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A RICCATI TECHNIQUE FOR PROVING OSCILLATION OF A HALF-LINEAR EQUATION

PAVEL ŘEHÁK

Abstract. In this paper we study the oscillation of solutions to the half-linear differential equation

$$(r(t)|y'|^{p-1}\operatorname{sgn} y)' + c(t)|y|^{p-1}\operatorname{sgn} y = 0,$$

under the assumptions $\int^\infty r^{1/(1-p)}(s)\,ds < \infty$, $r(t) > 0$, $p > 1$. Our main tool is a Riccati type transformation for using the so called "function sequence technique". This method leads to new and to known oscillation and comparison results. We also give an example that illustrates our results.

1. Introduction

The Riccati type transformation plays an important role in qualitative theory of the half-linear differential equation

$$(r(t)\Phi(y'))' + c(t)\Phi(y) = 0, \tag{1.1}$$

where r and c are continuous functions on $[a,\infty)$ with $r(t) > 0$, and $\Phi(u) = |u|^{p-1}\operatorname{sgn} u$ with $p > 1$. Monograph [1] presents a systematic and compact treatment of the qualitative theory of the above equation. Recall that (1.1) can be viewed at least in three ways: (1) as a natural generalization of a linear differential equation, (2) as a differential equation with one dimensional p-Laplacian, (3) as a special case of a generalized Emden-Fowler (quasilinear) differential equation.

If there exists a positive solution y of (1.1) on some interval $[t_0,\infty)$, then the function $w = r\Phi(y'/y)$ satisfies the generalized Riccati differential equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0 \tag{1.2}$$

on $[t_0,\infty)$. Here q is the conjugate number to p; i.e., $1/p + 1/q = 1$.

A nontrivial solution of (1.1) is said to be oscillatory if it has zeros of arbitrary large value, and non-oscillatory otherwise. An equation is said to be oscillatory if all its solutions are oscillatory, and non-oscillatory otherwise.

Note that one solution of (1.1) is oscillatory if and only if every solution of (1.1) is oscillatory, which follows from the Sturm type separation result. Further, if the

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generalized Riccati differential inequality $w'+c(t)+(p-1)r^{1-q}(t)|w|^q \le 0$ is solvable on some interval $[t_0,\infty)$, then (1.1) is non-oscillatory.

Methods based on these relations are referred as the Riccati technique. There are several refinements of this idea: Using a weighted Riccati type substitution; working with integral, instead of differential, Riccati type equations and inequalities using a function sequence technique; finding effective estimates for solutions of Riccati type equations; etc. See for example [1, Sections 2.2, 5.5].

It is known that many oscillation and asymptotical results for (1.1) substantially depend on the convergence or the divergence of the integral $\int^\infty r^{1-q}(s)\,ds$. In contrast to the linear case, a suitable transformation satisfactorily transfering one case into the other is not available for (1.1) and hence it is often necessary to examine these cases separately – by using different approaches. Note that usually the case with the convergent integral is more difficult than the convergent case, which can be modelled according to the case $r(t)\equiv 1$. We study the convergent case; i.e., we assume that

$$\int^\infty r^{1-q}(s)\,ds<\infty. \tag{1.3}$$

The principal aim of this paper is to establish the so-called function sequence technique for (1.1) under condition (1.3), and then to show some applications of this method. The function sequence techniques for (1.1) with $\int^\infty r^{1-q}(s)\,ds=\infty$ were studied in [1, 2, 4]. For this article [3] is a useful reference.

This paper is organized as follows. In the next section we present a modification of the Riccati technique involving a Riccati type integral inequality. These relations are then utilized in Section 3 to show the equivalence between nonoscillation of (1.1) and convergence of certain function sequence. In the last section, we apply this method to derive Hille-Nehari type oscillation criteria and a Hille-Wintner type comparison theorem for equation (1.1). We also give an example of an equation which, in particular, can be proved to be oscillatory using our new results, but other known criteria are inapplicable.

2. Modified Riccati Type Inequality

We start with showing that in the relation between non-oscillation of (1.1) and solvability of (1.2), under condition (1.3), the Riccati type differential equation or inequality can be replaced by certain Riccati type integral equation or inequality. For the first time, it was observed in [3]. Here we recall this result, we add some refinements, and also give two new proofs. Denote

$$R(t):=\int_t^\infty r^{1-q}(s)\,ds$$

and

$$\begin{aligned}\mathcal{S}(u)(t):=\int_t^\infty R^p(s)c(s)\,ds+p\int_t^\infty r^{1-q}(s)R^{p-1}(s)u(s)\,ds\\ +(p-1)\int_t^\infty r^{1-q}(s)R^p(s)|u(s)|^q ds\,.\end{aligned}$$

Theorem 2.1. *(i) Assume $c(t)\ge 0$ for large t. If (1.1) is non-oscillatory, then $\int^\infty R^p(s)c(s)\,ds<\infty$ and there is w satisfying $R^{p-1}(t)w(t)\ge -1$ and $R^p(t)w(t)=\mathcal{S}(w)(t)$ for large t. Moreover, $\limsup_{t\to\infty}R^{p-1}(t)w(t)\le 0$.*

(ii) Assume that $\infty > \int_t^\infty R^p(s)c(s)\,ds \geq 0$ *for large* t. *If there is* w *satisfying* $R^{p-1}(t)w(t) \geq -1$ *and* $R^p(t)w(t) \geq \mathcal{S}(w)(t)$ *for large* t, *then* (1.1) *is non-oscillatory.*

Proof. (i) See [3] or [1, Section 2.2]. (ii) Set $v(t) = R^{-p}(t)\mathcal{S}(w)(t)$. For convenience we skip the argument t sometimes in the computations. Differentiating the equality $R^p v = \mathcal{S}(w)$ we get

$$0 = R^p v' + R^p c - pR^{p-1}v^{1-q}v + pR^{p-1}r^{1-q}w + (p-1)r^{1-q}|R^{p-1}w|^q. \tag{2.1}$$

We will show that

$$pR^{p-1}r^{1-q}w + (p-1)r^{1-q}|R^{p-1}w|^q \geq pR^{p-1}r^{1-q}v + (p-1)r^{1-q}|R^{p-1}v|^q. \tag{2.2}$$

Observe that the function

$$x \to px + (p-1)|x|^q \text{ is strictly increasing for } x \geq -1. \tag{2.3}$$

From $R^p v = \mathcal{S}(w) \leq R^p w$, we have $v \leq w$. We know $R^{p-1}w \geq -1$. Next we show that also $R^{p-1}v \geq -1$. From $v = R^{-p}\mathcal{S}(w)$, we have that $R^{p-1}v \geq -1$ if and only if $\mathcal{S}(w) \geq -R$, i.e., $\int_t^\infty R^p(s)c(s)\,ds + \int_t^\infty r^{1-q}(s)[pR^{p-1}(s)w(s) + (p-1)|R^{p-1}(s)w(s)|^q + 1]\,ds \geq 0$. But the above inequality is satisfied because $\int_t^\infty R^p(s)c(s)\,ds \geq 0$ and $pR^{p-1}w + (p-1)|R^{p-1}w|^q + 1 \geq -p + (p-1) + 1 = 0$ which follows from (2.3) and $R^{p-1}w \geq -1$. Hence, $R^{p-1}v \geq -1$ which together with (2.3) and $v \leq w$ yields (2.2). Using (2.2) in (2.1) we obtain $0 \geq R^p v' + R^p c + (p-1)r^{1-q}R^p|v|^q$, or $0 \geq v' + c + (p-1)r^{1-q}|v|^q$. Thus (1.1) is non-oscillatory. □

Remark 2.2. (i) The part (ii) of the theorem was proved in [3] using a different technique, based on the Schauder-Tychonov fixed point theorem, under the stronger assumptions $c(t) \geq 0$ and $R^{p-1}(t)w(t)$ is bounded. A closer examination of that proof shows that these assumptions actually are not needed. Later, in this paper, we present another proof of the part (ii) of the theorem, which arises out as a by-product when deriving the function sequence technique.

(ii) From Theorem 2.1 (i), we immediately get the following simple criterion: If $c(t) \geq 0$ and $\int^\infty R^p(s)c(s)\,ds = \infty$, then (1.1) is oscillatory.

(iii) We conjecture that in the part (i) of the theorem, the condition $c(t) \geq 0$ can be relaxed, e.g., to $\int_t^\infty R^p(s)c(s)\,ds \geq 0$.

3. Function Sequence Technique

We are in a position to establish the function sequence technique for (1.1) under condition (1.3). Denote

$$H(t) = R^{-p}(t)\int_t^\infty R^p(s)c(s)\,ds,$$

$$\mathcal{G}(u)(t) = R^{-p}(t)\int_t^\infty r^{1-q}(s)[pR^{p-1}(s)u(s) + (p-1)|R^{p-1}(s)u(s)|^q]\,ds.$$

Observe that $H + \mathcal{G}(u) = R^{-p}\mathcal{S}(u)$. Further, $-R^{1-p}$ is a fixed point for $\mathcal{G}$, and for u with $uR^{p-1} \geq -1$, $\mathcal{G}(u)$ is increasing with respect to u, which follows from (2.3). Define the sequence $\{\varphi_k(t)\}$ as follows

$$\varphi_0 = -R^{1-p}, \quad \varphi_{k+1} = H + \mathcal{G}(\varphi_k), \quad k = 0, 1, 2, \dots.$$

It is easy to see that $\varphi_{k+1} \geq \varphi_k$, $k = 0, 1, 2, \dots$, provided $H \geq 0$.

Theorem 3.1. *Let $c(t) \ge 0$ for large t. Equation* (1.1) *is non-oscillatory if and only if there exists $t_0 \in [a, \infty)$ such that $\lim_{k\to\infty} \varphi_k(t) = \varphi(t)$ for $t \ge t_0$, i.e., $\{\varphi_k(t)\}$ is well defined and pointwise convergent.*

Proof. *Only if part:* If (1.1) is non-oscillatory then there is a function w satisfying $R^{p-1}(t)w(t) \ge -1$ and $R^p(t)w(t) = \mathcal{S}(w)(t)$ for large t, say $t \ge t_0$, by Theorem 2.1. In fact, instead of the equality $R^p(t)w(t) = \mathcal{S}(w)(t)$ we may take the inequality $R^p(t)w(t) \ge \mathcal{S}(w)(t)$, and the proof works as well. See also Remark 3.2 (i), why this is useful. For convenience we skip the argument t sometimes in the computations. Since $w \ge -R^{1-p}$, we have $w \ge \varphi_0$. Further $\varphi_1 = H + \mathcal{G}(\varphi_0) = H + \varphi_0 \ge \varphi_0$ and $\varphi_1 = H + \mathcal{G}(\varphi_0) \le H + \mathcal{G}(w) = w$. Hence, $\varphi_0 \le \varphi_1 \le w$ and $R^{p-1}\varphi_1 \ge -1$. Similarly, $w = H + \mathcal{G}(w) \ge H + \mathcal{G}(\varphi_1) = \varphi_2 \ge H + \mathcal{G}(\varphi_0) = \varphi_1$, hence, $\varphi_0 \le \varphi_1 \le \varphi_2 \le w$. By induction, $\varphi_k \le \varphi_{k+1} \le w$ for $k = 0, 1, 2, \dots$. Hence, $\lim_{k\to\infty} \varphi_k(t) = \varphi(t)$.

If part: If $\lim_{k\to\infty} \varphi_k(t) = \varphi(t)$, then from the monotonicity of $\{\varphi_k\}$ it follows $\varphi_k \le \varphi$ and $R^{p-1}\varphi_k \ge -1$ for $k = 0, 1, 2, \dots$, on $[t_0, \infty)$. Applying the Lebesgue monotone convergence theorem in $\varphi_{k+1} = H + \mathcal{G}(\varphi_k)$, we get $\varphi = H + \mathcal{G}(\varphi)$, or $R^p\varphi = \mathcal{S}(\varphi)$. Now it is easy to see that φ solves the generalized Riccati equation (1.2), and thus (1.1) is non-oscillatory. □

Remark 3.2. (i) A closer examination of the proof shows that, as a by-product, we have obtained another proof of Theorem 2.1 (ii). Indeed, if w satisfies $R^{p-1}w \ge -1$ and $R^p w \ge \mathcal{S}(w)$, then $\lim_{k\to\infty} \varphi_k(t) = \varphi(t)$, which implies non-oscillation of (1.1).

(ii) In the if part, $c(t) \ge 0$ can be relaxed to $\int_t^\infty R^p(s)c(s)\,ds \ge 0$. We conjecture that this is possible also in the only if part.

(iii) The approximating sequence $\{\varphi_k\}$ is not the only one that is available. Another possibility is, for instance, the sequence $\{\psi_k\}$, defined by $\psi_0 = \mathcal{G}(H - R^{1-p})$ and $\psi_{k+1} = \mathcal{G}(H + \psi_k)$.

Corollary 3.3. *Let $c(t) \ge 0$ for large t. Equation* (1.1) *is oscillatory if and only if either*

(i) *there is $m \in \mathbb{N}$ such that φ_k is defined for $k = 1, 2, \dots, m-1$, but φ_m does not exists, i.e.,*
$$\int_t^\infty r^{1-q}(s)[pR^{p-1}(s)\varphi_{m-1}(s) + (p-1)|R^{p-1}(s)\varphi_{m-1}(s)|^q]\,ds = \infty,$$
or

(ii) *φ_k is defined for $k = 1, 2, \dots$, but for arbitrarily large $t_0 \ge a$, there is $t_* \ge t_0$ such that $\lim_{k\to\infty} \varphi_k(t_*) = \infty$.*

4. Applications

In this section we show how the function sequence technique can be applied. By means of this method, we establish oscillation and comparison results for (1.1); some of them are known, some of them are new or improving known ones. We start with modified Hille-Nehari type criteria.

Theorem 4.1. *Let $c(t) \ge 0$ for large t. If*
$$\limsup_{t\to\infty} R^{-1}(t)\mathcal{S}(\varphi_k)(t) > 0 \tag{4.1}$$
for some $k \in \mathbb{N} \cup \{0\}$, then (1.1) *is oscillatory.*

Proof. If equation (1.1) is non-oscillatory, then as in the proof of Theorem 3.1, we have $\varphi_k(t) \le w(t)$, $k = 0, 1, 2, \dots$ for large t. Moreover, $R^{-1}(t)\mathcal{S}(w)(t) \le R^{p-1}(t)w(t)$ for large t and $\limsup_{t\to\infty} R^{p-1}(t)w(t) \le 0$ by Theorem 2.1. Hence,

$$\begin{aligned}\limsup_{t\to\infty} R^{-1}(t)\mathcal{S}(\varphi_k)(t) &\le \limsup_{t\to\infty} R^{-1}(t)\mathcal{S}(w)(t)\\ &\le \limsup_{t\to\infty} R^{p-1}(t)w(t) \le 0,\end{aligned}$$

which contradicts (4.1). □

Taking $k = 0$ in the previous theorem, we have the following statement, which was established also in [3].

Corollary 4.2. *Let $c(t) \ge 0$ for large t. If*

$$\limsup_{t\to\infty} R^{-1}(t)\int_t^\infty R^p(s)c(s)\,ds > 1,$$

then (1.1) *is oscillatory.*

Theorem 4.3. *Let $c(t) \ge 0$ for large t. If*

$$\liminf_{t\to\infty} R^{-1}(t)\int_t^\infty R^p(s)c(s)\,ds > q^{-p}, \tag{4.2}$$

then (1.1) *is oscillatory.*

Proof. Condition (4.2) can be rewritten as

$$\int_t^\infty R^p(s)c(s)\,ds \ge \gamma R(t) \tag{4.3}$$

for large t, say $t \ge t_0$, where $\gamma > q^{-p}$. Then

$$\varphi_1(t) = H(t) + \mathcal{G}(\varphi_0)(t) \ge R^{-p}(t)\gamma R(t) - R^{1-p}(t) = \gamma_1 R^{1-p}(t), \tag{4.4}$$

$t \ge t_0$, where $\gamma_1 = \gamma - 1$.

Note that $\gamma_1 > -1$ and $R^{p-1}(t)\varphi_1(t) > -1$. Hence, in view of (2.3), (4.3), and (4.4), $\varphi_2(t) = H(t) + \mathcal{G}(\varphi_1)(t) \ge \gamma R^{1-p}(t) + R^{-p}(t)\int_t^\infty r^{1-q}(s)[p\gamma_1 + (p-1)|\gamma_1|^q]\,ds = \gamma_2 R^{1-p}(t)$, where $\gamma_2 = \gamma + p\gamma_1 + (p-1)|\gamma_1|^q$. Since $\gamma_1 > -1$, we have $\gamma_2 > \gamma - p + p - 1 = \gamma - 1 = \gamma_1$ by (2.3), and so $\gamma_2 > \gamma_1 > -1$ and $R^{p-1}(t)\varphi_2(t) > -1$. Arguing as above, by induction,

$$\varphi_k(t) \ge \gamma_k R^{1-p}(t), \quad k = 1, 2, \dots, \tag{4.5}$$

where $\{\gamma_k\}$ is defined by

$$\gamma_{k+1} = \gamma + p\gamma_k + (p-1)|\gamma_k|^q, \quad k = 1, 2, \dots. \tag{4.6}$$

Moreover, $\gamma_{k+1} > \gamma_k > -1$, $k = 1, 2, \dots$. Hence the limit $\lim_{k\to\infty}\gamma_k = L \in (-1, \infty) \cup \{\infty\}$ exists. We claim that $L = \infty$. If not, then (4.6) yields

$$|L|^q + L + \gamma/(p-1) = 0. \tag{4.7}$$

We show that this equation has no solution in $(-1, \infty)$. We distinguish two cases. If $L \in [0, \infty)$, then $|L|^q + L + \gamma/(p-1) \ge \gamma/(p-1) > 0$, a contradiction. To show that also $L \in (-1, 0)$ is impossible, it is sufficient to examine the problem $x = g(x; \lambda)$, $x \in (-1, 0)$, where $g(x; \lambda) = \lambda + px + (p-1)|x|^q$ and λ is a parameter. It is easy to see that $-q^{1-p}$ is a fixed point of $g(\cdot; q^{-p})$, and the parabola-like curve $x \to g(x; q^{-p})$ touches the line $x \to x$ at $x = -q^{1-p}$. Since $\gamma > q^{-p}$, the problem $x = g(x; \gamma)$ has no solution in $(-1, 0)$. But this problem is equivalent to (4.7), and so $\lim_{k\to\infty}\gamma_k=\infty$. Hence, from (4.5), we have $\lim_{k\to\infty}\varphi_k(t)=\infty$ for $t\geq t_0$. Equation (1.1) is oscillatory by Corollary 3.3. □

Theorem 4.4. *Let $c(t)\geq 0$ for large t. If*

$$R^{-1}(t)\int_t^\infty R^p(s)c(s)\,ds\leq q^{-p}\quad \textit{for large } t, \tag{4.8}$$

then (1.1) *is non-oscillatory.*

Proof. Condition (4.8) can be rewritten as $\int_t^\infty R^p(s)c(s)\,ds\leq \delta R(t)$ for large t, say $t\geq t_0$, where $0<\delta\leq q^{-p}$. Similarly as in the previous part, with a wide utilization of (2.3), we get

$$\varphi_k(t)\leq \delta_k R^{1-p}(t),\quad k=1,2,\dots, \tag{4.9}$$

where $\{\delta_k\}$ is defined by

$$\delta_{k+1}=\delta+p\delta_k+(p-1)|\delta_k|^q,\quad k=1,2,\dots \tag{4.10}$$

and $\delta_1=\delta-1$. Moreover, $\delta_{k+1}>\delta_k>-1$, $k=1,2,\dots$. We claim that $\{\delta_k\}$ converges. Consider the fixed point problem $x=g(x;\lambda)$, where g is defined as above. In addition to the already mentioned properties of g, we remark that $g(\cdot;\lambda)$ has the minimum at $x=-1$, $g(-1;\lambda)=\lambda-1$, and $g:[-1,-q^{1-p}]\to[q^{-p}-1,-q^{1-p}]$. Hence, if we choose $x_1=q^{-p}-1$, then the approximating sequence $x_{k+1}=g(x_k;q^{-p})$ is strictly increasing and converges to $-q^{1-p}$. Consequently, $\{\delta_k\}$ defined by (4.10) with $\delta_1=\delta-1$ converges as well, and permits $\delta_k\leq x_k<-q^{1-p}$. Thus $\{\varphi_k\}$ converges by (4.9), and so (1.1) is non-oscillatory by Theorem 3.1. □

Remark 4.5. Theorems 4.3 and 4.4 were proved also in [3], using a different technique. See also [1, Section 2.3.1].

Now we give an example of an equation involving parameters which, in particular, can be proved to be oscillatory using our new results, but other known criteria are inapplicable.

Example 4.6. Let $r(t)=t^{(1-q)t}(1+\log t)^{q-1}$ and $c(t)=t^{pt}[\lambda t^{-t}(1+\log t)+\gamma t^{-t}(1+\log t)\sin t+\gamma t^{-t}\cos t]$ in equation (1.1), where $\lambda>\gamma>0$. It is easy to see that $c(t)>0$ for large t and $R(t)=t^{-t}$. Further,

$$\begin{aligned}
&R^{-1}(t)\int_t^\infty R^p(s)c(s)\,ds\\
&=t^t\int_t^\infty \big[\lambda s^{-s}(1+\log s)+\gamma s^{-s}(1+\log s)\sin s+\gamma s^{-s}\cos s\big]\,ds\\
&=t^t(\lambda t^{-t}+\gamma t^{-t}\sin t)\\
&=\lambda+\gamma\sin t.
\end{aligned}$$

If $\lambda+\gamma\leq q^{-p}$, then (1.1) is non-oscillatory by Theorem 4.4. If $\lambda-\gamma>q^{-p}$, then (1.1) is oscillatory by Theorem 4.3. Thus next we assume $\lambda-\gamma\leq q^{-p}$ and $\lambda+\gamma>1$. Then Theorem 4.3 cannot be applied, but (1.1) is oscillatory by Corollary 4.2. Now assume that $\lambda+\gamma\leq 1$ and $\lambda+\gamma+f(\lambda+\gamma-1)>0$, where $f(x)=px+(p-1)|x|^q$. Then Corollary 4.2 cannot be applied, but (1.1) is oscillatory by Theorem 4.1 with

$k = 1$. Indeed, this follows from the equality

$$R^{-1}(t)\mathcal{S}(\varphi_1)(t) = \lambda + \gamma \sin t + t^t \int_t^\infty s^{-s}(1+\log s)\big[p(\lambda + \gamma \sin s - 1) + (p-1)|\lambda + \gamma \sin s - 1|^q\big]\, ds.$$

It is easy to see that the sets of λ's and γ's, which satisfy these requirements, are nonempty. Using Theorem 4.1 with $k \geq 2$ we can similarly handle the cases where $\lambda + \gamma + f(\lambda + \gamma - 1)$ is nonpositive, but is not "too negative".

Next we prove a Hille-Wintner type comparison theorem. Along with (1.1) consider the equation

$$(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0, \tag{4.11}$$

where $\tilde{c}$ is continuous on $[a, \infty)$.

Theorem 4.7. *Let $c(t) \geq 0$ and*

$$\int_t^\infty R^p(s)c(s)\, ds \geq \int_t^\infty R^p(s)\tilde{c}(s)\, ds \geq 0 \tag{4.12}$$

for large t. If (1.1) is non-oscillatory, then (4.11) is non-oscillatory.

Proof. If (1.1) is non-oscillatory, then $\{\varphi_k\}$ is well defined and $\lim_{k\to\infty} \varphi_k(t) = \varphi(t)$ by Theorem 3.1. The following computations hold for large t. From condition (4.12), we have $H(t) \geq R^{-p}(t)\int_t^\infty R^p(s)\tilde{c}(s)\, ds =: \tilde{H}(t)$. Then $\varphi_1(t) = H(t) + \mathcal{G}(\varphi_0)(t) \geq \tilde{H}(t) + \mathcal{G}(\varphi_0)(t) =: \tilde{\varphi}_1(t)$. Clearly, $\tilde{\varphi}_1(t) \geq \varphi_0(t) =: \tilde{\varphi}_0(t)$. By induction, $\varphi_{k+1}(t) \geq \tilde{H}(t) + \mathcal{G}(\tilde{\varphi}_k)(t) =: \tilde{\varphi}_{k+1}(t)$, $k = 0, 1, 2, \dots$. Moreover, $\tilde{\varphi}_k(t) \leq \varphi(t)$ and $\tilde{\varphi}_k(t) \leq \tilde{\varphi}_{k+1}(t)$, $k = 0, 1, 2, \dots$. Consequently, (4.11) is non-oscillatory by Theorem 3.1 and Remark 3.2 (ii). $\square$

Remark 4.8. (i) This theorem was established also in [3] by direct using of the Riccati technique. See also [1, Section 2.3.1]. Notice however that here we do not require $\tilde{c}$ to be nonnegative.

(ii) Under the conditions of the theorem, oscillation of (4.11) implies oscillation of (1.1).

(iii) From Hille-Nehari type criteria (Theorem 4.3 and Theorem 4.4) we get that the generalized Euler differential equation

$$(r(t)\Phi(y'))' + \lambda r^{1-q}(t)R^{-p}(t)\Phi(y) = 0 \tag{4.13}$$

is oscillatory if and only if $\lambda > q^{-p}$. Note that $y = R^{(p-1)/p}$ is a nonoscillatory solution of (4.13) with $\lambda = q^{-p}$. Observe that, conversely, knowing this result, Theorems 4.3 and 4.4 can be alternatively obtained by the Hille-Wintner type result comparing equation (1.1) with equation (4.13). Similar but a little bit more complicated approach to establish these theorems was used in [3]: The proofs there are based on a knowledge of oscillation behavior of certain generalized Euler differential equation (which a special case of (4.13)), Hille-Wintner type comparison theorem, and a transformation of independent variable. At any rate, we believe that the approach via the function sequence technique has an advantage over this comparison method in cases where a transformation is not available or guessing a solution is difficult. This may concern, e.g., a discrete counterpart of (1.1), a half-linear difference equation.

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Pavel Řehák
Institute of Mathematics, Academy of Sciences, Žižkova 22, CZ61662 Brno, Czech Republic
E-mail address: rehak@math.muni.cz