Electronic Journal of Differential Equations, Vol. 2008(2008), No. 101, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

INFINITELY MANY SOLUTIONS FOR THE p-LAPLACE EQUATIONS WITH NONSYMMETRIC PERTURBATIONS

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Abstract. In this article, we study Dirichlet problems involving the p-Laplacian with a nonsymmetric term. By using the large Morse index of the corresponding Laplace equation, we establish an estimate on the growth of the min-max values for a functional associated with the problem. The estimate is better than the given result in some range. We show that the problem possesses infinitely many weak solutions.

1. Introduction and statement of main results

In this paper, we investigate the existence of infinitely many weak solutions for the Dirichlet problem, involving p-Laplacian,

$$\begin{gathered} -\Delta_p u = |u|^{q-2}u + f(x), \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $N > p > 1$; $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is so-called p-Laplace operator and $p < q < p^* = Np/(N-p)$.

Rabinowitz [6], Bahri & Berestycki [2] and Struwe [7] discussed this type problem in the special case of $p = 2$ using perturbation method for even symmetric functional. Bahri & Lions [3] and Tanaka [8] employed the Morse index theory and obtained the best result up to now. Garcia Azorero & Peral Alonso [1] generalized the multiple solutions results as in [2, 6] into p-Laplacian equation. In the case of p-Laplacian with $p \neq 2$, not as in Hilbert space $H_0^1(\Omega)$, because of the lack of Hilbert space framework, in the Banach space $W_0^{1,p}(\Omega)$, Morse index theory cannot directly be applied into this class of equation, yet corresponding multiple solutions problem had not reached the optimal result as in the situation of Laplace equation. In this note, we compare the variational functional of problem (1.1) involving p-Lapacian ($p > 2$) with that involving Laplacian, establish some estimate on growth of critical values of p-Laplace equation with the large Morse index of Laplace equation and show existence of infinitely many solutions of p-Laplace problem with nonsymmetric perturbation (1.1). On the some range, the result we obtain improves the known conclusion of Garcia Azorero and Peral Alonso [1].

2000 *Mathematics Subject Classification.* 35J70, 35D50.
Key words and phrases. p-Laplacian; large Morse index; nonsymmetric perturbation; infinitely many solutions.

Submitted February 28, 2008. Published July 30, 2008.
Supported by grant 7005795 from Guangdong Provincial Natural Science Foundation of China.

The main result in this paper is the following theorem.

Theorem 1.1. *Suppose $f(x) \in C(\bar{\Omega})$ and*

$$2 < p < q < \max\big\{\bar{q},\, \min\{\bar{\bar{q}}, \frac{Np-2N+4}{Np-2N+2p}\frac{Np}{N-2}\}\big\},$$

where $\bar{q}$ and $\bar{\bar{q}}$ are the largest roots of the equations

$$\frac{q}{q-1} = \frac{Np-q(N-p)}{(q-p)N} \quad \text{and} \quad \frac{q}{q-1} = \frac{p}{N}\frac{q(Np-2N+2p)-pN(p-2)}{(q-p)(Np-2N+2p)}, \tag{1.2}$$

respectively. Then the problem (1.1) *possesses infinitely many (weak) solutions $\{u_k\} \subset W_0^{1,p}(\Omega)$, and the corresponding critical values of variational functional tend to the positive infinity.*

The weak solutions of (1.1) are the critical points of the C^1 functional for $\theta = 1$ as follows:

$$J_\theta(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{q}\int_\Omega |u|^q\, dx - \theta\int_\Omega f(x)u dx, \quad (u,\theta) \in W_0^{1,p}(\Omega)\times[0,1]. \tag{1.3}$$

The equivalent norm in $W_0^{1,p}(\Omega)$ is $\|\nabla u\|_p$. If $\theta = 0$, $J_0(u)$ is an even functional on $W_0^{1,p}(\Omega)$. It is clear that, if $p > 2$, $\frac{Np-2N+4}{Np-2N+2p}\frac{Np}{N-2} < p^*$. Therefore, under the assumptions of Theorem 1.1, the functional $J_\theta(u)$ satisfies Palais-Smale condition in $W_0^{1,p}(\Omega)$ (see, for instance, [7]).

2. Preliminaries

To seek a series of critical values of $J_1(u)$, we introduce the following facts: In the Sobolev space $W_0^{1,p}(\Omega)$, we know that (see, for example, Triebel [9]) there is Schauder basis. And in the Sobolev space $W_0^{1,p}(\Omega)$, the Sobolev inequality is valid.

Lemma 2.1 (Triebel [9])**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of cone-type. There exists a Schauder basis $\{w_k\}_{k=1}^\infty$ in $W^{1,p}(\Omega)$, such that for $q \in [p, p^*)$, there is some positive constant $C_1 > 0$, it holds for $u \in \overline{\operatorname{span}}\{w_k, w_{k+1}, \dots\}$*

$$C_1 k^{\frac{1}{N}+\frac{1}{q}-\frac{1}{p}}\|u\|_{L_q(\Omega)} \le \|u\|_{W^{1,p}(\Omega)}. \tag{2.1}$$

Denote the Schauder basis in $W_0^{1,p}(\Omega)$ satisfied the above lemma by $\{w_k\}$. The basis $\{w_k\}$ is also Schauder basis in $H_0^1(\Omega)$. Denote by

$$E_k = \operatorname{span}\{w_1, w_2, \dots, w_k\}, \quad E_k^\perp = \overline{\operatorname{span}}\{w_k, w_{k+1}, \dots\}.$$

Define family of maps and series of min-max values of $J_0(u)$ as follows:

$$\Gamma_k(p) = \{\gamma \in C(E_k \cap \bar{B}_{R_k}(0), W_0^{1,p}(\Omega)); \gamma \text{ is odd, and } \gamma|_{E_k\cap\partial B_{R_k}(0)} = \mathrm{id}\};$$
$$c_k = \inf_{\gamma\in\Gamma_k(p)}\max_{u\in E_k\cap B_{R_k}(0)} J_0(\gamma(u)),$$

where R_k is a series of positive constants tend to positive infinite, such that $J_0(u) < 0$ for all $u \in E_k$ and $\|u\| \ge R_k$.

Lemma 2.2. *For every k large enough, there exists $\rho_k > 0$ such that*

$$J_\theta(u) \ge C_2 k^{\frac{pN-q(N-p)}{N(q-p)}}, \quad \text{when } u \in E_k^\perp \cap \partial B_{\rho_k}(0),$$

where C_2 is independent of k.

Proof. For $u \in E_k^\perp$, applying Lemma 2.1, we estimate the functional $J_\theta(u)$:

$$\begin{aligned} J_\theta(u) &= \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\|u\|_q^q - \theta \int_\Omega f(x) u dx \\ &\geq \frac{1}{p}\|\nabla u\|_p^p - Ck^{\frac{q}{p^*}-1}\|\nabla u\|_p^q - C', \end{aligned}$$

where C' is a constant depending only on f.

By letting $\rho_k = (k^{1-\frac{q}{p}+\frac{q}{N}}/(2qC))^{1/(q-p)}$, one have, for all k large enough and $u \in E_k^\perp \cap \partial B_{\rho_k}(0)$,

$$J_\theta(u) \geq \big[\frac{1}{2p}(2qC)^{-\frac{p}{q-p}} - C(2qC)^{\frac{-q}{q-p}}\big]k^{(\frac{q}{p^*}-1)\frac{p}{q-p}} - C' \geq C_2 k^{\frac{qp}{N(q-p)}-1},$$

where C_2 is a positive constant independent of k. □

From the above lemma, using the Borsuk theorem, we obtain

$$c_k \geq Ck^{\frac{pN-q(N-p)}{N(q-p)}}. \tag{2.2}$$

This estimate on c_k in the case $p = 2$ was obtained by Rabinowitz with the growth of eigenvalue of Laplacian due to Hilbert-Courant; Garcia Azorero and Peral Alonso [1] got the similar result in the special case $\Omega = [0, 1]^N$ for $p \neq 2$.

3. Proof of the main theorem

As a matter of convenience, we write the functional $J_0(u)$ as $I_{p,q}(u)$; that is,

$$I_{p,q}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{C}{q}\int_\Omega |u|^q dx,$$

where C is a positive constant depends only on p, q and $|\Omega|$.

Lemma 3.1. *Suppose $q < p^* = pN/(N-p)$. There exits a positive constant C_3 dependent only on p and q, such that*

$$I_{p,q}(u) \geq C_3(I_{2,r}(u))^{p/2}, \quad u \in W^{1,p}(\Omega), \tag{3.1}$$

where $r = (q(Np - 2N + 2p) - pN(p-2))/p^2$; moreover if

$$q < \frac{Np - 2N + 4}{Np - 2N + 2p}\frac{Np}{N-2}, \tag{3.2}$$

then $r < 2^*$.

Proof. By the assumption $q < p^* = pN/(N-p)$ and the interpolation inequality and Sobolev inequality, one can estimate the second term in $I_{p,q}(u)$ as follows:

$$\begin{aligned} \int_\Omega |u|^q dx &\leq \Big(\int_\Omega |u|^{p^*}dx\Big)^{q\frac{(1-\alpha)}{p^*}}\Big(\int_\Omega |u|^r dx\Big)^{q\frac{\alpha}{r}} \\ &\leq \Big(S\int_\Omega |\nabla u|^p dx\Big)^{q\frac{1-\alpha}{p}}\Big(\int_\Omega |u|^r dx\Big)^{q\frac{\alpha}{r}} \\ &\leq \frac{q}{2p}\int_\Omega |\nabla u|^p dx + C\Big(\int_\Omega |u|^r dx\Big)^{q\frac{\alpha}{r}\frac{p}{p-(1-\alpha)q}}, \end{aligned} \tag{3.3}$$

where the parameters r and α satisfy: $q > r > 1$, $0 < \alpha < 1$, $p > q(1-\alpha)$ and

$$q\frac{1-\alpha}{p^*} + q\frac{\alpha}{r} = 1. \tag{3.4}$$

When $u \in W_0^{1,p}(\Omega)$ and $J_0(u) = I_{p,q}(u) \gg 0$, we also have

$$\begin{aligned} I_{p,q}(u) &\geq \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{q}\Big[\frac{q}{2p}\int_\Omega |\nabla u|^p dx + C\Big(\int_\Omega |u|^r dx\Big)^{q\frac{\alpha}{r}\frac{p}{p-(1-\alpha)q}}\Big] \\ &\geq \frac{1}{2p}\int_\Omega |\nabla u|^p dx - \frac{C}{q}\Big(\int_\Omega |u|^r dx\Big)^{q\frac{\alpha}{r}\frac{p}{p-(1-\alpha)q}} \\ &\geq \Big[\Big(\frac{1}{2p}\int_\Omega |\nabla u|^p dx\Big)^{2/p} - \Big(\frac{C}{q}\int_\Omega |u|^r dx\Big)^{q\frac{\alpha}{r}\frac{p}{p-(1-\alpha)q}\frac{2}{p}}\Big]^{p/2} \\ &\geq \Big[C(|\Omega|)\int_\Omega |\nabla u|^2 dx - \Big(\frac{C}{q}\int_\Omega |u|^r dx\Big)^{q\frac{\alpha}{r}\frac{2}{p-(1-\alpha)q}}\Big]^{p/2}. \end{aligned} \tag{3.5}$$

In addition to (3.4), we require that r and α satisfy

$$q\frac{\alpha}{r}\frac{2}{p-(1-\alpha)q} = 1. \tag{3.6}$$

Solving the simultaneous equations (3.4) and (3.6), we obtain

$$\alpha = 1 - \frac{1}{q}\frac{\frac{p}{2}-1}{\frac{1}{2}-\frac{1}{p^*}}, \quad r = \frac{q(\frac{1}{2}-\frac{1}{p^*}) - (\frac{p}{2}-1)}{\frac{p}{2}\Big(\frac{1}{p}-\frac{1}{p^*}\Big)} = \frac{q(Np-2N+2p)-pN(p-2)}{p^2}.$$

Therefore, the inequality (3.5) becomes

$$\begin{aligned} I_{p,q}(u) &\geq \Big[C(|\Omega|)\int_\Omega |\nabla u|^2 dx - C\int_\Omega |u|^r dx\Big]^{p/2} \\ &\geq C_3\Big[\frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{C}{r}\int_\Omega |u|^r dx\Big]^{p/2} = C_3 I_{2,r}(u)^{p/2}, \end{aligned} \tag{3.7}$$

where

$$I_{2,r}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{C}{r}\int_\Omega |u|^r dx,$$

and C_3 and C are positive constants, without loss of generality, we suppose that $C \geq 1$ and $C_3 \leq 1$.

As regards (3.2), which is a simple fact, we skip over the detail. □

Next, we present an estimate on c_k under the condition $p \geq 2$ which is superior to (2.2) in a certain extent.

Lemma 3.2. *Suppose that $2 < p < q < \frac{Np-2N+4}{Np-2N+2p}\frac{Np}{N-2}$. There exists positive constant C_4 independent of k, such that for all k large enough,*

$$c_k \geq C_4 k^{\frac{p}{N}\frac{q(Np-2N+2p)-pN(p-2)}{(q-p)(Np-2N+2p)}}.$$

Proof. We define a series of min-max values of the functional $I_{2,r}(u)$ in the previous lemma as follows:

$$\tilde{c}_k = \inf_{\gamma\in\Gamma_k(p)} \max_{u\in E_k\cap B_{R_k}(0)} I_{2,r}(\gamma(u)).$$

Obviously, (3.7) indicates that if $\tilde{c}_k$ is sufficiently large that for every $\gamma \in \Gamma_k(p)$, and w_k satisfying

$$I_{2,r}(w_k) = \max_{u\in E_k\cap B_{R_k}(0)} I_{2,r}(\gamma(u)),$$

there holds $I_{2,r}(w_k) \gg 0$. Therefore, the inequality (3.7) leads to

$$c_k \geq C_3\tilde{c}_k^{p/2}. \tag{3.8}$$

On the other hand, for every γ in $\Gamma_k(p)$, $\gamma(E_k \cap \bar{B}_{R_k}(0)) \subset W_0^{1,p}(\Omega) \subset H_0^1(\Omega)$, which implies $\Gamma_k(p) \subset \Gamma_k(2)$, hence we have

$$\tilde{c}_k \geq b_k = \inf_{\gamma \in \Gamma_k(2)} \max_{u \in E_k \cap B_{R_k}(0)} I_{2,r}(\gamma(u)). \tag{3.9}$$

Moreover, from (3.2) it follows that $q < \frac{Np-2N+4}{Np-2N+2p} \frac{Np}{N-2}$ implies $r < 2^*$. Therefore, the functional $I_{2,r}(u)$ fulfill the Palais-Smale condition in $H_0^1(\Omega)$. By the results due to Tanaka [8] or Bahri-Lions [3], we know that b_k are the critical values of $I_{2,r}(u)$. And the large Morse index of $I_{2,r}(u)$ at the critical point u corresponding to b_k implies that

$$b_k \geq Ck^{\frac{2}{N}\frac{r}{r-2}} = Ck^{\frac{2}{N}\frac{q(Np-2N+2p)-pN(p-2)}{q(Np-2N+2p)-pN(p-2)-2p^2}}, \tag{3.10}$$

where C is a positive constant independent of k. Thus $\tilde{c}_k \to +\infty$, that is, (3.8) holds for all k large enough. Again by (3.7),

$$c_k \geq C_3 b_k^{p/2} \geq C_4 k^{\frac{2}{N}\frac{q(Np-2N+2p)-pN(p-2)}{q(Np-2N+2p)-pN(p-2)-2p^2}\frac{p}{2}}, \tag{3.11}$$

the conclusion of the lemma follows. □

By Hölder inequality, if $u \in W_0^{1,p}(\Omega)$ is a critical point of $J_\theta(u)$, we obtain

$$\big|\frac{\partial}{\partial\theta} J_\theta(u)\big| = \big|\int_\Omega f(x)u dx\big| \leq C_5(|J_\theta(u)|+1)^{1/q}, \tag{3.12}$$

where C_5 is a positive constant depending only on $\|f\|_{C(\bar{\Omega})}$.

The method in Hilbert spaces developed by Bolle [4] can be generalized into Banach spaces (just replace the gradient vector fields with the pseudo-gradient vector fields), which can be used to deduce the following fact, if the functional $J_1(u)$ has at most finite critical values, by using (3.12) and the proof of [5, Theorem 2.2], we get

$$c_{k+1} - c_k \leq C(c_{k+1}^{1/q} + 1). \tag{3.13}$$

With the facts above, we can establish the following estimates on c_k.

Lemma 3.3. *If there are at most finite critical values for the functional $J_1(u)$, there exits a positive constant C_6 independent of k, such that*

$$c_k \leq C_6 k^{\frac{q}{q-1}}.$$

Proof. Without loss of generality, one can suppose $c_k \geq 1$, then from (3.13) it follows that

$$\begin{gathered} c_{k+1} - c_k \leq 2C \cdot c_{k+1}^{1/q}; \\ c_k - c_{k-1} \leq 2C \cdot c_k^{1/q} \leq 2C \cdot c_{k+1}^{1/q}; \\ \dots \\ c_2 - c_1 \leq 2C \cdot c_2^{1/q} \leq 2C \cdot c_{k+1}^{1/q}. \end{gathered}$$

Adding the two sides, respectively, we have $c_{k+1} - c_1 \leq 2k \cdot C \cdot c_{k+1}^{1/q}$, which implies

$$c_{k+1} \leq 2k \cdot C \cdot c_{k+1}^{1/q} + c_1 < 2(k+1) \cdot C \cdot c_{k+1}^{1/q},$$

where the constant C may be changed. And then the conclusion follows. □

Proof of the main theorem. Combining the above lemmas and the method developed by Bolle-Ghoussoub-Tehrani [5], we can deduce the existence of infinitely many solutions of (1.1). In fact, if $J_1(u)$ has at most finite critical values, the conclusions of Lemma 2.2, Lemma 3.1 and Lemma 3.2 imply that, for all sufficiently large k,

$$\max\{C_4 k^{\frac{p}{N}\frac{q(Np-2N+2p)-pN(p-2)}{(q-p)(Np-2N+2p)}}, C_3 k^{\frac{p}{N}\frac{2q-N(p-2)}{2(q-p)}}\} \le c_k \le C_6 k^{\frac{q}{q-1}}.$$

However, according to the assumptions of Theorem 1.1, we have

$$\frac{q}{q-1} < \max\Big\{\frac{p}{N}\frac{2q-N(p-2)}{2(q-p)},\ \frac{p}{N}\frac{q(Np-2N+2p)-pN(p-2)}{(q-p)(Np-2N+2p)}\Big\},$$

which yields contradiction. □

Remark. The conclusion of infinitely many solutions based on (3.1) is better than that of Garcia Azorero and Peral Alonso [1] in some range. In fact, according to the conclusions in [1], we know that, when $q \in (p, \bar{q})$, the problem (1.1) has infinitely many solutions, where $\bar{q} = \bar{q}(p)$ is the largest root of the first equation in (1.2) as follows:

$$\frac{q}{q-1} = \frac{pN - q(N-p)}{N(q-p)}, \tag{3.14}$$

the expression on the right hand of the equation (3.14) is also the exponent in Lemma 2.1. Notice that if we denote the largest root of the second equation in (1.2) by $\bar{\bar{q}} = \bar{\bar{q}}(p)$, it is clear that

$$\bar{\bar{q}}(2) = \frac{2(N-1)}{N-2} > \bar{q}(2),$$

so there exits some $p_0 \in (2, +\infty]$, such that $\bar{\bar{q}}(p) > \bar{q}(p)$ for all $p \in [2, p_0)$. Setting $\bar{q}(p) = \bar{\bar{q}}(p)$, we can find out p_0. Since $\bar{q}(p)$ and $\bar{\bar{q}}(p)$ satisfy the equations in (1.2), we have

$$\frac{p}{N}\frac{q(Np-2N+2p)-pN(p-2)}{(q-p)(Np-2N+2p)} = \frac{pN - q(N-p)}{N(q-p)},$$

that is, $q = (Np - 2N + p^2)p/(Np - 2N + 2p)$. From the equation $\bar{q}(p)$ or $\bar{\bar{q}}(p)$ satisfies, it follows that p_0 meets a quartic equation as follows:

$$p^4 - (2 + N + N^2)p^2 + (2N + 4N^2)p - 4N^2 = 0. \tag{3.15}$$

Analysis on this quartic equation with Mathematica yields some interesting facts: when $N = 3, 4, 5, 6$, the equation has no real root greater than 2, that is, in those cases, our result under the hypothesis in Theorem 1.1 is better than that in [1]; when $N \ge 7$, the equation (3.15) has two real roots greater than 2, the first one is p_0, which is in the interval $(2, 3)$. Therefore, we can conclude that, under the conditions of Theorem 1.1, if $p \in (2, p_0)$ and $q \in (p, \bar{\bar{q}}(p))$, the problem (1.1) possesses infinitely many solutions. The conclusion is better than that in [1], since $(p, \bar{q}(p)) \subsetneq (p, \bar{\bar{q}}(p))$.

Figure 1 illustrates the relationship among $\bar{q}$, $\bar{\bar{q}}$, p^* and $\bar{r}(p) = \frac{Np-2N+4}{Np-2N+2p}\frac{Np}{N-2}$ in the cases $N = 6$ and $N = 8$. In each figure, two dashed curves are $q = p^*(p)$ and $q = \bar{r}(p)$; the two solid curves represent $q = \bar{q}(p)$ and $q = \bar{\bar{q}}(p)$. Notice that the curve $q = \bar{\bar{q}}(p)$ is always over $q = \bar{q}(p)$ for all $p \ge 2$ when $N = 6$, and the curve $q = \bar{\bar{q}}(p)$ is over $q = \bar{q}(p)$ near $p = 2$ when $N = 8$. The two figures were produced with Mathematica.

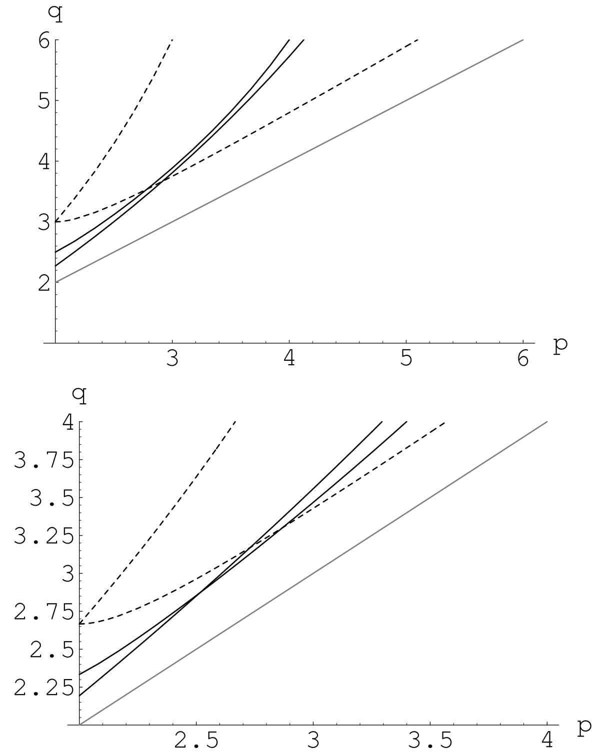

FIGURE 1. Graphs of $\bar{q}$, $\bar{\bar{q}}$, p^* for $N = 6$ and $N = 8$

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