \le \widetilde{C} + \beta\big\}.$$

From the condition (H3), there exist constants $\lambda > 0, T > 0$, such that
$$(\lambda - a_{ij})\xi_{ij} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} M_f^* \xi_{ij} + \sum_{C_{kl}\in N_r(i,j)} C^* C_{ij}^{kl} \mu P_{ij}(\lambda) \xi_{kl} < 0, \tag{3.6}$$
for $i = 1, \dots, m$; $j = 1, \dots, n$, where
$$C^* = \Big(\sum_{ij} L_{ij}^2 \frac{1}{a^2} + T\Big)^{1/2} < \beta, \quad M_f^* = \sup\{f(u) : \quad |u| \le C^* + \beta\}.$$
The next theorem deals with the uniqueness and locally exponential stability of periodic solution of (1.1).

Theorem 3.4. *Assume that hypotheses* (H1)–(H3) *hold. Then* (1.1) *has a unique ω-periodic solution $x^*(t, \varphi^*)$ in the region $B = \{\varphi \in BC : \|\varphi\| < \frac{\beta}{2}\}$, which is locally exponentially stable. Moreover the attractive domain of x^* is given as $D(\varphi^*) = \{\varphi \in BC : \quad \|\varphi - \varphi^*\| \le \beta\}$.*

Proof. By Theorem 3.1, there exists a ω-periodic solution of (1.1) $x^*(t) = x^*(t, \varphi^*)$ satisfies $\|x^*(t)\| < C^*, \quad t \in \mathbb{R}$. Let $x(t)$ is a arbitrary solution of (1.1) with initial function φ satisfies $\|\varphi - \varphi^*\| \le \beta$. It follows that $\|\varphi\| \le \|\varphi^*\| + \beta < C^* + \beta$ and from Lemma 3.3, we have
$$|x_{ij}(t)| \le N_{ij} < C^* + \beta, \quad \forall t \in \mathbb{R}.$$
Setting $z(t) = x(t) - x^*(t)$, the we have
$$\begin{aligned} \dot z_{ij}(t) = &- a_{ij}(t) z_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f\Big(\int_0^\infty K_{ij}(u) x_{kl}(t-u)du\Big) x_{ij}(t) \\ &+ \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f\Big(\int_0^\infty K_{ij}(u) x_{kl}^*(t-u)du\Big) x_{ij}^*(t). \end{aligned} \tag{3.7}$$
Consider the Lyapunov functionals
$$V_{ij}(t) = |z_{ij}(t)| e^{\lambda t}, \quad i = 1, \dots, m;\ j = 1, \dots, n,$$
where $\lambda > 0$ is determined from (3.6).

Putting $A = \frac{(1+\alpha)}{\xi_{\min}} \|\varphi - \varphi^*\|$, $\xi_{min} = \min_{i,j} \xi_{ij}$, $\alpha > 0$. We will show that
$$V_{ij}(t) \le \xi_{ij} A, \quad \text{for } i = 1, \dots, m;\ j = 1, \dots n;\ t \in \mathbb{R}^+.$$
Indeed, if this is not true, then there exists i, j and $t_{ij} > 0$ such that: $V_{ij}(t) \le \xi_{ij} A$ and $V_{kl}(t) \le \xi_{kl} A, (k, l) \ne (i, j)$, for all $t < t_{ij}$ and $V_{ij}(t_{ij}) = \xi_{ij} A$, $D^+ V_{ij}(t_{ij}) \ge 0$.

Taking Dini derivative of $V_{ij}(t)$ along trajectories of (1.1), we have

$$\begin{aligned}
D^+V_{ij}(t_{ij}) &\le e^{\lambda t_{ij}}(\lambda - a_{ij}(t_{ij}))|z_{ij}(t_{ij})| \\
&\quad + e^{\lambda t_{ij}} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} \Big| f\Big(\int_0^\infty K_{ij}(u)x_{kl}(t_{ij}-u)du\Big)x_{ij}(t_{ij}) \\
&\quad - f\Big(\int_0^\infty K_{ij}(u)x_{kl}^*(t_{ij}-u)du\Big)x_{ij}^*(t_{ij})\Big| \\
&\le e^{\lambda t_{ij}}(\lambda - a_{ij}(t_{ij}))|z_{ij}(t_{ij})| \\
&\quad + e^{\lambda t_{ij}} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} \Big| f\Big(\int_0^\infty K_{ij}(u)x_{kl}(t_{ij}-u)du\Big)x_{ij}^*(t_{ij}) \\
&\quad - f\Big(\int_0^\infty K_{ij}(u)x_{kl}^*(t_{ij}-u)du\Big)x_{ij}^*(t_{ij})\Big| \\
&\quad + e^{\lambda t_{ij}} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f\Big(\int_0^\infty K_{ij}(u)x_{kl}(t_{ij}-u)du\Big)|x_{ij}(t_{ij}) - x_{ij}^*(t_{ij})| \\
&\le e^{\lambda t_{ij}}\Big((\lambda - a_{ij}(t_{ij}))|z_{ij}(t_{ij})| + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} M_f^* |z_{ij}(t_{ij})| \\
&\quad + \sum_{C_{kl}\in N_r(i,j)} C^* C_{ij}^{kl} \mu \int_0^\infty K_{ij}(u)|z_{kl}(t_{ij}-u)|du\Big) \\
&= (\lambda - a_{ij}(t_{ij}))V_{ij}(t_{ij}) + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} M_f^* V_{ij}(t_{ij}) \\
&\quad + \sum_{C_{kl}\in N_r(i,j)} C^* C_{ij}^{kl} \mu \int_0^\infty K_{ij}(u)V_{kl}(t_{ij}-u)e^{\lambda u}du \\
&\le \Big((\lambda - a_{ij}(t_{ij}))\xi_{ij} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} M_f^* \xi_{ij} \\
&\quad + \sum_{C_{kl}\in N_r(i,j)} C^* C_{ij}^{kl} \mu P_{ij}(\lambda)\xi_{kl}\Big)A.
\end{aligned}$$

From (3.6) we have $D^+V_{ij}(t_{ij}) < 0$. This is a contradiction. Hence we have, $V_{ij}(t) \le \xi_{ij}A$ for all $i = 1, \dots, m$; $j = 1, \dots, n$, $t \in \mathbb{R}^+$ and therefore

$$|z_{ij}(t)| \le \frac{(1+\alpha)\xi_{ij}}{\xi_{\min}} e^{-\lambda t}\|\varphi - \varphi^*\|, \quad \forall t \ge 0.$$

This inequality shows that the periodic solution of (1.1) is exponentially stable. The proof is completed. □

For the global exponential stability, in [4] the authors consider (SICNNs) with bounded activation function with constant time delay. When the activation function f is assume to be bounded, condition (H3) is replaced by

(H3') There exists $\mu > 0$ such that $|f(x) - f(y)| \le \mu|x - y|$ for all $x, y \in \mathbb{R}$ and also there are constants $\xi_{ij} > 0$ such that

$$-a_{ij}\xi_{ij} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} M_f \xi_{ij} + \sum_{C_{kl}\in N_r(i,j)} \widetilde{C} C_{ij}^{kl} \mu \xi_{kl} < 0, \tag{3.8}$$

for $i = 1, \dots, m$; $j = 1, \dots, n$, where

$$\widetilde{C} = \Big(\sum_{ij} L_{ij}^2 \frac{1}{a^2}\Big)^{1/2}, \quad M_f = \sup\{f(u) : u \in \mathbb{R}\}.$$

In this case, we have the following results.

Corollary 3.5. *Assume that the hypotheses* (H1), (H2) *and* (H3') *hold. Then* (1.1) *has a unique ω-periodic solution $x^*(t, \varphi^*)$ which is globally exponentially stable.*

4. Example

In this section, we give an numerical example to illustrate our obtained results. Consider (SICNNs) described by the system

$$\begin{aligned} \dot{x}_{ij}(t) = -a_{ij}(t)x_{ij}(t) - &\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f\left(\int_0^\infty K_{ij}(u)x_{kl}(t-u)du\right) x_{ij}(t) \\ &+ L_{ij}(t), \quad i, j = 1, 2, 3, \end{aligned} \tag{4.1}$$

where

$$\begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{pmatrix} = \begin{pmatrix} 2+\cos^2 t & 3+|\cos t| & 2+|\sin t| \\ 2+0.5\sin^2 t & 2+|\sin t\cos t| & 3+|\cos 2t| \\ 3+|\sin t| & 2+|\sin 2t| & 2+\cos^2 2t \end{pmatrix},$$

and

$$\begin{pmatrix} L_{11}(t) & L_{12}(t) & L_{13}(t) \\ L_{21}(t) & L_{22}(t) & L_{23}(t) \\ L_{31}(t) & L_{32}(t) & L_{33}(t) \end{pmatrix} = \begin{pmatrix} 0.6\sin t & 0.1\cos 3t & 0.4\cos t \\ 0.3\sin 2t & 0.2\sin 3t & 0.3\cos 3t \\ 0.5\cos 2t & 0.5|\sin t| & 0.2|\cos t| \end{pmatrix}$$

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} 0.2 & 0.1 & 0.1 \\ 0.15 & 0.3 & 0 \\ 0.1 & 0.25 & 0.1 \end{pmatrix},$$

Taking $r = 1, K_{ij}(u) = e^{-u}$ and $f(x) = \frac{1}{6}|x-1|$, we have

$$\mu = \frac{1}{6}; \quad \widetilde{C} = \Big(\sum_{ij} \frac{L_{ij}^2}{a^2}\Big)^{1/2} < 1$$

$$\sum_{C_{kl}\in N_1(1,1)} C_{11}^{kl} = 0.75, \quad \sum_{C_{kl}\in N_1(1,2)} C_{12}^{kl} = 0.85, \quad \sum_{C_{kl}\in N_1(1,3)} C_{13}^{kl} = 0.5;$$
$$\sum_{C_{kl}\in N_1(2,1)} C_{21}^{kl} = 1.1, \quad \sum_{C_{kl}\in N_1(2,2)} C_{22}^{kl} = 1.3, \quad \sum_{C_{kl}\in N_1(2,3)} C_{23}^{kl} = 0.85;$$
$$\sum_{C_{kl}\in N_1(3,1)} C_{31}^{kl} = 0.8, \quad \sum_{C_{kl}\in N_1(3,2)} C_{32}^{kl} = 0.9, \quad \sum_{C_{kl}\in N_1(3,3)} C_{33}^{kl} = 0.65.$$

We choose $\beta = 6$, $\xi_{ij} = 1$ then $M_f = 4/3$ and we have

$$\begin{aligned} &-a_{ij}\xi_{ij} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} M_f \xi_{ij} + \sum_{C_{kl}\in N_r(i,j)} \widetilde{C} C_{ij}^{kl} \mu \xi_{kl} \\ &< -2 + \frac{4}{3} \times 1.3 + \frac{1}{6} \times 1.3 < 0, \quad i, j = 1, 2, 3. \end{aligned}$$

According to Theorem 3.4, System (1.1) has a unique periodic solution $x^*(t, \varphi^*)$ in the region $B = \{\varphi : \|\varphi\| < 3\}$ which is locally exponentially stable with the attractive domain $D(\varphi^*) = \{\varphi \in BC : \|\varphi - \varphi^*\| \le 6\}$.

Note that, the activation function $f(x)$ is not bounded and $f(0) \neq 0$, so the results in [17, 4] are not applicable for this example.

Conclusions. This paper addressed the existence and local exponential stability of periodic solutions of (SICNNs) with continuously distributed delays. By using coincidence degree theory, we give new sufficient conditions for the existence and locally exponential stability of periodic solutions of (SICNNs) without assumption of boundedness on the activation function. The results are new and complement previously known results.

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