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ON NON-ABSOLUTE FUNCTIONAL VOLTERRA INTEGRAL EQUATIONS AND IMPULSIVE DIFFERENTIAL EQUATIONS IN ORDERED BANACH SPACES

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Abstract. In this article we derive existence and comparison results for discontinuous non-absolute functional integral equations of Volterra type in an ordered Banach space which has a regular order cone. The obtained results are then applied to first-order impulsive differential equations.

1. Introduction

In [6] a theory for HL integrable functions with values in ordered Banach spaces was developed, and applied to Fredholm integral equations and concrete boundary value problems of second order ordinary differential equations. In this paper we apply that theory and a fixed point result in abstract spaces to prove existence and comparison results for non-absolute functional Volterra integral equations in an ordered Banach space E, and give applications to first-order impulsive initial value problems involving discontinuities and functional dependencies.

The main features of this paper are:

– The E-valued functions in considered equations are discontinuous and depend functionally on the unknown function. Thus integro-differential equations are included.

– Integrals in integral equations are non-absolute integrals, and differential equations of impulsive problems may be singular.

– Impulses are allowed to occur in well-ordered sets, in particular, in finite sets or in increasing sequences.

The main tools are:

– Fixed point results in partially ordered sets, proved in [7] by generalized iteration methods.

– Dominated and monotone convergence theorems for HL integrable mappings and results on the existence of supremum and infimum of chains of locally HL integrable mappings from a real interval to E, proved in [6].

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2. Preliminaries

In this section we study properties of HL integrable, a.e. differentiable and locally HL integrable functions from a real interval to a Banach space E.

A *K-partition of* a compact real interval I is formed by a finite collection of closed subintervals $[t_{i-1}, t_i]$ of I whose union is I, and tags $\xi_i \in [t_{i-1}, t_i]$. A function $u : I \to E$ is *HL integrable* if there is a function $F : I \to E$, called a *primitive of* u, which has the following property: If $\epsilon > 0$, there is such a function $\delta : I \to (0, \infty)$ that

$$\sum_i \big\| u(\xi_i)(t_i - t_{i-1}) - (F(t_i) - F(t_{i-1}))\big\| < \epsilon$$

for every K-partition $\{(\xi_i, [t_{i-1}, t_i])\}$ of I with $[t_{i-1}, t_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all i. If u is HL integrable on I, it is HL integrable on every closed subinterval $J = [a, b]$ of I, and $F(b) - F(a)$ is the *Henstock-Kurzweil* integral of u over J, i.e.,

$$F(b) - F(a) = {}^K\!\!\int_J u(s)\, ds = {}^K\!\!\int_a^b u(s)\, ds. \tag{2.1}$$

The proofs for the results of the next Lemma can be found, e.g. in [10].

Lemma 2.1. (a) *The Henstock-Kurzweil integrals of a.e. equal HL integrable functions are equal.*
(b) *Every HL integrable function is strongly measurable.*
(c) *A Bochner integrable function $u : I \to E$ is HL integrable, and $\int_J u(s)\, ds = {}^K\!\!\int_J u(s)\, ds$ whenever J is a closed subinterval of I.*

The set $H(I, E)$ of all HL integrable functions $u : I \to E$ is a vector space with respect to the usual addition and scalar multiplication of functions. Identifying a.e. equal functions it follows that the space $L^1(I, E)$ of all Bochner integrable functions $u : I \to E$ is a subset of $H(I, E)$.

A function $u : I \to E$ is called *absolutely continuous (AC)* on I if for each $\epsilon > 0$ there corresponds such a $\delta > 0$, that for any sequence $[a_j, b_j]$, $j = 1, \dots, n$ of disjoint subintervals of I with $\sum_{j=1}^n (b_j - a_j) < \delta$ we have $\sum_{j=1}^n \|u(b_j) - u(a_j)\| < \epsilon$.

We say that a function $u : I \to E$ is *generalized absolutely continuous in the restricted sense (ACG*)* on I if I can be expressed as such a countable union of its subsets B_n, $n \in \mathbb{N}$, that for all $\epsilon > 0$ and $n \in \mathbb{N}$ there exists such a $\delta_n > 0$ that

$$\sum_i \sup\{\|u(d) - u(c)\| : [c, d] \subseteq [c_i, d_i]\} < \epsilon$$

whenever $\{[c_i, d_i]\}$ is a finite sequence of non-overlapping intervals which have end-points in B_n and satisfy $\sum_i (d_i - c_i) < \delta_n$. If u is AC on I, it is continuous and ACG^* on I.

A function $v : I \to E$ is said to be *a.e. (strongly) differentiable*, if the strong derivative $v'(t) = \lim_{h\to 0} \frac{v(t+h) - v(t)}{h}$ exists for a.e. $t \in I$.

As for the proof of the following result, see, e.g., [10, subsection 7.4.1].

Theorem 2.2. *Given $u, v : I \to E$ and $(t_0, x_0) \in I \times E$, then the following conditions are equivalent.*

(a) *u is continuous and ACG^* on I, $u'(t) = v(t)$ for a.e. $t \in I$ and $u(t_0) = x_0$.*
(b) *v is HL-integrable and $u(t) = x_0 + {}^K\!\!\int_{t_0}^t v(s) ds$ for all $t \in I$.*

If $u : I \to E$ is a.e. differentiable, define $u'(t) = 0$ at those points $t \in I$ where u is not differentiable.

The next result is a consequence of Theorem 2.2.

Corollary 2.3. *If $u : I \to E$ is a.e. differentiable, then u is continuous and ACG^* on I if and only if u' is HL-integrable, and*

$$u(t) - u(t_0) = {}^K\!\!\int_{t_0}^{t} u'(s)ds \quad \text{for all } t_0,\, t \in I.$$

The following result is needed in section 4.

Lemma 2.4. *If $u : I \to \mathbb{R}$ is absolutely continuous, and $v : I \to E$ is continuous, ACG^* and a.e. differentiable, then*

$$u(t)v(t) - u(t_0)v(t_0) = {}^K\!\!\int_{t_0}^{t} (u(s)v'(s) + u'(s)v(s))ds \quad \text{for all } t_0,\, t \in I.$$

Proof. Let $t,\, t + h \in I$, $h \neq 0$ be given. Since u and v are continuous on a compact interval I, they are also bounded, whence

$$u(t + h)v(t + h) - u(t)v(t) = (u(t + h) - u(t))v(t + h) + u(t)(v(t + h) - v(t)).$$

implies when $M = \max\{\|v(t)\| : t \in I\}$ and $m = \max\{|u(t)| : t \in I\}$ that

$$\|u(t + h)v(t + h) - u(t)v(t)\| \leq M\,|u(t + h) - u(t)| + m\,\|v(t + h) - v(t)\|.$$

Because u is an absolutely continuous real-valued function, it is ACG^* on I. It then follows from the above inequality that $u \cdot v$ is continuous and ACG^* on I. Moreover, u and v are a.e. differentiable, whence $u \cdot v$ is a.e. differentiable and

$$(u \cdot v)'(t) = u(t)v'(t) + u'(t)v(t) \quad \text{for a.e. } t \in I.$$

The assertion follows then from Corollary 2.3. □

The following result is adapted from [8].

Proposition 2.5. *If $v : I \to E$ is HL-integrable and $u : I \to \mathbb{R}$ is of bounded variation, then $u \cdot v$ is HL-integrable.*

Given an interval J of $\mathbb{R}$, not necessarily closed or bounded, denote by $H_{loc}(J, E)$ the space of all strongly measurable functions $u : J \to E$ which are HL integrable on each compact subinterval of J. We assume that $H_{\rm loc}(J, E)$ is ordered a.e. pointwise; i.e.,

$$u \leq v \quad \text{if and only if } u(s) \leq v(s) \text{ for a.e. } s \in J. \tag{2.2}$$

The results of the next Lemma follow from [6, Proposition 2.1 and Lemma 2.5].

Lemma 2.6. *Given an ordered Banach space, let $u,\, v : J \to E$ be strongly measurable, $u_\pm \in H_{\rm loc}(J, E)$, and assume that $u_-(s) \leq u(s) \leq v(s) \leq u_+(s)$ for a.e. $s \in J$. Then $u \in H_{\rm loc}(J, E)$. Moreover,*

$$ {}^K\!\!\int_{a}^{t} u(s)\,ds \leq {}^K\!\!\int_{a}^{t} v(s)\,ds \quad \text{for all } a,\, t \in J,\ a \leq t.$$

Next we present Dominated and Monotone Convergence Theorems for locally HL-integrable functions, which are needed in applications.

Theorem 2.7. *Given a real interval J and a Banach space E ordered by a normal order cone, let $(u_n)_{n=1}^\infty$ be a sequence of strongly measurable functions from J to E, let $u_\pm \in H_{\rm loc}(J,E)$, and assume that $u_- \le u_n \le u_+$ for each $n = 1,2,\dots$, and that $u_n(s) \to u(s)$ for a.e. $s \in J$. Then u, $u_n \in H_{\rm loc}(J,E)$, $n = 1,2,\dots$, and ${}^K\!\!\int_a^t u_n(s)ds \to {}^K\!\!\int_a^t u(s)ds$ for all a, $t \in J$, $a < t$.*

Proof. The given hypotheses imply by Lemma 2.6 that $u_n \in H_{\rm loc}(J,E)$, $n = 1,2,\dots$. If a, $t \in J$, $a < t$, are fixed, then $u_\pm \in H([a,t],E)$ and $u_n \in H([a,t],E)$, $n = 1,2,\dots$, and $u_n(s) \to u(s)$ for a.e. $s \in [a,t]$. Thus $u \in H([a,t],E)$ and ${}^K\!\!\int_a^t u_n(s)ds \to {}^K\!\!\int_a^t u(s)ds$ by [6, Theorem 3.1]. □

As an easy consequence of Theorem 2.7 we obtain the following result.

Theorem 2.8. *Given a real interval J and a Banach space E ordered by a regular order cone, let $(u_n)_{n=1}^\infty$ be a monotone sequence of strongly measurable functions from a real interval J to E. Assume that $u_\pm \in H_{\rm loc}(J,E)$, and that $u_- \le u_n \le u_+$ for each $n = 1,2,\dots$. Then there exists a function $u \in H_{\rm loc}(J,E)$ such that $u(t) = \lim_n u_n(t)$ for a.e. $t \in J$, and ${}^K\!\!\int_a^t u_n(s)ds \to {}^K\!\!\int_a^t u(s)ds$ for all a, $t \in J$, $a < t$.*

Proof. Since $(u_n(s))$ is monotone and $u_-(s) \le u_n(s) \le u_+(s)$ for a.e. $s \in [a,b)$, and since the order cone of E is regular, then (u_n) converges a.e. pointwise to a function $u : J \to E$. The conclusions follow then from Theorem 2.7. □

In our study of Volterra integral equations we need the following result, which is proved in [6, Proposition 3.2].

Lemma 2.9. *Assume that W is a nonempty set in an order interval $[w_-, w_+]$ of $H_{\rm loc}(J,E)$, where J is a real interval J and E a Banach space ordered by a regular order cone.*

(a) *If W is well-ordered, it contains an increasing sequence which converges a.e. pointwise to* $\sup W$.
(b) *If W is inversely well-ordered, it contains a decreasing sequence which converges a.e. pointwise to* $\inf W$.

Since each increasing sequence of $H_{\rm loc}(J,E)$ is well-ordered and each decreasing sequence of $H_{\rm loc}(J,E)$ is inversely well-ordered, we obtain as a consequence of Lemma 2.9 and [7, Proposition 1.1.3, Corollary 1.1.3], the following results.

Corollary 2.10. *Given a real interval J and a Banach space E ordered by a normal order cone, assume that (u_n) is a sequence of $H_{\rm loc}(J,E)$, and that there exist functions $w_\pm \in H_{\rm loc}(J,E)$ such that $u_n \in [w_-, w_+]$ for each n.*

(a) *If (u_n) is increasing, it converges a.e. pointwise to $u_* = \sup_n u_n$ in the space $H_{\rm loc}(J,E)$, and u_* belongs to $[w_-, w_+]$.*
(b) *If (u_n) is decreasing, it converges a.e. pointwise to $u^* = \inf_n u_n$ in the space $H_{\rm loc}(J,E)$, and u^* belongs to $[w_-, w_+]$.*

The following fixed point result is a consequence of [1, Theorem A.2.1], [7, Theorem 1.2.1 and Proposition 1.2.1].

Lemma 2.11. *Given a partially ordered set $P = (P, \le)$ and its order interval $[w_-, w_+] = \{w \in P \mid w_- \le u \le w_+\}$, assume that $G : P \to [w_-, w_+]$ is increasing, i.e., $Gu \le Gv$ whenever $u \le v$ in P, and that each well-ordered chain of the range $G[P]$ of G has a supremum in P and each inversely well-ordered chain of $G[P]$*

has an infimum in P. Then G has least and greatest fixed points, and they are increasing with respect to G.

3. Existence and comparison results for a functional Volterra integral equation

Throughout this section $E = (E, \leq, \|\cdot\|)$ is an ordered Banach space with a regular order cone, which means by [7, Lemma 1.3.3], that all order bounded and monotone sequences of E converge.

In this section we study the functional Volterra integral equation

$$u(t) = q(t,u) + {}^K\!\!\int_a^t k(t,s) f(s,u(s),u)\,ds, \quad t \in J = [a,b), \tag{3.1}$$

where $q : J \times H_{\rm loc}((a,b),E) \to E$, $f : J \times E \times H_{\rm loc}((a,b),E) \to E$ and $k : \Lambda \to \mathbb{R}_+$, where $\Lambda = \{(t,s) \in J \times J : s \leq t\}$ and $-\infty < a < b \leq \infty$.

Assuming that $H_{\rm loc}((a,b),E)$ is equipped with a.e. pointwise ordering (2.2), we impose the following hypotheses on the functions q, f and k.

- (q0) $q(t,\cdot)$ is increasing for a.e. $t \in J$, $q(\cdot,u)$ is strongly measurable for all $u \in H_{\rm loc}((a,b),E)$, and there exist $\alpha_\pm \in H_{\rm loc}((a,b),E)$ such that $\alpha_- \leq q(\cdot,u) \leq \alpha_+$ for all $u \in H_{\rm loc}((a,b),E)$.
- (f0) There exist functions $u_\pm \in H_{\rm loc}((a,b),E)$ such that $u_- \leq f(\cdot,x,u) \leq u_+$ for all $x \in E$ and $u \in H_{\rm loc}((a,b),E)$.
- (f1) The mapping $f(\cdot,u(\cdot),u)$ is strongly measurable for each $u \in H_{\rm loc}((a,b),E)$.
- (f2) $f(s,z,u)$ is increasing with respect to z and u for a.e. $s \in J$.
- (k0) k is continuous and the mappings $s \mapsto k(t,s)u_\pm(s)$ belong to $H_{\rm loc}(J,E)$ for each $t \in J$.

Our main existence and comparison result for the integral equation (3.1) reads as follows.

Theorem 3.1. *Assume that the hypotheses* (q0), (f0), (f1), (f2), (k0) *are satisfied. Then the equation* (3.1) *has least and greatest solutions in $H_{\rm loc}((a,b),E)$. Moreover, these solutions u_* and u^* are increasing with respect to q and f.*

Proof. The hypotheses (q0), (k0) and (f0) ensure that the equations

$$w_\pm(t) = \alpha_\pm(t) + {}^K\!\!\int_a^t k(t,s) u_\pm(s)\,ds, \quad t \in J, \tag{3.2}$$

define functions $w_\pm : J \to E$. Noticing that the integral on the right-hand side of (3.2) is continuous in its upper limit t, and that the integrand is continuous in t for fixed s, one can show by applying also Theorem 2.7, that the second term on the right-hand side of (3.2) is continuous in t. Thus the functions $w_\pm$ belong to the set $P := H_{\rm loc}((a,b),E)$. By using the hypotheses (q0), (k0), (f0)–(f2), Lemmas 2.1 and 2.6 and Theorem 2.7 it can be shown that the equation

$$Gu(t) = q(t,u) + {}^K\!\!\int_a^t k(t,s) f(s,u(s),u)\,ds, \qquad t \in J, \tag{3.3}$$

defines an increasing mapping $G : P \to [w_-, w_+]$. Since $G[P] \subset [w_-, w_+]$, it follows from Lemma 2.9 that each well-ordered chain of $G[P]$ has a supremum in P and each inversely well-ordered chain of $G[P]$ has an infimum in P.

The above proof shows that all the hypotheses of Lemma 2.11 are valid for the operator G defined by (3.3). Thus G has least and greatest fixed points u_* and u^*.

Noticing that fixed points of G defined by (3.3) are solutions of (3.1) and vice versa, then u_* and u^* are least and greatest solutions of (3.1). It follows from (3.3), by Lemma 2.6, that G is increasing with respect to q and f, whence the last assertion of Theorem follows from the last assertion of Lemma 2.11. $\square$

Next we consider a case when the extremal solutions of the integral equation (3.1) can be obtained by ordinary iterations.

Proposition 3.2. *Assume that the hypotheses* (q0), (f0), (f1), (f2), (k0) *hold, and let G be defined by* (3.3).

(a) *The sequence $(u_n)_{n=0}^\infty := (G^n w_-)_{n=0}^\infty$ is increasing and converges a.e. pointwise to a function $u_* \in H_{\rm loc}((a,b),E)$. Moreover, u_* is the least solution of* (3.1) *if $q(t,u_n) \to q(t,u_*)$ for a.e. $t \in J$ and $f(s,u_n(s),u_n) \to f(s,u_*(s),u_*)$ for all $t \in J$ and for a.e. $s \in [a,t]$;*

(b) *The sequence $(v_n)_{n=0}^\infty := (G^n w_+)_{n=0}^\infty$ is decreasing and converges a.e. pointwise to a function $u^* \in H_{\rm loc}((a,b),E)$. Moreover, u^* is the greatest solution of* (3.1) *if $q(t,v_n) \to q(t,u^*)$ for a.e. $t \in J$ and $f(s,v_n(s),v_n) \to f(s,u^*(s),u^*)$ for a.e. $s \in J$.*

Proof. (a) The sequence $(u_n) := (G^n w_-)$ is increasing and contained in the order interval $[w_-,w_+]$. Hence the asserted a.e. pointwise limit $u_* \in H_{\rm loc}((a,b),E)$ exists by Corollary 2.10 (a). Moreover, (u_n) equals to the sequence of successive approximations $u_n : J \to E$ defined by

$$u_{n+1}(t) = q(t,u_n) + {}^K\!\!\int_a^t k(t,s) f(s,u_n(s),u_n)\,ds, \quad u_0(t) = w_-(t),\ t \in J,\ n \in \mathbb{N}. \tag{3.4}$$

In view of these results, the hypotheses of (a) and Theorem 2.8, it follows from (3.4) as $n \to \infty$ that u_* is a solution of (3.1).

If u is any solution of (3.1), then $u = Gu \in [w_-,w_+]$. By induction one can show that $u_n = G^n w_- \in [w_-,u]$ for each n. Thus $u_* = \sup_n u_n \le u$, which proves that u_* is the least solution of (3.1).

The proof of part (b) is similar to that of (a) and is omitted. $\square$

Example 3.3. Let E be the space c_0 of all sequences $(c_n)_{n=1}^\infty$ of real numbers converging to zero, ordered componentwise and equipped with the sup-norm. Define h_n, $\alpha_n : [0,\infty) \to \mathbb{R}$ and $k : \Lambda \to \mathbb{R}_+$ by equations

$$\begin{gathered} h_n(t) = \frac{2}{\sqrt{n}} \cos\big(\frac{1}{t^2}\big) + \frac{2}{\sqrt{n}t^2} \sin\big(\frac{1}{t^2}\big), \quad t > 0,\ h_n(0) = 0, \\ \alpha_n(t) = \frac{1}{\sqrt{n}t} H\Big(t - \frac{2n-1}{2n}\Big), \quad n = 1,2,\dots, \\ k(t,s) = \frac{s}{t}, \quad t > 0, \quad \alpha_n(0) = k(0,\cdot) = 0, \end{gathered} \tag{3.5}$$

The solutions of the infinite system of integral equations

$$w_n(t) = \pm\alpha_n(t) + {}^K\!\!\int_0^t k(t,s)\Big(h_n(s) \pm \frac{1}{\sqrt{n}}\Big)\,ds, \quad n = 1,2,\dots, \tag{3.6}$$

in $H_{\rm loc}((0,\infty),c_0)$ are

$$w_\pm(t) = (w_{n\pm}(t))_{n=1}^\infty = \Big(\pm\frac{1}{\sqrt{n}t}H\Big(t-\frac{2n-1}{2n}\Big)+\frac{t}{\sqrt{n}}\cos\big(\frac{1}{t^2}\big)\pm\frac{t}{2\sqrt{n}}\Big)_{n=1}^\infty. \tag{3.7}$$

In particular, Theorem 3.1 can be applied to show that the infinite system of integral equations

$$u_n(t) = q_n(u)\alpha_n(t) + {}^K\!\!\int_0^t k(t,s)\Big(h_n(s)+\frac{1}{\sqrt{n}}g_n(u)\Big)\,ds, \quad n=1,2,\dots, \tag{3.8}$$

where $u=(u_n)_{n=1}^\infty$ has least and greatest solutions $u_*=(u_{*n})_{n=1}^\infty$ and $u^*=(u_n^*)_{n=1}^\infty$ in $H_{\rm loc}((0,\infty),c_0)$, if all the functions q_n, $g_n : H_{\rm loc}([0,\infty),c_0)\to\mathbb{R}$ are increasing, and if $-1\le g_n(u)$, $q_n(u)\le 1$ for all $u\in H_{\rm loc}((0,\infty),c_0)$ and $n=1,2,\dots$. Moreover, both u_* and u^* belong to the order interval $[w_-,w_+]$ of $H_{\rm loc}(0,\infty),c_0)$, where the functions $w_\pm$ are given by (3.7).

Remarks 3.4. The functions h_n in Example 3.1 do not belong to $H([0,t_1],\mathbb{R})$ for any $t_1>0$. However, $k(t,s)=\frac{s}{t}$ is continuous and the functions $k(t,\cdot)h_n$ belong to $H_{\rm loc}([0,\infty),\mathbb{R})$, whence the hypothesis (k0) is valid.

Continuity of k and Theorem 2.7 ensure that the integral on the right-hand side of equation (3.1) is continuous in t. If also the function q is continuous in t in that equation, then its solutions are continuous.

4. An application to an impulsive IVP

Let E be a Banach space ordered by a regular order cone. The result of Theorem 3.1 will now be applied to the following impulsive initial value problem (IIVP)

$$\begin{gathered} u'(t)+p(t)u(t) = f(t,u(t),u) \quad \text{a.e. on } J=[a,b),\\ u(a)=x_0, \quad \Delta u(\lambda)=D(\lambda,u), \quad \lambda\in W, \end{gathered} \tag{4.1}$$

where $p\in L^1(J,\mathbb{R})$, $f: J\times E\times H_{\rm loc}(J,E)\to E$, $x_0\in E$, $\Delta u(\lambda)=u(\lambda+0)-u(\lambda)$, $D: W\times H_{\rm loc}(J,E)\to E$, and W is a well-ordered (and hence countable) subset of (a,b).

Denoting $W^{<t}=\{\lambda\in W \mid \lambda<t\}$, $t\in J$, and by $ACG^*_{\rm loc}(J,E)$ the set of all continuous functions from J to E which are ACG^* on every compact subinterval of J, we say that $u:J\to E$ is a solution of the IIVP (4.1) if it satisfies the equations of (4.1), and if it belongs to the set

$$\begin{aligned} V=\{u:J\to E \mid \sum_{\lambda\in W}\|\Delta u(\lambda)\|<\infty \quad\text{and}\\ t\mapsto u(t)-\sum_{\lambda\in W^{<t}}\Delta u(\lambda)\in ACG^*_{\rm loc}(J,E)\}. \end{aligned}$$

It is easy to verify that V is a subset of $H_{\rm loc}(J,E)$.

The following result, which is a generalization to [2, Lemma 3.1], allows us to convert the IIVP (4.1) to an improper Volterra integral equation.

Lemma 4.1. *If $p\in L^1(J,\mathbb{R})$, $g\in H_{\rm loc}(J,E)$, $x_0\in E$ and $c: W\to E$, and if $\sum_{\lambda\in W}\|c(\lambda)\|<\infty$, then the problem*

$$\begin{gathered} u'(t)+p(t)u(t)=g(t) \quad \text{a.e. on } J,\\ u(a)=x_0, \quad \Delta u(\lambda)=c(\lambda), \quad \lambda\in W, \end{gathered} \tag{4.2}$$

has a unique solution u. This solution can be represented as

$$u(t) = e^{-\int_a^t p(s)ds}x_0 + \sum_{\lambda\in W^{<t}} e^{-\int_\lambda^t p(s)ds}c(\lambda) + {}^K\!\!\int_a^t e^{-\int_s^t p(\tau)d\tau}g(s)ds \tag{4.3}$$

for $t \in J$. Moreover, u is increasing with respect to g, c and x_0.

Proof. Let $u : J \to E$ be defined by (4.3). Given a compact subinterval $I = [a, t_1]$ of J, define a mapping $\Gamma : I \to I$ by

$$\Gamma(s) = \min\{t \in W \cup \{t_1\} \mid s < t\}, \quad s \in [a, t_1), \quad \Gamma(t_1) = t_1.$$

Denote by C the well-ordered chain of Γ-iterations of a, i.e. (cf. [7, Theorem 1.1.1]) C is the only well-ordered subset of J with the following properties: $a = \min C$, and if $s > a$, then $s \in C$ if and only if $s = \sup \Gamma\{t \in C | t < s\}$.

It follows from [7, Corollary 1.1.1] that $W \subset C$, and I is a disjoint union of C and open intervals $(s, \Gamma(s))$, $s \in C$. Moreover, C is countable as a well-ordered set of real numbers. Hence, rewriting (4.3) as

$$u(t) = e^{-\int_a^t p(s)ds}\Big[x_0 + \sum_{\alpha\in W^{<t}} e^{-\int_\alpha^a p(s)ds}c(\alpha) + {}^K\!\!\int_a^t e^{-\int_s^a p(\tau)d\tau}g(s)ds\Big],$$

it is easy to verify that

$$u'(t) + p(t)u(t) = g(t) \quad \text{for a.e. } t \in I, \quad u(a) = x_0. \tag{4.4}$$

For each $\alpha \in W$ the open interval $(\alpha, \Gamma(\alpha))$ does not contain any point of W, so that

$$\Delta u(\alpha) = u(\alpha + 0) - u(\alpha) = \lim_{t\to\alpha+0} e^{-\int_\alpha^t p(s)ds}c(\alpha) = c(\alpha), \quad \alpha \in W. \tag{4.5}$$

It follows from (4.3) and (4.5) that

$$u(t) - \sum_{\alpha\in W^{<t}} \Delta u(\alpha) = u(t) - \sum_{\alpha\in W^{<t}} c(\alpha) = v(t) + w(t), \tag{4.6}$$

where

$$\begin{gathered} v(t) = e^{-\int_a^t p(s)ds}x_0 + {}^K\!\!\int_a^t e^{-\int_s^t p(\tau)d\tau}g(s)ds, \quad t \in I, \\ w(t) = \sum_{\alpha\in W^{<t}} (e^{-\int_\alpha^t p(s)ds} - 1)c(\alpha), \quad t \in I. \end{gathered}$$

Thus, for $a \le \bar t < t \le t_1$ we obtain

$$\begin{aligned} &w(t) - w(\bar t) \\ &= \sum_{\alpha\in W\cap(a,\bar t)} (e^{-\int_\alpha^t p(s)ds} - e^{-\int_\alpha^{\bar t} p(s)ds})c(\alpha) + \sum_{\alpha\in W\cap[\bar t,t)} (e^{-\int_\alpha^t p(s)ds} - 1)c(\alpha) \\ &= \sum_{\alpha\in W\cap(a,\bar t)} \int_{\bar t}^t -p(s)e^{-\int_\alpha^s p(\tau)d\tau}ds\, c(\alpha) + \sum_{\alpha\in W\cap[\bar t,t)} \int_\alpha^t -p(s)e^{-\int_\alpha^s p(\tau)d\tau}ds\, c(\alpha). \end{aligned}$$

Applying this representation and denoting $M = e^{\int_a^{t_1} |p(s)|ds} \sum_{\alpha\in W} \|c(\alpha)\|$, it follows that

$$\|w(t) - w(\bar t)\| \le M \int_{\bar t}^t |p(s)|ds \quad \text{for } a \le \bar t < t \le t_1.$$

This implies that w is absolutely continuous. Obviously, w is a.e. differentiable and the function v is continuous and belongs to $ACG^*(I,E)$ by Theorem 2.2 and Proposition 2.5.

The above result holds for every $t_1 \in (a,b)$, so that $u \in V$ by (4.6). This, (4.4) and (4.5) imply that u is a solution of problem (4.2).

If $v \in V$ is a solution of (4.2), then $w = u - v$ is a function of V and $\Delta w(\alpha) = 0$ for each $\alpha \in W$, whence $w \in ACG^*_{\rm loc}(J,E)$ and w is a solution of the initial value of problem

$$w'(t) + p(t)w(t) = 0 \quad \text{a.e. on } J, \quad w(a) = 0. \tag{4.7}$$

For every fixed $t \in J$ the function

$$h(s) = e^{\int_a^s p(\tau)d\tau}, \quad s \in I = [a,t],$$

is absolutely continuous on I and real-valued. It then follows from Lemma 2.4 that

$$h(t)w(t) - h(a)w(a) = {}^K\!\!\int_a^t (h'(s)w(s) + h(s)w'(s))\,ds, \quad t \in J,$$

or equivalently,

$$h(t)w(t) - h(a)w(a) = {}^K\!\!\int_a^t (e^{\int_a^s p(\tau)d\tau}(p(s)w(s) + w'(s))\,ds, \quad t \in J.$$

This equation and (4.7) imply that $h(t)w(t) \equiv 0$, so that $w(t) \equiv 0$, whence $u = v$.

The last assertion of Lemma is a direct consequence from the representation (4.3) and Lemma 2.6. □

We shall impose the following hypotheses on the function D.

- (D0) $D(\lambda,\cdot)$ is increasing for all $\lambda \in W$, and there exist $c_\pm : W \to E$ such that $c_-(\lambda) \le D(\lambda,u) \le c_+(\lambda)$ for all $\lambda \in W$ and $u \in H_{\rm loc}(J,E)$, and that $\sum_{\lambda\in W}\|c_\pm(\lambda)\| < \infty$.

As an application of Theorem 3.1 we get the following existence and comparison result for the IIVP (4.1).

Theorem 4.2. *Let the functions f and D in* (4.1) *satisfy the hypotheses* (f0)–(f2), (D0)*. If $p \in L^1(J,\mathbb{R})$, and if the improper integrals ${}^K\!\!\int_a^t e^{{}^K\!\!\int_a^s p(\tau)d\tau} h_\pm(s)ds$ exist for some $t \in J$, then the IIVP* (4.1) *has for each $x_0 \in E$ least and greatest solutions u_* and u^* in V. Moreover, these solutions are increasing with respect to x_0, D and f.*

Proof. The hypotheses given for D and p ensure that for each $x_0 \in E$ the relations

$$\begin{gathered} q(t,u) = e^{-\int_a^t p(s)ds}x_0 + \sum_{\lambda\in W^{<t}} e^{-\int_\lambda^t p(s)ds} D(\lambda,u), \\ \text{for } t \in J,\ u \in H_{\rm loc}(J,E); \end{gathered} \tag{4.8}$$

$$k(t,s) = e^{-\int_s^t p(\tau)d\tau}, \quad (t,s) \in \Lambda = \{(t,s) \in J \times J \mid s \le t\},$$

define mappings $q : J \times H_{\rm loc}(J,E) \to E$ and $k : \Lambda \to \mathbb{R}_+$ which satisfy the hypotheses (q0), and (k0) of Theorem 3.1. Then the integral equation (4.1), which by (4.8) can be rewritten as a fixed point equation

$$\begin{aligned} u(t) = Gu(t) := {}& e^{-\int_a^t p(s)ds}x_0 + \sum_{\lambda\in W^{<t}} e^{-\int_\lambda^t p(s)ds} D(\lambda,u) \\ &+ {}^K\!\!\int_a^t e^{-\int_s^t p(\tau)d\tau} f(s,u(s),u)ds, \end{aligned} \tag{4.9}$$

has by Theorem 3.1 least and greatest solutions u_* and u^*, and they are increasing with respect to q and f. Because by Lemma 4.1 the solutions of the IIVP (4.1) are the same as the solutions of the integral equation (4.9), then u_* and u^* are least and greatest solutions of the (IIVP) (4.1), and they are increasing with respect to x_0, D and q. □

The next result is a consequence of Proposition 3.2.

Proposition 4.3. *Assume that the hypotheses of Theorem 4.1 hold, and let G be defined by (4.9).*

(a) *The sequence $(u_n)_{n=0}^\infty = (G^n w_-)_{n=0}^\infty$ is increasing and converges a.e. pointwise to a function $u_* \in H_{\rm loc}(J,E)$. Moreover, u_* is the least solution of (4.1) if $D(\lambda,u_n) \to D(\lambda,u_*)$ for each $\lambda \in W$ and $f(s,u_n(s),u_n) \to f(s,u_*(s),u_*)$ for a.e. $s \in J$;*
(b) *The sequence $(v_n)_{n=0}^\infty = (G^n w_+)_{n=0}^\infty$ is decreasing and converges a.e. pointwise to a function $u^* \in H_{\rm loc}(J,E)$. Moreover, u^* is the greatest solution of (4.1) if $D(\lambda,v_n) \to D(\lambda,u^*)$ for each $\lambda \in W$ and $f(s,v_n(s),v_n) \to f(s,u^*(s),u^*)$ for a.e. $s \in J$.*

Example 4.4. Let E be, as in Example 3.3, the space c_0 of the sequences of real numbers converging to zero, ordered componentwise and equipped with the sup-norm. The solutions of the infinite system of IIVP's

$$\begin{gathered} w_n'(t) + \frac{1}{1+t}w_n(t) = \frac{2}{\sqrt{n}(1+t)}\Big(\cos\big(\frac{1}{t^2}\big) + \frac{2}{t}\sin\big(\frac{1}{t^2}\big)\Big) \pm \frac{1}{\sqrt{n}}, \\ w_n(0+) = 0, \quad \Delta w_n\Big(t - \frac{2n-1}{2n}\Big) = \pm\frac{1}{\sqrt{n}}, \quad n = 1,2,\dots, \end{gathered} \tag{4.10}$$

in $H_{\rm loc}((0,2),c_0)$ are

$$\begin{aligned} &(w_{n\pm}(t))_{n=1}^\infty \\ &= \Big(\frac{1}{2\sqrt{n}(1+t)}\Big(\pm\frac{4n-1}{n}H\Big(t-\frac{2n-1}{2n}\Big) + 2t^2\cos\big(\frac{1}{t^2}\big) \pm 2t \pm t^2\Big)\Big)_{n=1}^\infty . \end{aligned} \tag{4.11}$$

Thus Theorem 4.2 can be applied to show that least and greatest solutions $u_* = (u_{*n})_{n=1}^\infty$ and $u^* = (u_n^*)_{n=1}^\infty$ of infinite system of IIVP's

$$\begin{gathered} u_n'(t) + \frac{1}{1+t}u_n(t) = \frac{1}{\sqrt{n}(1+t)}\Big(\cos\big(\frac{1}{t^2}\big) + \frac{2}{t}\sin\big(\frac{1}{t^2}\big)\Big) + \frac{1}{\sqrt{n}}g_n(u), \\ w_n(0+) = 0, \quad \Delta w_n\Big(t - \frac{2n-1}{2n}\Big) = \frac{1}{\sqrt{n}}D_n(u), \quad n = 1,2,\dots, \end{gathered} \tag{4.12}$$

exist in $H_{\rm loc}((0,2),c_0)$ and belong to its order interval $[w_-,w_+]$, if we assume that all the functions D_n, $g_n : H_{\rm loc}([0,2),c_0) \to \mathbb{R}$, are increasing, and if $-1 \le D_n(u)$, $g_n(u) \le 1$ for all $u \in H_{\rm loc}((0,2),c_0)$ and $n = 1,2,\dots$.

Remarks 4.5. The functional dependence on the last argument u of q, f and D can be formed, e.g., by bounded, linear and positive operators, such as integral operators of Volterra and/or Fredholm type with nonnegative kernels. Thus the results derived in this paper can be applied also to integro-differential equations.

If $a > -\infty$, then $H_{\rm loc}([a,b),E)$ contains those functions $u : [a,b) \to E$ which are HL integrable on every compact subinterval of (a,b) and for which the improper integral

$$ {}^K\!\!\int_{a+}^{t} u(s)\,ds = \lim_{c\to a+} {}^K\!\!\int_{c}^{t} u(s)\,ds $$

exists for some $t \in (a, b)$ (cf. [3, Theorem 2.1]). Noticing also that Bochner integrable functions are HL integrable, the results of Sections 3 and 4 generalize the corresponding results of [5] in the case when $a > -\infty$.

As for other papers dealing with functional Volterra integral equations and differential equations via non-absolute integrals; see, e.g. [3, 4, 9, 11, 12].

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