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POSITIVE SOLUTIONS FOR SINGULAR THREE-POINT BOUNDARY-VALUE PROBLEMS

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Abstract. Using the theory of fixed point index, this paper discusses the existence of at least one positive solution and the existence of multiple positive solutions for the singular three-point boundary value problem:

$$y''(t) + a(t)f(t, y(t), y'(t)) = 0, \quad 0 < t < 1,$$
$$y'(0) = 0, \quad y(1) = \alpha y(\eta),$$

where $0 < \alpha < 1$, $0 < \eta < 1$, and f may be singular at $y = 0$ and $y' = 0$.

1. Introduction

In this paper, we consider the singular three-point boundary-value problem (BVP):

$$y''(t) + a(t)f(t, y(t), y'(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$
$$y'(0) = 0, \quad y(1) = \alpha y(\eta), \tag{1.2}$$

where $0 < \alpha < 1$, $0 < \eta < 1$, f may be singular at $y = 0$ and $y' = 0$, and $a \in C((0, 1), (0, \infty))$.

When $f(t, x, z)$ has no singularity at $x = 0$ and $z = 0$, there are many results on the existence of solutions to (1.1)-(1.2) with different boundary conditions such as $x(0) = 0$, $x(1) = \delta x(\eta)$, or $x(0) = x_0$, $x(\eta) - x(1) = x_1$ (see [4, 5, 7, 8, 9]). Also when $f(t, x, z) = f(t, x)$ has no singularity at $x = 0$, using Krasnoselkii's fixed point theorem, Liu citel1 discussed the existence of positive solutions to (1.1)-(1.2). In [3, 14], the authors obtained the existence of at least one positive solutions to (1.1)-(1.2) when $f(t, x, z)$ is singular at $x = 0$ and $z = 0$.

The features in this article, that differ from those in [3, 14], are as follows. Firstly, the nonlinearity $f(t, x, z)$ may be sublinear in x at $x = +\infty$ and the degree of singularity in x and z may be arbitrary; i.e., $f(t, x, z)$ contains $\frac{1}{x^\alpha}$, x^β and $\frac{1}{(-z)^{-\gamma}}$ for any $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. Secondly, (1.1)-(1.2) may have at least two positive solutions. Thirdly, (1.1)-(1.2) may have no positive solutions.

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There are main five sections in our paper. In sections 2, we discuss a special Banach space and define a new cone in this space, and some lemmas are proved for convenience. In section 3, we discuss the nonexistence of positive solutions to (1.1)-(1.2). In section 4, the existence of at least one positive solution to (1.1)-(1.2) is presented when $f(t,x,z)$ is singular at $x=0$ and $z=0$. In section 5, we consider the existence of at least two positive solutions to (1.1)-(1.2) when $f(t,x,z)$ is singular at $x=0$ and $z=0$ and f is suplinear at $x=+\infty$. Some of the ideas in this paper were motivated from [1, 2, 12, 13].

2. Preliminaries

Let

$$C^1[0,1]=\{y:[0,1]\to R:\ y(t) \text{ and } y'(t) \text{are continuous on } [0,1]\}$$

with norm $\|y\|=\max\{\max_{t\in[0,1]}|y(t)|,\max_{t\in[0,1]}|y'(t)|\}$ and

$$P=\{y\in C^1[0,1]:y(t)\geq 0,\forall t\in[0,1]\}.$$

Obviously $C^1[0,1]$ is a Banach space and P is a cone in $C^1[0,1]$. The following lemmas are needed later.

Lemma 2.1 (citeg3)**.** *Let Ω be a bounded open set in real Banach space E, P be a cone of E, $\theta\in\Omega$ and $A:\bar{\Omega}\cap P\to P$ be continuous and compact. Suppose*

$$\lambda Ax\neq x,\quad \forall x\in\partial\Omega\cap P,\ \lambda\in(0,1], \tag{2.1}$$

then $i(A,\Omega\cap P,P)=1$.

Lemma 2.2 ([6])**.** *Let Ω be a bounded open set in real Banach space E, P be a cone of E, $\theta\in\Omega$ and $A:\bar{\Omega}\cap P\to P$ be continuous and compact. Suppose*

$$Ax\neq x,\quad \forall\ x\in\partial\Omega\cap P, \tag{2.2}$$

then $i(A,\Omega\cap P,P)=0$.

Lemma 2.3 ([11])**.** *Let $0<\alpha<1$, a, $h\in C((0,1),(0,\infty))$, $a,h\in L^1[0,1]$ and*

$$\begin{aligned} y(t)=\frac{1}{1-\alpha}\int_0^1\int_0^s a(\tau)h(\tau)\,d\tau\,ds-\frac{\alpha}{1-\alpha}\int_0^\eta\int_0^s a(\tau)h(\tau)\,d\tau\,ds\\ -\int_0^t\int_0^s a(\tau)h(\tau)\,d\tau\,ds. \end{aligned}$$

Then

$$\min_{t\in[0,1]}y(t)\geq\frac{\alpha(1-\eta)}{1-\alpha\eta}\max_{t\in[0,1]}|y(t)|. \tag{2.3}$$

Lemma 2.4. *Assume that $f\in C((0,1)\times(0,+\infty)\times(-\infty,0),(0,+\infty))$, that $a\in C((0,1),(0,+\infty))$, and that for any constant $H>0$ there exists a function $\Psi_H(t)$ continuous on $(0,1)$, and positive on $(0,1)$ and a constant $0\leq\gamma<1$ such that*

$$f(t,x,z)\geq\Psi_H(t)(-z)^\gamma,\quad \forall t\geq 0,\ 0<x\leq H,\ z<0, \tag{2.4}$$

where $\int_0^1 a(s)\Psi_H(s)ds<+\infty$. Then there is a $c_0>0$ such that for any positive solution $x\in C[0,1]$ with $x'(t)<0$ for all $t\in(0,1)$ to (1.1)-(1.2) we have

$$x(t)\geq c_0,\quad t\in[0,1]. \tag{2.5}$$

Moreover, if $x_0 \in C[0,1]$ *is a positive solution to*

$$y''(t) + a(t)f(t, \max\{c_0, y(t)\}, -|y'(t)| - \frac{1}{n}) = 0, \quad 0 < t < 1,$$
$$y'(0) = 0, \quad y(1) = \alpha y(\eta),$$

where $\frac{\alpha}{1-\alpha}(1-\eta)\frac{1}{n} < c_0$*,* x_0 *is a positive solution to*

$$y''(t) + a(t)f(t, y(t), -|y'(t)| - \frac{1}{n}) = 0, \quad 0 < t < 1,$$
$$y'(0) = 0, \quad y(1) = \alpha y(\eta).$$

Proof. Assume that x is a positive solution to (1.1)-(1.2) with $x'(t) < 0$ for $t \in (0,1)$. Then Lemma 2.3 implies $\min_{t\in[0,1]} x(t) \ge \frac{\alpha(1-\eta)}{1-\alpha\eta} \max_{t\in[0,1]} |x(t)| > 0$.

Let $H = 1$. Then there exists a function $\Psi_1(t)$ continuous on $[0,1]$, and positive on $(0,1)$ a constant $0 \le \gamma < 1$ such that

$$f(t,x,z) \ge \Psi_1(t)(-z)^\gamma, \quad \forall t \ge 0,\ 0 < x \le 1,\ z < 0.$$

There are two cases to be considered: (1) $x(t) \ge 1$ for all $t \in [0,1]$. (2) $x(1) < 1$. Let $t_* = \inf\{t | x(t) < 1 \text{ for all } s \in [t,1]\}$. If $t_* > 0$, we have $x(t_*) = 1$ and $x(0) \ge 1$. Then, Lemma 2.3 yields

$$\min_{t\in[0,1]} |x(t)| \ge \frac{\alpha(1-\eta)}{1-\alpha\eta} \max_{t\in[0,1]} |x(t)| \ge \frac{\alpha(1-\eta)}{1-\alpha\eta}. \tag{2.6}$$

If $t_* = 0$ and $x(t_*) = 1$, Lemma 2.3 implies

$$\min_{t\in[0,1]} |x(t)| \ge \frac{\alpha(1-\eta)}{1-\alpha\eta} \max_{t\in[0,1]} |x(t)| \ge \frac{\alpha(1-\eta)}{1-\alpha\eta} \tag{2.7}$$

also. If $t_* = 0$ and $x(t_*) < 1$, from (2.4), we have

$$-x''(t) = a(t)f(t, x(t), x'(t)) \ge a(t)\Psi_1(t)(-x'(t))^\gamma, \quad t \in (0,1).$$

Also note

$$-\frac{x''(t)}{(-x'(t))^\gamma} \ge a(t)\Psi_1(t), \quad t \in (0,1).$$

Integrating from 0 to t, we have

$$\frac{1}{1-\gamma}(-x'(t))^{1-\gamma} \ge \int_0^t a(s)\Psi_1(s)ds, t \in (0,1),$$

which implies

$$-x'(t) \ge [(1-\gamma)\int_0^t a(s)\Psi_1(s)ds]^{\frac{1}{1-\gamma}}, \quad t \in (0,1).$$

Integration from η to 1 yields

$$x(\eta) - x(1) \ge \int_\eta^1 [(1-\gamma)\int_\eta^1 \int_0^t a(s)\Psi_1(s)ds]^{\frac{1}{1-\gamma}} dt.$$

Since $x(1) = \alpha x(\eta)$, we have

$$x(1) \ge \frac{\alpha}{1-\alpha} \int_\eta^1 [(1-\gamma)\int_0^t a(s)\Psi_1(s)ds]^{\frac{1}{1-\gamma}} dt. \tag{2.8}$$

Let $c_0 = \frac{1}{2}\min\{1, \frac{\alpha(1-\eta)}{1-\alpha\eta}, \frac{\alpha}{1-\alpha}\int_\eta^1 [(1-\gamma)\int_0^t a(s)\Psi_1(s)ds]^{\frac{1}{1-\gamma}}dt\}$. Combining (2.6), (2.7) and (2.8), we have

$$\min_{t\in[0,1]} x(t) \geq c_0.$$

Suppose that x_0 satisfies

$$\begin{gathered} x_0''(t) + a(t)f(t, \max\{c_0, x_0(t)\}, -|x_0'(t)| - \frac{1}{n}) = 0, \quad t \in (0,1), \\ x_0'(0) = 0, \quad x_0(\eta) = \alpha x_0(1). \end{gathered}$$

Then $x_0''(t) < 0$ and so $x_0(t) < 0$ for $t \in (0,1)$. Then x_0 satisfies

$$\begin{gathered} x_0''(t) + a(t)f(t, \max\{c_0, x_0(t)\}, x_0'(t) - \frac{1}{n}) = 0, \quad t \in (0,1), \\ x_0'(0) = 0, \quad x_0(\eta) = \alpha x_0(1). \end{gathered}$$

There are are two cases to be considered:
(1) $x_0(t) \geq 1$ for all $t \in [0,1]$. In this case, since $c_0 \leq 1$, we have

$$x_0''(t) + a(t)f(t, \max\{c_0, x_0(t)\}, x_0'(t) - \frac{1}{n}) = x_0''(t) + a(t)f(t, x_0(t), x_0'(t) - \frac{1}{n}) = 0,$$

for $0 < t < 1$.
(2) $x_0(1) < 1$. Let $t_* = \inf\{t | x_0(t) < 1 \text{ for all } s \in [t,1]\}$. If $t_* > 0$, we have $x_0(t_*) = 1$ and $x_0(0) \geq 1$. Then

$$\min_{t\in[0,1]} x_0(t) \geq \frac{\alpha(1-\eta)}{1-\alpha\eta} \max_{t\in[0,1]} |x_0(t)| \geq \frac{\alpha(1-\eta)}{1-\alpha\eta}.$$

If $t_* = 0$ and $x_0(t_*) = 1$, we have

$$\min_{t\in[0,1]} x_0(t) \geq \frac{\alpha(1-\eta)}{1-\alpha\eta} \max_{t\in[0,1]} |x_0(t)| \geq \frac{\alpha(1-\eta)}{1-\alpha\eta}$$

also. If $t_* = 0$ and $x_0(t_*) < 1$, from (2.4), we have

$$-x_0''(t) = a(t)f(t, \max\{c_0, x_0(t)\}, x_0'(t) - \frac{1}{n}) \geq a(t)\Psi_1(t)(-x_0'(t) + \frac{1}{n})^\gamma, \quad t \in (0,1).$$

Also note

$$-\frac{x_0''(t)}{(-x_0'(t) + \frac{1}{n})^\gamma} \geq a(t)\Psi_1(t), \quad t \in (0,1).$$

Integrating from 0 to t, we have

$$\frac{1}{1-\gamma}[(-x_0'(t) + \frac{1}{n})^{1-\gamma} - (\frac{1}{n})^{1-\gamma}] \geq \int_0^t a(s)\Psi_1(s)ds, t \in (0,1),$$

which implies

$$-x_0'(t) + \frac{1}{n} \geq [(1-\gamma)\int_0^t a(s)\Psi_1(s)ds]^{\frac{1}{1-\gamma}}, t \in (0,1).$$

Integration from η to 1 yields

$$x_0(\eta) - x_0(1) \geq \int_\eta^1 [(1-\gamma)\int_\eta^1 \int_0^t a(s)\Psi_1(s)ds]^{\frac{1}{1-\gamma}}]dt - (1-\eta)\frac{1}{n}.$$

Since $x_0(1) = \alpha x_0(\eta)$, we have

$$x_0(1) \geq \frac{\alpha}{1-\alpha}\int_\eta^1 [(1-\gamma)\int_\eta^1 \int_0^t a(s)\Psi_1(s)ds]^{\frac{1}{1-\gamma}}]dt - \frac{\alpha}{1-\alpha}(1-\eta)\frac{1}{n} \geq c_0.$$

Consequently, the definition of c_0 implies that $x_0(t) \geq c_0$ for all $t \in [0,1]$. Therefore,

$$x_0''(t) + a(t)f(t, \max\{c_0, x_0(t)\}, x_0'(t) - \frac{1}{n}) = x_0''(t) + a(t)f(t, x_0(t), x_0'(t) - \frac{1}{n}) = 0,$$

for $0 < t < 1$. The proof is complete. □

To discuss the existence of multiple positive solutions, we construct a new space. Let $q(t) = 1 - t$, $t \in [0,1]$ and

$$C_q^1[0,1] = \{y : [0,1] \to R : y(t) \text{ and } q(t)y'(t) \text{ are continuous on } [0,1]\}$$

with norm $\|y\|_q = \max\{\max_{t\in[0,1]} |y(t)|, \max_{t\in[0,1]} q(t)|y'(t)|\}$ and

$$P_q = \{y \in C_q^1[0,1] : \min_{t\in[0,1]} y(t) \geq \frac{\alpha(1-\eta)}{1-\alpha\eta} \max_{t\in[0,1]} |y(t)| \text{ and } y(0) \geq \max_{t\in[0,1]} q(t)|y'(t)|\}.$$

Lemma 2.5. *The set $C_q^1[0,1]$ is a Banach space and P_q is cone in $C_q^1[0,1]$*

Proof. It is easy to see that $\|\cdot\|_q$ is a norm of the space C_q^1. Now we show that C_q^1 is a Banach space. Assume that $\{x_n\}_{n=1}^{\infty} \subseteq C_q^1$ is a Cauchy sequence; i.e., for each $\varepsilon > 0$, there is a $N > 0$ such that

$$\|x_n - x_m\|_q < \varepsilon, \quad \forall n > N,\ m > N\,. \tag{2.9}$$

Then

$$\max_{t\in[0,1]} |x_n(t) - x_m(t)| < \varepsilon, \quad \forall n > N,\ m > N.$$

Thus, there is a $x_0 \in C[0,1]$ such that

$$\lim_{n\to+\infty} \max_{t\in[0,1]} |x_n(t) - x_0(t)| = 0. \tag{2.10}$$

For $1 > \delta > 0$, since $(1-\delta)\max_{t\in[0,1-\delta]} |x_n'(t) - x_m'(t)| \leq \max_{t\in[0,1-\delta]} q(t)|x_n'(t) - x_m'(t)|$, we have

$$\max_{t\in[0,1-\delta]} |x_n'(t) - x_m'(t)| \leq \frac{1}{1-\delta} \max_{t\in[0,1-\delta]} q(t)|x_n'(t) - x_m'(t)| < \frac{1}{1-\delta}\varepsilon,$$

which implies that for any $\delta > 0$, $x_n'(t)$ is uniformly convergent on $[0, 1-\delta]$. Hence, $x_0(t)$ is continuously differentiable on $[0,1)$. And since $q(t)x_{N+1}'(t)$ is uniformly continuous on $[0,1]$, there exists a $\delta' > 0$ such that

$$|q(t_1)x_{N+1}'(t_1) - q(t_2)x_{N+1}'(t_2)| < \varepsilon \quad \text{for } |t_1 - t_2| < \delta, t_1, t_2 \in [0,1).$$

Then

$$\begin{aligned}
&|q(t_1)x_0'(t_1) - q(t_2)x_0'(t_2)| \\
&= |q(t_1)x_0'(t_1) - q(t_1)x_{N+1}'(t_1) \\
&\quad + q(t_1)x_{N+1}'(t_1) - q(t_2)x_{N+1}'(t_2) + q(t_2)x_{N+1}'(t_2) - q(t_2)x_0'(t_1)| \\
&\leq |q(t_1)x_0'(t_1) - q(t_1)x_{N+1}'(t_1)| \\
&\quad + |q(t_1)x_{N+1}'(t_1) - q(t_2)x_{N+1}'(t_2)| + |q(t_2)x_{N+1}'(t_2) - q(t_2)x_0'(t_1)| \\
&< 3\varepsilon, \quad \text{for } |t_1 - t_2| < \delta',\ t_1, t_2 \in [0,1),
\end{aligned}$$

which implies that $\lim_{t\to1^-} q(t)x_0'(t)$ exists. Let $q(1)x_0(1) = \lim_{t\to1^-} q(t)x_0'(t)$. Now from (2.9), we have for any $t \in [0,1]$,

$$q(t)|x_n'(t) - x_m'(t)| < \varepsilon, \quad \forall n > N, m > N.$$

Letting $m \to +\infty$, for all $t \in [0,1]$, we have

$$q(t)|x_n'(t) - x_0'(t)| \le \varepsilon, \quad \forall n > N. \tag{2.11}$$

Combining (2.10) and (2.11) shows $C_q^1[0,1]$ is a Banach space.

Clearly P_q is a cone of $C_q^1[0,1]$. The proof is complete. □

Lemma 2.6. *For each $y \in P_q$, $\|y\|_q = \max_{t\in[0,1]} |y(t)|$.*

Proof. For $y \in P$, obviously $\|y\|_q \ge \max_{t\in[0,1]} |y(t)|$. On the other hand, since $y \in P_q$,

$$\max_{t\in[0,1]} |y(t)| \ge y(0) \ge \max_{t\in[0,1]} q(t)|y'(t)|.$$

Then

$$\begin{aligned}\|y\|_q &= \max\{\max_{t\in[0,1]} |y(t)|, \max_{t\in[0,1]} q(t)|y'(t)|\}\\ &\le \max\{\max_{t\in[0,1]} |y(t)|, y(0)\} = \max_{t\in[0,1]} |y(t)|.\end{aligned}$$

Consequently, $\|y\|_q = \max_{t\in[0,1]} |y(t)|$. The proof is complete. □

Now we list the following conditions to be used in this article.

(H) $f \in C((0,1)\times(0,\infty)\times(-\infty,0),(0,\infty))$ and there are three functions $g, h \in C((0,+\infty),(0,+\infty))$, $\Phi \in C((0,1),[0,+\infty))$, with $\Phi(t) > 0$ for all $t \in (0,1)$, and

$$f(t,x,z) \le \Phi(t)h(x)g(|z|) \quad \forall\,(t,x,z) \in (0,1)\times(0,+\infty)\times(-\infty,0). \tag{2.12}$$

(H') For any constant $H > 0$ there exists a function $\Psi_H(t)$ continuous on $(0,1)$ and positive on $(0,1)$, and a constant $0 \le \gamma < 1$ such that

$$f(t,x,z) \ge \Psi_H(t)(-z)^\gamma, \quad \forall t \in (0,1),\ 0 < x \le H,\ z < 0, \tag{2.13}$$

where $\int_0^1 a(s)\Psi_H(s)ds < +\infty$.

For each $n \in N = \{1,2,\dots\}$, for $y \in P$ (or $y \in P_q$), define operators

$$\begin{aligned}(A_n y)(t) = &\frac{1}{1-\alpha}\int_0^1\int_0^s a(\tau)f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)| - \frac{1}{n})\,d\tau\,ds\\ &- \frac{\alpha}{1-\alpha}\int_0^\eta\int_0^s a(\tau)f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)| - \frac{1}{n})\,d\tau\,ds\\ &- \int_0^t\int_0^s a(\tau)f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)| - \frac{1}{n})\,d\tau\,ds,\end{aligned} \tag{2.14}$$

for $t \in [0,1]$ and $c_0 > 0$.

Suppose that (H) and (H') hold. A standard argument (see [6, 7]) applied to (2.14) yields that $A_n : P \to P$ is continuous and completely continuous for each $n \in N$.

Lemma 2.7. *Suppose* (H) *and* (H') *holds and $\int_0^1 a(t)\Phi(t)\sup_{\frac{1}{c}\le u\le \frac{1}{c}+\frac{1}{1-t}c} g(u)dt < +\infty$ for all $c > 1$. Then $A_n : P_q \to P_q$ is a continuous and completely continuous for each $n \in N$.*

Proof. For $y \in P_q$, it is easy to see that $|y'(t)| \le \frac{1}{1-t}\|y\|_q$ for all $t \in [0,1)$. Also (H) and Lemma 2.3 yield

$$\frac{\alpha(1-\eta)}{1-\alpha\eta}\max_{t\in[0,1]} |(A_n y)(t)| \le (A_n y)(t) < +\infty, \quad \forall t \in [0,1]$$

and $(A_ny)'(t) > -\infty$ for all $t \in [0,1)$. Moreover, since

$$\begin{aligned}(A_ny)(0) &= \frac{1}{1-\alpha}\int_0^1\int_0^s a(\tau)f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})\,d\tau\,ds\\ &\quad -\frac{\alpha}{1-\alpha}\int_0^\eta\int_0^s a(\tau)f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})\,d\tau\,ds\\ &\geq \int_0^1\int_0^s a(\tau)f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|)\,d\tau\,ds\\ &= \int_0^1(1-s)a(s)f(s,\max\{c_0,y(s)\},-|y'(s)|-\frac{1}{n})ds\end{aligned}$$

and

$$\begin{aligned}q(t)|(A_nx)'(t)| &= (1-t)\int_0^t a(s)f(s,\max\{c_0,y(s)\},-|y'(s)|-\frac{1}{n})ds\\ &\leq \int_0^1(1-s)a(s)f(s,\max\{c_0,y(s)\},-|y'(s)|-\frac{1}{n})ds,\end{aligned}$$

we have

$$(A_nx)(0) \geq \max_{t\in[0,1]} q(t)|(A_nx)'(t)|.$$

Consequently, $A_nP_q \subseteq P_q$ for each $n \in N = \{1,2,\dots\}$. Moreover, since

$$\lim_{t\to 1^-}|(A_ny)'(t)| = \int_0^1 a(s)f(s,\max\{c_0,y(s)\},|y'(s)|-\frac{1}{n})ds,$$

we can assume that $A_ny \in C^1[0,1]$.

Next we show that $A_n : P_q \to P_q$ is continuous and completely continuous. Suppose that $\{y_m\} \subseteq P_q$, $y_0 \in P_q$ with $\lim_{m\to+\infty}\|y_m - y_0\|_q = 0$. Then, there is an $M > c_0$ such that

$$\|y_m\|_q \leq M, \|y_0\|_q \leq M, \quad m \in N.$$

Then $|y_m'(t)| \leq M/(1-t)$ for $m \in \{1,2,\dots\}$ and so

$$f(t,\max\{c_0,y_m(t)\},-|y_m'(t)|-\frac{1}{n}) \leq \Phi(t)\max_{c_0\leq u\leq M}h(u)\sup_{\frac{1}{n}\leq u\leq\frac{1}{n}+\frac{1}{1-t}M}g(u),$$

for $t \in (0,1)$. Moreover, since

$$\lim_{m\to+\infty} f(t,\max\{c_0,y_m(t)\},-|y_m'(t)|-\frac{1}{n}) = f(t,\max\{c_0,y_0(t)\},-|y_0'(t)|-\frac{1}{n}),$$

for $t \in (0,1)$, the Lebesgue Dominated Convergence Theorem guarantees that

$$\begin{aligned}&\max_{t\in[0,1]}|(A_ny_m)(t)-(A_ny_0)(t)|\\ &\leq \frac{1}{1-\alpha}\int_0^1\int_0^s a(\tau)\Big|f(\tau,\max\{c_0,y_m(\tau)\},-|y_m'(\tau)|-\frac{1}{n})\\ &\quad -f(\tau,\max\{c_0,y_0(\tau)\},-|y_0'(\tau)|-\frac{1}{n})\Big|\,d\tau\,ds\\ &\quad +\frac{\alpha}{1-\alpha}\int_0^\eta\int_0^s a(\tau)\Big||f(\tau,\max\{c_0,y_m(\tau)\},-|y_m'(\tau)|-\frac{1}{n})\\ &\quad -f(\tau,\max\{c_0,y_0(\tau)\},-|y_0'(\tau)|-\frac{1}{n})Big|\,d\tau\,ds\end{aligned}$$

$$+\int\int_0^1\int_0^s a(\tau)\Big|f(\tau,\max\{c_0,y_m(\tau)\},-|y_m'(\tau)| \\ -\frac{1}{n})-f(\tau,\max\{c_0,y_0(\tau)\},-|y_0'(\tau)|-\frac{1}{n})\Big|\,d\tau\,ds\to 0,\quad \text{as } m\to+\infty.$$

Since A_ny_m, $A_ny_0\in P_q$, Lemma 2.6 yields

$$\lim_{m\to+\infty}\|A_ny_m-A_ny_0\|_q=\max_{t\in[0,1]}|(A_ny_m)(t)-(A_ny_0)(t)|=0,$$

which implies that $A_n:P_q\to P_q$ is continuous.

Suppose $D\subseteq P_q$ is bounded. Then, there is an $M>c_0$ such that $\|y\|_q\le M$ for all $y\in D$. Then $|y'(t)|\le M/(1-t)$ for all $y\in D$, and so

$$f(t,\max\{c_0,y(t)\},-|y'(t)|-\frac{1}{n})\le\Phi(t)\max_{c_0\le u\le M}h(u)\sup_{\frac{1}{n}\le u\le\frac{1}{n}+\frac{1}{1-t}M}g(u),\quad t\in(0,1).$$

Thus

$$\begin{aligned}
&\max_{t\in[0,1]}|(A_ny)(t)|\\
&\le\frac{1}{1-\alpha}\int_0^1\int_0^s a(\tau)|f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})|\,d\tau\,ds\\
&\quad+\frac{\alpha}{1-\alpha}\int_0^\eta\int_0^s a(\tau)|f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})|\,d\tau\,ds\\
&\quad+\int_0^1\int_0^s a(\tau)|f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})|\,d\tau\,ds\\
&\le\frac{1}{1-\alpha}\int_0^1\int_0^s a(\tau)\Phi(s)\max_{c_0\le u\le M}h(u)\sup_{\frac{1}{n}\le u\le\frac{1}{n}+\frac{1}{1-\tau}M}g(u)\,d\tau\,ds\\
&\quad+\frac{\alpha}{1-\alpha}\int_0^\eta\int_0^s\Phi(s)a(\tau)\max_{c_0\le u\le M}h(u)\sup_{\frac{1}{n}\le u\le\frac{1}{n}+\frac{1}{1-\tau}M}g(u)\,d\tau\,ds\\
&\quad+\int_0^1\int_0^s a(\tau)\Phi(\tau)\max_{c_0\le u\le M}h(u)\sup_{\frac{1}{n}\le u\le\frac{1}{n}+\frac{1}{1-\tau}M}g(u)\,d\tau\,ds
\end{aligned}$$

and

$$\begin{aligned}
&\max_{t\in[0,1]}|(A_ny)'(t)|\\
&\le\max_{t\in[0,1]}\int_0^t a(\tau)|f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})|d\tau\\
&\le\max_{t\in[0,1]}\int_0^t a(\tau)\Phi(\tau)\max_{c_0\le u\le M}h(u)\sup_{\frac{1}{n}\le u\le\frac{1}{n}+\frac{1}{1-\tau}M}g(u)d\tau.
\end{aligned}$$

Also A_nD is bounded in the norm $\|x\|_0=\max\{\max_{t\in[0,1]}|x(t)|,\max_{t\in[0,1]}|x'(t)|\}$. For t_1, $t_2\in[0,1]$, $y\in D$, we have

$$\begin{aligned}
&|(A_ny)(t_1)-(A_ny)(t_2)|\\
&=|\int_{t_1}^{t_2}\int_0^s a(\tau)\Phi(\tau)|f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})|\,d\tau\,ds|\\
&\le|\int_{t_1}^{t_2}\int_0^s a(\tau)\Phi(\tau)\max_{c_0\le u\le M}h(u)\sup_{\frac{1}{n}\le u\le\frac{1}{n}+\frac{1}{1-\tau}M}g(u)\,d\tau\,ds|
\end{aligned}$$

and

$$\begin{aligned}|(A_ny)'(t_1)-(A_ny)'(t_2)| &= |\int_{t_1}^{t_2} a(\tau)|f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})|d\tau| \\ &\le |\int_{t_1}^{t_2} a(\tau)\Phi(\tau)\max_{c_0\le u\le M} h(u) \sup_{\frac{1}{n}\le u\le \frac{1}{n}+\frac{1}{1-\tau}M} g(u)d\tau|,\end{aligned}$$

which implies that $\{(A_ny)(t)|y\in D\}$ and $\{(A_ny)'(t)|y\in D\}$ are equicontinuous on $[0,1]$.

The Arzela-Ascoli Theorem guarantees that A_nD and $(A_nD)'$ are relatively compact in $C[0,1]$. Since

$$\begin{aligned}\|A_ny\|_q &= \max\{\max_{t\in[0,1]}|(A_ny)(t)|, \max_{t\in[0,1]}(1-t)|(A_ny)'(t)|\} \\ &\le \max\{\max_{t\in[0,1]}|(A_ny)(t)|, \max_{t\in[0,1]}|(A_ny)'(t)|\},\end{aligned}$$

the set A_nD is relatively compact in $C_q^1[0,1]$. Consequently, $A_n: P_q\to P_q$ is continuous and completely continuous for each $n\in\{1,2,\dots\}$. The proof is complete. □

3. Nonexistence of positive solutions to (1.1)-(1.2)

In this section, we notice that the presence of z in $f(t,x,z)$ can lead to the nonexistence of positive solutions to (1.1)-(1.2).

Theorem 3.1. *Suppose* (H) *holds and* $\int_0^z \frac{1}{g(r)}dr=+\infty$ *for all* $z\in(0,+\infty)$ *and* $\int_0^1 a(s)\Phi(s)ds<+\infty$. *Then* (1.1)-(1.2) *has no positive solution.*

Proof. Suppose $x_0(t)$ is a positive solution to (1.1)-(1.2). Then

$$\begin{gathered}x_0''(t)+a(t)f(t,x_0(t),x_0'(t))=0,\quad t\in(0,1)\\ x_0'(0)=0,\quad x_0(1)=\alpha x_0(\eta),\end{gathered}$$

which means that there is a $t_0\in(0,1)$ with $x_0'(t_0)<0$, $x_0(t_0)>0$ (otherwise $x'(t)\ge 0$ for all $t\in(0,1)$ which would contradict $x(1)=\alpha x(\eta)<x(\eta)$). Let $t_*=\inf\{t<t_0|x_0'(s)<0 \text{ for all } s\in[t,t_0]\}$. Obviously, $t_*\ge 0$ and $x_0'(t_*)=0$ and $x_0'(t)<0$ for all $t\in(t_*,t_0]$. Condition (H) implies

$$-x_0''(t)\le a(t)f(t,x_0(t),x_0'(t))\le a(t)\Phi(t)h(x_0(t))g(|x_0'(t)|),\quad \forall t\in(t_*,t_0),$$

and so

$$\frac{-x_0''(t)}{g(-x_0'(t))}\le a(t)\Phi(t)h(x_0(t))\le a(t)\Phi(t)h(x_0(t)),\quad \forall t\in(t_*,t_0).$$

Integration from t to t_0 yields

$$\int_{-x_0'(t)}^{-x_0'(t_0)}\frac{1}{g(r)}dr=\int_t^{t_0}\frac{1}{g(-x_0'(s))}d(-x_0'(s))\le \max_{u\in[x_0(t_0),x_0(t)]}h(u)\int_0^1 a(s)\Phi(s)ds.$$

Letting $t\to t_*$, we have

$$+\infty=\int_0^{-x_0'(t_0)}\frac{1}{g(r)}dr\le \max_{u\in[x_0(t_0),x_0(t_*)]}h(u)\int_0^1 a(s)\Phi(s)ds<+\infty,$$

a contradiction. Consequently, (1.1)-(1.2) has no positive solution. □

Example 3.2. Consider the boundary-value problem

$$x'' + (1-t)^a(|x'|)^a[x^b + (x+1)^{-d} + 1] = 0, \quad t \in (0,1),$$
$$x(0) = 0, \quad x(1) = \frac{1}{2}x(\frac{1}{2}),$$

where $a \geq 1$, $b > 1$, $d > 0$. This problem has no positive solution. It is easy to see that $f(t,x,z) = (1-t)^a(|z|)^a[x^b + (x+1)^{-d} + 1]$ for all $(t,x,z) \in [0,1] \times [0,+\infty) \times (-\infty,+\infty)$. Obviously, $g(r) = r^a$ and $\int_0^z \frac{1}{g(r)}dr = +\infty$ for all $z \in (0,+\infty)$. Then Theorem 3.1 guarantees that (3.2)-(3.2) has no positive solution.

4. Existence of at least one positive solution to (1.1)-(1.2)

In this section our nonlinearity f may be singular at $y' = 0$ and $y = 0$ and Φ . Throughout this section we will assume that the following conditions hold:

- (H1) $a(t) \in C(0,1)$, $a(t) > 0$ for all $t \in (0,1)$;
- (H2) Conditions (H) and (H') hold and $I(z) = \int_0^z \frac{1}{g(r)}dr < +\infty$ for all $z \in [0,+\infty)$ with
$$\sup_{c_0 \leq r \leq c} h(r) \int_0^1 a(s)\Phi(s)ds < \int_0^\infty \frac{dr}{g(r)}$$
for all $c \in [c_0,+\infty)$ and suppose
$$\sup_{c_0 \leq c < +\infty} \frac{c}{\frac{1-\alpha\eta}{1-\alpha} I^{-1}(\sup_{c_0 \leq r \leq c} h(r) \int_0^1 a(s)\Phi(s)ds)} > 1,$$
where c_0 is defined in Lemma 2.4.

Theorem 4.1. *Suppose that* (H1)–(H2) *hold. Then* (1.1)-(1.2) *has at least one positive solution $y_0 \in C[0,1] \cap C^2(0,1)$ with $y_0(t) > 0$ on $[0,1]$ and $y_0'(t) < 0$ on $(0,1)$.*

Proof. Choose $R_1 > 0$ with

$$\frac{R_1}{\frac{1-\alpha\eta}{1-\alpha} I^{-1}(\sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s)\Phi(s)ds)} > 1. \tag{4.1}$$

From the continuity of I^{-1} and I, we can choose $\varepsilon > 0$ and $\varepsilon < R_1$ with

$$\frac{R_1}{\frac{1-\alpha\eta}{1-\alpha} I^{-1}(\sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s))\Phi(s)ds) + I(\varepsilon))} > 1. \tag{4.2}$$

Let $n_0 \in \{1,2,\dots\}$ with $\frac{1}{n_0} < \min\{\varepsilon, \frac{1}{2}\frac{1-\alpha}{\alpha(1-\eta)}c_0\}\}$ and let $N_0 = \{n_0, n_0+1, \dots\}$. Now (H1)–(H2) guarantee that for each $n \in N_0$, $A_n : P \to P$ is a continuous and completely continuous operator.

Let $\Omega_1 = \{y \in C^1[0,1] : \|y\| < R_1\}$. We now show that

$$y \neq \mu A_n y, \quad \forall y \in P \cap \partial\Omega_1, \mu \in (0,1], n \in N_0. \tag{4.3}$$

Suppose there exists a $y_0 \in P \cap \partial\Omega_1$ and a $\mu_0 \in (0,1]$ such that $y_0 = \mu_0 A_n y_0$. It is easy to see that $y_0'(t) \leq 0$ and

$$y_0'(t) = -\mu_0 \int_0^t a(s)f(s, \max\{c_0, y_0(s)\}, y_0'(s) - \frac{1}{n})ds, t \in (0,1). \tag{4.4}$$

Also

$$y_0''(t) + \mu_0 a(t) f(t, \max\{c_0, y_0(t)\}, y_0'(t) - \frac{1}{n}) = 0, \quad 0 < t < 1, \tag{4.5}$$

$$y_0'(0) = 0, y_0(1) = \alpha y_0(\eta). \tag{4.6}$$

Therefore,

$$\begin{aligned} -y_0''(t) &= \mu_0 a(t) f(t, \max\{c_0, y_0(t)\}, y_0'(t) - \frac{1}{n}) \\ &\le a(t)\Phi(t) h(\max\{c_0, y_0(t)\}) g(-y_0'(t) + \frac{1}{n}), \forall t \in (0,1), \end{aligned}$$

which yields

$$\frac{-y_0''(t)}{g(-y_0'(t) + \frac{1}{n})} \le a(t)\Phi(t) h(\max\{c_0, y_0(t)\}), \quad \forall t \in (0,1).$$

Integration from 0 to t yields

$$\begin{aligned} I(-y_0'(t) + \frac{1}{n}) - I(\frac{1}{n}) &\le \int_0^t a(s)\Phi(s) h(\max\{c_0, y_0(s)\}) ds \\ &\le \sup_{c_0 \le r \le R_1} h(r) \int_0^t a(s)\Phi(s) ds, \end{aligned}$$

and so

$$I(-y_0'(t) + \frac{1}{n}) \le \sup_{c_0 \le r \le R_1} h(r) \int_0^t a(s)\Phi(s) ds + I(\varepsilon).$$

Thus

$$-y_0'(t) \le I^{-1}\Big(\sup_{c_0 \le r \le R_1} h(r) \int_0^1 a(s)\Phi(s) ds + I(\varepsilon)\Big), \quad t \in (0,1). \tag{4.7}$$

Therefore,

$$y_0(t) - y_0(1) \le (1-t) I^{-1}\Big(\sup_{c_0 \le r \le R_1} h(r) \int_0^1 a(s)\Phi(s) ds + I(\varepsilon)\Big). \tag{4.8}$$

Let $t = \eta$ in (4.8). Then

$$y_0(\eta) - y_0(1) \le (1-\eta) I^{-1}\Big(\sup_{c_0 \le r \le R_1} h(r) \int_0^1 a(s)\Phi(s) ds + I(\varepsilon)\Big).$$

Since $y_0(1) = \alpha y_0(\eta)$, one has

$$(\frac{1}{\alpha} - 1) y_0(1) \le (1-\eta) I^{-1}\Big(\sup_{c_0 \le r \le R_1} h(r) \int_0^1 a(s)\Phi(s) ds + I(\varepsilon)\Big),$$

which yields

$$y_0(1) \le \frac{\alpha}{1-\alpha}(1-\eta) I^{-1}\Big(\sup_{c_0 \le r \le R_1} h(r) \int_0^1 a(s)\Phi(s) ds + I(\varepsilon)\Big).$$

Then (4.8) implies

$$\begin{aligned} y_0(0) &\le y_0(1) + I^{-1}\Big(\sup_{c_0 \le r \le R_1} h(r) \int_0^1 a(s)\Phi(s) ds + I(\varepsilon)\Big) \\ &= \frac{1-\alpha\eta}{1-\alpha} I^{-1}\Big(\sup_{c_0 \le r \le R_1} h(r) \int_0^1 a(s)\Phi(s) ds + I(\varepsilon)\Big). \end{aligned} \tag{4.9}$$

Now (4.7) and (4.9) guarantee that

$$\begin{aligned} R_1 &= \max\{\max_{t\in[0,1]}|y_0(t)|, \max_{t\in[0,1]}|y_0'(t)|\} \\ &\le \frac{1-\alpha\eta}{1-\alpha} I^{-1}\Big(\sup_{c_0\le r\le R_1} h(r)\int_0^1 a(s)\Phi(s)ds + I(\varepsilon)\Big) \end{aligned}$$

which implies

$$\frac{R_1}{\frac{1-\alpha\eta}{1-\alpha} I^{-1}(\sup_{c_0\le r\le R_1} h(r)\int_0^1 a(s)\Phi(s)ds + I(\varepsilon))} \le 1.$$

This contradicts (4.2). Thus (4.3) is true.

From Lemma 2.1, for each $n \in N_0$, we have $i(A_n, \Omega_1\cap P, P) = 1$. As a result, for each $n \in N_0$, there exists a $y_n \in \Omega_1 \cap P$, such that $y_n = A_n y_n$; i.e.,

$$\begin{aligned} y_n(t) &= \frac{1}{1-\alpha}\int_0^1\int_0^s a(\tau) f(\tau, \max\{c_0, y_n(\tau)\}, -|y_n'(\tau)| - \frac{1}{n})\, d\tau\, ds \\ &\quad - \frac{\alpha}{1-\alpha}\int_0^\eta\int_0^s a(\tau) f(\tau, \max\{c_0, y_n(\tau)\}, -|y_n'(\tau)| - \frac{1}{n})\, d\tau\, ds \\ &\quad - \int_0^t\int_0^s a(\tau) f(\tau, \max\{c_0, y_n(\tau)\}, -|y_n'(\tau)| - \frac{1}{n})\, d\tau\, ds. \end{aligned}$$

It is easy to see that $y_n'(t) < 0$, and

$$y_n'(t) = -\int_0^t a(s) f(s, \max\{c_0, y_n(s)\}, y_n'(s) - \frac{1}{n}) ds, \quad n \in N_0,\ t \in (0,1).$$

Now we consider $\{y_n(t)\}_{n\in N_0}$ and $\{y_n'(t)\}_{n\in N_0}$. Since $\|y_n\| \le R_1$, one has

the functions belonging to $\{y_n\}$ are uniformly bounded on $[0,1]$, (4.10)

the functions belonging to $\{y_n'\}$ are uniformly bounded on $[0,1]$. (4.11)

Thus

the functions belonging to $\{y_n\}$ are equicontinuous on $[0,1]$. (4.12)

A similar argument to that used to show (4.5) yields

$$\begin{gathered} y_n''(t) + a(t) f(t, \max\{c_0, y_n(t)\}, y_n'(t) - \frac{1}{n}) = 0, \quad 0 < t < 1, \\ y_n'(0) = 0, \quad y_n(1) = \alpha y_n(\eta). \end{gathered} \tag{4.13}$$

By Lemma 2.4, we have

$$y_n(t) \ge c_0, \quad \forall n \in N_0. \tag{4.14}$$

Now we claim that for any $t_1, t_2 \in [0,1]$,

$$|I(y_n'(t_2) - \frac{1}{n}) - I(y_n'(t_1) - \frac{1}{n})| \le \sup_{c_0\le r\le R_1} h(r))|\int_{t_1}^{t_2} a(t)\Phi(t))dt|. \tag{4.15}$$

Notice that

$$\begin{aligned} -y_n''(t) &= a(t) f(t, \max\{c_0, y_n(t)\}, y_n'(t) - \frac{1}{n}) \\ &\le a(t)|f(t, \max\{c_0, y_n(t)\}, y_n'(t) - \frac{1}{n})| \\ &\le a(t)\Phi(t) h(\max\{c_0, y_n(t)\}) g(y_n'(t) - \frac{1}{n}), \quad \forall t \in (0,1), \end{aligned}$$

and

$$\begin{aligned} y_n''(t) &= -a(t)f(t, \max\{c_0, y_n(t)\}, y_n'(t) - \frac{1}{n}) \\ &\le a(t)|f(t, \max\{c_0, y_n(t)\}, y_n'(t) - \frac{1}{n})| \\ &\le a(t)\Phi(t)h(\max\{c_0, y_n(t)\})g(y_n'(t) - \frac{1}{n}), \forall t \in (0,1), \end{aligned}$$

which yields

$$\frac{-y_n''(t)}{g(y_n'(t) - \frac{1}{n})} \le a(t)\Phi(t)h(\max\{c_0, y_n(t)\}), \forall t \in (0,1), \tag{4.16}$$

$$\frac{y_n''(t)}{g(y_n'(t) - \frac{1}{n})} \le a(t)\Phi(t)h(\max\{c_0, y_n(t)\}), \forall t \in (0,1). \tag{4.17}$$

Note that the right hand sides are always positive in (4.16) and (4.17). For any t_1, $t_2 \in [0,1]$ with $t_1 < t_2$, we have

$$|\int_{t_1}^{t_2} \frac{1}{g(-y_n'(s) + \frac{1}{n})} d(-y_n'(s) + \frac{1}{n})| \le \sup_{c_0 \le r \le R_1} h(r)| \int_{t_1}^{t_2} a(t)\Phi(t)dt|;$$

i.e., (4.15) is true.

Since I^{-1} is uniformly continuous on $[0, I(R_1)]$, for any $\bar{\varepsilon} > 0$, there is a $\varepsilon' > 0$ such that

$$|I^{-1}(s_1) - I^{-1}(s_2)| < \bar{\varepsilon}, \forall\ |s_1 - s_2| < \varepsilon', \quad s_1, s_2 \in [0, I(R_1)]. \tag{4.18}$$

Also (4.15) guarantees that, for $\varepsilon' > 0$, there is a $\delta' > 0$ such that

$$|I(y_n'(t_2) - \frac{1}{n}) - I(y_n'(t_1) - \frac{1}{n})| < \varepsilon', \forall\ |t_1 - t_2| < \delta', \quad t_1, t_2 \in [0,1]. \tag{4.19}$$

Now (4.18) and (4.19) yield

$$\begin{aligned} |y_n'(t_2) - y_n'(t_1)| &= | - y_n'(t_2) + \frac{1}{n} + y_n'(t_1) - \frac{1}{n}| \\ &= |I^{-1}(I(-y_n'(t_2) + \frac{1}{n})) - I^{-1}(I(-y_n'(t_1) + \frac{1}{n}))| \\ &< \bar{\varepsilon}, \quad \forall\ |t_1 - t_2| < \delta',\ t_1, t_2 \in [0,1], \end{aligned}$$

which implies

$$\text{the functions belonging to } \{y_n'\} \text{ are equicontinuous on } [0,1]. \tag{4.20}$$

Consequently (4.10), (4.11), (4.12) and (4.20), the Arzela-Ascoli Theorem guarantees that $\{y_n\}$ and $\{y_n'\}$ are relatively compact in $C[0,1]$; i.e., there is a function $y_0 \in C^1[0,1]$, and a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\lim_{j \to +\infty} \max_{t \in [0,1]} |y_{n_j}(t) - y_0(t)| = 0, \quad \lim_{j \to +\infty} \max_{t \in [0,1]} |y_{n_j}'(t) - y_0'(t)| = 0.$$

Since $y_{n_j}'(0) = 0$, $y_{n_j}(1) = \alpha y_{n_j}(\eta)$, $y_{n_j}'(t) < 0$, $y_{n_j}(t) > 0$, $t \in (0,1)$, $j \in \{1, 2, \dots\}$, then one has

$$y_0'(0) = 0, y_0(1) = \alpha y_0(\eta), y_0'(t) \le 0, y_0(t) \ge 0, t \in (0,1). \tag{4.21}$$

Now since $\sup_{n \ge 1} \|y_n\| \le R_1$, (H') guarantees that there exists a $\Psi_{R_1}(t)$ continuous and $\Psi_{R_1}(t) > 0$ on $(0,1)$ such that

$$f(t, x, z) \ge \Psi_{R_1}(t)(-z)^{\gamma}, \quad t \in (0,1),\ x \in (0, R_1],\ z < 0.$$

Then

$$-y''_{n_j}(t) = a(t)f(t, \max\{c_0, y_{n_j}(t)\}, y'_{n_j}(t) - \frac{1}{n_j})) \geq a(t)\Psi_{R_1}(t)(-y'_{n_j}(t) + \frac{1}{n_j})^\gamma,$$

for $t \in (0,1)$. Also note that

$$-\frac{y''(t)}{(-y'_{n_j}(t) + \frac{1}{n_j})^\gamma} \geq a(t)\Psi_{R_1}(t), \quad t \in (0,1).$$

Integrating from 0 to t, we have

$$\frac{1}{1-\gamma}(-y'_{n_j}(t) + \frac{1}{n_j})^{1-\gamma} - \frac{1}{1-\gamma}(\frac{1}{n_j})^{1-\gamma} \geq \int_0^t a(s)\Psi_1(s)ds, \quad t \in (0,1),$$

which implies

$$-y'_{n_j}(t) + \frac{1}{n_j} \geq [(1-\gamma)(\int_0^t a(s)\Psi_1(s)ds + \frac{1}{1-\gamma}(\frac{1}{n_j})^{1-\gamma})]^{\frac{1}{1-\gamma}}, \quad t \in (0,1).$$

Letting $j \to +\infty$, we have

$$-y'_0(t) \geq [(1-\gamma)(\int_0^t a(s)\Psi_1(s)ds)]^{\frac{1}{1-\gamma}}, \quad t \in (0,1).$$

Consequently, $y'_0(t) < 0$ for all $t \in (0,1)$, which together with $y_0(1) > 0$ guarantees that $y_0(t) > 0$ for all $t \in [0,1]$. Therefore,

$$\min\{\min_{s\in[\frac{1}{2},t]} y_0(s), \min_{s\in[\frac{1}{2},t]} |y'_0(s)|\} > 0, \quad \text{for all } t \in [\frac{1}{2}, 1),$$

$$\min\{\min_{s\in[t,\frac{1}{2}]} y_0(s), \min_{s\in[t,\frac{1}{2}]} |y'_0(s)|\} > 0, \quad \text{for all } t \in (0, \frac{1}{2}].$$

Since

$$y'_{n_j}(t) - y'_{n_j}(\frac{1}{2}) = -\int_{\frac{1}{2}}^t a(s)f(s, \max\{c_0, y_{n_j}(s)\}, y'_{n_j}(s) - \frac{1}{n_j})ds, \quad t \in (0,1),$$

letting $j \to +\infty$, one has

$$y'_0(t) - y'_0(\frac{1}{2}) = -\int_{\frac{1}{2}}^t a(s)f(s, \{c_0, y_0(s)\}, y'_0(s))ds, t \in (0,1).$$

Now by direct differentiation, we have

$$y''_0(t) + a(t)f(t, \{c_0, y_0(t)\}, y'_0(t)) = 0, 0 < t < 1.$$

Now (4.14) guarantees that $y_0(t) \geq c_0$ for all $t \in [0,1]$ and so

$$y''_0(t) + a(t)f(t, y_0(t), y'_0(t)) = 0, \quad 0 < t < 1.$$

From (4.21), we have $y_0 \in C[0,1] \cap C^2(0,1)$ and y_0 is a positive solution to (1.1)-(1.2). □

Example 4.2. Consider the three-point boundary value problems

$$y'' + \alpha[(-y')^{\frac{1}{2}} + (-y')^{-a}][y^b + (\frac{1}{\alpha})^{\frac{1}{2}d} y^{-d}] = 0, t \in (0,1),$$

$$y'(0) = 0, y(1) = \frac{1}{2}y(\frac{1}{2}),$$

where $\alpha > 0$, $a > 0$, $1 > \gamma \geq 0$, $b \geq 0$ and $d > 0$. Then, there is a $\alpha_0 > 0$ such that (4.2)-(4.2) has one positive solution $y_0 \in C[0,1] \cap C^2(0,1)$ with $y_0(t) > 0$ on $[0,1]$ and $y_0'(t) < 0$ on $(0,1)$ for all $0 < \alpha < \alpha_0$.

Let $a(t) \equiv \mu$, $\Phi(t) \equiv 1$ for all $t \in [0,1]$, $h(x) = x^b + (\frac{1}{\alpha})^{\frac{1}{2}d} x^{-d}$ for $x \in (0,+\infty)$ and $g(z) = z^{\frac{1}{2}} + z^{-a}$ for $z \in (0,+\infty)$. From the proof of Lemma 2.4, we have $c_0 = \frac{7}{192}\alpha^{\frac{1}{2}}$ with $\alpha \leq 1$, and then $\alpha[y^b + (\frac{1}{\alpha})^{\frac{1}{2}d} y^{-d}] \leq \alpha y^b + (\frac{192}{7})^d$ for all $y \in [c_0,+\infty)$. Let $I(z) = \int_0^z \frac{1}{r^{\frac{1}{2}} + r^{-a}} dr$. Thus there exists an α_0 such that

$$\frac{I(1/3)}{\alpha \sup_{c_0 \leq r \leq 1} h(r)} > 1, \quad \forall \alpha \in (0,\alpha_0]$$

and then

$$\sup_{c_0 \leq c < +\infty} \frac{c}{3I^{-1}(\sup_{c_0 \leq r \leq c} h(r)\alpha)} > 1.$$

Hence, the conditions (H1) and (H2) hold. Thus Theorem 4.1 guarantees that (4.2) and (4.2) has at least one positive solution.

5. Multiple positive solutions to (1.1)-(1.2)

In this section our nonlinearity f may be singular at $y' = 0$ and $y = 0$. Throughout this section we will assume that the following conditions hold:

- (P1) $a(t) \in C(0,1)$, $a(t) > 0$ for all $t \in (0,1)$;
- (P2) Conditions (H) and (H') hold and $I(z) = \int_0^z \frac{1}{g(r)} dr < +\infty$ for all $z \in [0,+\infty)$ with $\sup_{c_0 \leq r \leq c} h(r) \int_0^1 a(s)\Phi(s)ds < \int_0^\infty \frac{dr}{g(r)}$ for all $c \in [c_0,+\infty)$ and suppose
$$\sup_{c_0 \leq c < +\infty} \frac{c}{\frac{1-\alpha\eta}{1-\alpha} I^{-1}(\sup_{c_0 \leq r \leq c} h(r) \int_0^1 a(s)\Phi(s)ds)} > 1,$$
where c_0 is defined by Lemma 2.4;
- (P3) $\lim_{u \to +\infty} f(t,u,z)/u = +\infty$ uniformly for $(t,z) \in [\frac{1}{4}, \frac{3}{4}] \times (0,+\infty)$.

Theorem 5.1. *Suppose that* (P1)–(P3) *hold. Then* (1.1)-(1.2) *has at least two positive solutions* $y_{1,0}$, $y_{1,0} \in C[0,1] \cap C^2(0,1)$ *with* $y_{1,0}(t) > 0$, $y_{2,0}(t) > 0$ *on* $[0,1]$ *and* $y_{1,0}'(t) < 0$, $y_{2,0}'(t) < 0$ *on* $(0,1)$.

Proof. Choose $R_1 > 0$ with

$$\frac{R_1}{\frac{1-\alpha\eta}{1-\alpha} I^{-1}\big(\sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s)\Phi(s)ds\big)} > 1. \tag{5.1}$$

From the continuity of I^{-1} and I, we can choose $\varepsilon > 0$ and $\varepsilon < R_1$ with

$$\frac{R_1}{\frac{1-\alpha\eta}{1-\alpha} I^{-1}(\sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s)\Phi(s)ds + I(\varepsilon))} > 1. \tag{5.2}$$

Let $n_0 \in \{1,2,\dots\}$ so that $\frac{1}{n_0} < \min\{\varepsilon, \frac{1}{2}\frac{1-\alpha}{\alpha(1-\eta)}c_0\}$ and let $N_0 = \{n_0, n_0+1, \dots\}$. Lemma 2.7 guarantees that for each $n \in N_0$, $A_n : P_q \to P_q$ is a continuous and completely continuous operator. From (P_3), there is a $R' > R_1$ such that

$$f(t,x,y) \geq N^* x, \quad \forall x \geq R',$$

where $N^* > (\int_{1/4}^{3/4}(1-s)a(s)ds\frac{\alpha(1-\eta)}{1-\alpha\eta})^{-1}$. Let

$$R_2 > \max\{R', \frac{1-\alpha\eta}{\alpha(1-\eta)}R'\}.$$

Now let

$$\Omega_1 = \{y \in C_q^1[0,1] : \|y\|_q < R_1\}, \quad \Omega_2 = \{y \in C_q^1[0,1] : \|y\|_q < R_2\}.$$

We now show that

$$y \neq \mu A_n y, \quad \forall y \in P \cap \partial\Omega_1,\ \mu \in (0,1],\ n \in N_0, \tag{5.3}$$

and

$$A_n x \not\leq x, \quad \forall\ x \in \partial\Omega_2 \cap P, n \in N_0. \tag{5.4}$$

Suppose there exists a $y_0 \in P \cap \partial\Omega_1$ and a $\mu_0 \in (0,1]$ such that $y_0 = \mu_0 A_n y_0$. It is easy to see that $y_0'(t) \leq 0$ and

$$y_0'(t) = -\mu_0 \int_0^t a(s)f(s, \max\{c_0, y_0(s)\}, y_0'(s) - \frac{1}{n})ds, t \in (0,1). \tag{5.5}$$

Also

$$\begin{gathered} y_0''(t) + \mu_0 a(t)f(t, \max\{c_0, y_0(t)\}, y_0'(t) - \frac{1}{n}) = 0, \quad 0 < t < 1, \\ y_0'(0) = 0, y_0(1) = \alpha y_0(\eta). \end{gathered}$$

Therefore,

$$\begin{aligned} -y_0''(t) &= \mu_0 a(t)f(t, \max\{c_0, y_0(t)\}, y_0'(t) - \frac{1}{n}) \\ &\leq a(t)\Phi(t)h(\max\{c_0, y_0(t)\})g(-y_0'(t) + \frac{1}{n}), \quad \forall t \in (0,1). \end{aligned}$$

which yields

$$\frac{-y_0''(t)}{g(-y_0'(t) + \frac{1}{n})} \leq a(t)\Phi(t)h(\max\{c_0, y_0(t)\}), \quad \forall t \in (0,1).$$

Integration from 0 to t yields

$$\begin{aligned} I(-y_0'(t) + \frac{1}{n}) - I(\frac{1}{n}) &\leq \int_0^t a(s)\Phi(s)h(\max\{c_0, y_0(s)\})ds \\ &\leq \sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s)\Phi(s)ds, \end{aligned}$$

and so

$$I(-y_0'(t) + \frac{1}{n}) \leq \sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s)\Phi(s)ds + I(\varepsilon).$$

Thus

$$-y_0'(t) \leq I^{-1}\Big(\sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s)\Phi(s)ds + I(\varepsilon)\Big), \quad t \in (0,1). \tag{5.6}$$

Integration from t to 1 yields

$$y_0(t) - y_0(1) \leq (1-t)I^{-1}\Big(\sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s)\Phi(s)ds + I(\varepsilon)\Big), \quad t \in (0,1). \tag{5.7}$$

Let $t = \eta$ in (5.7). Then

$$y_0(\eta) - y_0(1) \le (1-\eta)I^{-1}\Big(\sup_{c_0 \le r \le R_1} h(r) \int_0^1 a(s)\Phi(s)ds + I(\varepsilon)\Big).$$

Since $y_0(1) = \alpha y_0(\eta)$, one has

$$(\frac{1}{\alpha} - 1)y_0(1) \le (1-\eta)I^{-1}\Big(\sup_{c_0 \le r \le R_1} h(r) \int_0^1 a(s)\Phi(s)ds + I(\varepsilon)\Big),$$

which yields

$$y_0(1) \le \frac{\alpha}{1-\alpha}(1-\eta)I^{-1}\Big(\sup_{c_0 \le r \le R_1} h(r) \int_0^1 a(s)\Phi(s)ds + I(\varepsilon)\Big).$$

Then (5.7) implies

$$\begin{aligned} y_0(0) &\le y_0(1) + I^{-1}(\sup_{c_0 \le r \le R_1} h(r) \int_0^1 a(s)\Phi(s)ds + I(\varepsilon)) \\ &= \frac{1-\alpha\eta}{1-\alpha} I^{-1}(\sup_{c_0 \le r \le R_1} h(r) \int_0^1 a(s)\Phi(s)ds + I(\varepsilon)). \end{aligned} \tag{5.8}$$

Now (5.6) and (5.8) guarantees

$$\begin{aligned} R_1 &= \max\{\max_{t\in[0,1]} |y_0(t)|, \max_{t\in[0,1]} (1-t)|y_0'(t)|\} \\ &\le \frac{1-\alpha\eta}{1-\alpha} I^{-1}\Big(\sup_{c_0 \le r \le R_1} h(r) \int_0^1 a(s)\Phi(s)ds + I(\varepsilon)\Big) \end{aligned}$$

which implies

$$\frac{R_1}{\frac{1-\alpha\eta}{1-\alpha} I^{-1}(\sup_{c_0 \le r \le R_1} h(r) \int_0^1 a(s)\Phi(s)ds + I(\varepsilon))} \le 1.$$

This contradicts (5.2). Thus (5.3) is true.

From Lemma 2.1, for each $n \in N_0$, we have

$$i(A_n, \Omega_1 \cap P, P) = 1, \quad n \in N_0. \tag{5.9}$$

Suppose there is a $x_0 \in \partial\Omega_2 \cap P$ such that $A_n x_0 \le x_0$. Then $\|x_0\|_q = R_2$. Also Lemma 2.3 implies

$$\min_{t\in[0,1]} x_0(t) \ge \frac{\alpha(1-\eta)}{1-\alpha\eta} \max_{t\in[0,1]} |x_0(t)| = \frac{\alpha(1-\eta)}{1-\alpha\eta}\|x_0\|_q = \frac{\alpha(1-\eta)}{1-\alpha\eta} R_2 > R'.$$

Then, we have

$$\begin{aligned} x_0(0) &\ge (Ax_0)(0) \\ &= \frac{1}{1-\alpha}\int_0^1 \int_0^s a(\tau) f(\tau, \max\{c_0, x_0(\tau)\}, -|x_0'(\tau)| - \frac{1}{n})\, d\tau\, ds \\ &\quad - \frac{\alpha}{1-\alpha}\int_0^\eta \int_0^s a(\tau) f(\tau, \max\{c_0, x_0(\tau)\}, -|x_0'(\tau)| - \frac{1}{n})\, d\tau\, ds \\ &\ge \int_{1/4}^{3/4} (1-s)a(s) f(\tau, \max\{c_0, x_0(s)\}, -|x_0'(s)| - \frac{1}{n}) ds \\ &\ge \int_{1/4}^{3/4} (1-s)a(s) N * \max\{c_0, x_0(s)\} ds \end{aligned}$$

$$
\begin{aligned}
&\geq \int_{1/4}^{3/4}(1-s)a(s)dsN * \frac{\alpha(1-\eta)}{1-\alpha\eta}R_2 \\
&> \|x_0\|_q,
\end{aligned}
$$

which is a contradiction. Thus, (5.4) is true. Then Lemma 2.2 implies

$$
i(A_n, \Omega_2 \cap P, P) = 0, \quad n \in N_0. \tag{5.10}
$$

From (5.9) and (5.10), we have

$$
i(A_n, (\Omega_2 - \overline{\Omega}_1) \cap P, P) = -1, \quad n \in N_0. \tag{5.11}
$$

By (5.9), (5.11), there is a $x_{1,n} \in \Omega_1 \cap P$ and another $x_{2,n} \in \Omega_2 \cap P$ such that

$$
A_n x_{1,n} = x_{1,n}, \quad A_n x_{2,n} = x_{2,n}, \quad n \in N_0.
$$

Now we consider $\{x_{1,n}\}_{n\in N_0}$ and $\{x_{2,n}\}_{n\in N_0}$. By Lemma 2.4, we have $x_{1,n}(t) \geq c_0$ and $x_{2,n} \geq c_0$.

We consider $\{x_{1,n}\}_{n\in N_0}$. Obviously $\max_{t\in[0,1]}|x_{1,n}(t)| \leq R_1$ for all $n \in N_0$ and $\max_{t\in[0,1]}(1-t)|x'_{1,n}(t)| \leq R_1$ for all $n \in N_0$. Also $|x'_{1,n}(t)| \leq \frac{1}{1-t}R_1$ for all $t \in [0,1)$ and $n \in N_0$. Hence, the functions belonging to $\{x_{1,n}\}$ are uniformly bounded on $[0,1]$.

Since $x_{1,n}(t)$ satisfies

$$
\begin{gathered}
x''_{1,n}(t) + a(t)f(t, \max\{c_0, x_{1,n}(t)\}, x'_{1,n}(t) - \frac{1}{n}) = 0, \quad 0 < t < 1, \\
x'_{1,n}(0) = 0, x_{1,n}(1) = \alpha x_{1,n}(\eta).
\end{gathered}
$$

A similar argument to that used to show (5.6) yields that

$$
-x'_{1,n}(t) \leq I^{-1}\Big(\sup_{c_0\leq r\leq R_1} h(r)\int_0^1 a(s)\Phi(s)ds + I(\varepsilon)\Big), \quad t \in (0,1),
$$

which implies that the functions belonging to $\{x'_{1,n}\}$ are uniformly bounded on $[0,1]$ and so the functions belonging to $\{x_{1,n}\}$ are equicontinuous on $[0,1]$.

A similar argument to that to used to show (4.15) yields the functions belonging to $\{x'_{1,n}\}$ are equicontinuous on $[0,1]$.

Consequently, the Arzela-Ascoli Theorem guarantees that $\{x_{1,n}(t)\}$ and $\{x'_{1,n}(t)\}$ are relatively compact in $C[0,1]$; i.e., there is a function $x_{1,0} \in C^1[0,1]$, and a subsequence $\{x_{1,n_j}\}$ of $\{x_{1,n}\}$ such that

$$
\lim_{j\to+\infty}\max_{t\in[0,1]}|x_{1,n_j}(t) - x_{1,0}(t)| = 0, \quad \lim_{j\to+\infty}\max_{t\in[0,1]}|x'_{1,n_j}(t) - x'_{1,0}(t)| = 0.
$$

Similar reasoning as in the proof of Theorem 4.1 establishes that $x_{1,0}$ is a positive solution to (1.1) and (1.2).

Similarly, there is a convergent subsequence $\{x_{2,n_k}\}$ of $\{x_{2,n}\}$ such that

$$
\lim_{k\to+\infty}|x_{2,n_k}(t) - x_{2,0}(t)| = 0, \quad \lim_{k\to+\infty}|x'_{2,n_j}(t) - x'_{2,0}(t)| = 0
$$

and $x_{2,0}$ satisfies (1.1)-(1.2).

Since $\|x_{1,0}\|_q = \max_{t\in[0,1]}|x_{1,0}(t)| \leq R_1$ and $\|x_{2,0}\|_q = \max_{t\in[0,1]}|x_{2,0}(t)| \geq R_1$, a similar argument to that used to show (5.3) yields that $x_{1,0}$, $x_{2,0} \notin P \cap \partial\Omega_1$; i.e.,

$$
\|x_{1,0}\|_q = \max_{t\in[0,1]}|x_{1,0}(t)| < R_1, \quad \|x_{2,0}\|_q = \max_{t\in[0,1]}|x_{2,0}(t)| > R_1.
$$

Consequently, $x_{1,0}$ and $x_{2,0}$ are different positive solutions to (1.1)-(1.2). $\square$

Example 5.2. Consider the three-point boundary value problems

$$y'' + \alpha(1-t)^a[1 + (-y')^e + (-y')^{-a}][1 + y^b + y^{-d}] = 0, \quad t \in (0,1),$$
$$y'(0) = 0, y(1) = \frac{1}{2}y(\frac{1}{2})$$

where $1 \geq e \geq 0$, $a > 0$, $b \geq 0$, $d > 0$ and $\alpha > 0$. Then there is a $\alpha_0 > 0$ such that (5.11)-(5.2) has at least two positive solutions $y_{1,0}$, $y_{2,0} \in C[0,1] \cap C^2(0,1)$ with $y_{1,0}(t) > 0$, $y_{2,0}(t) > 0$ on $[0,1]$ and $y'_{1,0}(t) < 0$, $y'_{2,0}(t) < 0$ on (0,1) for all $0 < \alpha \leq \alpha_0$.

Let $a(t) \equiv \mu$, $\Phi(t) = (1-t)^a$ for all $t \in [0,1]$, $h(x) = 1 + x^b + x^{-d}$ for $x \in (0,+\infty)$ and $g(z) = 1 + z^e + z^{-a}$ for $z \in (0,+\infty)$. From the proof of Lemma 2.4, we have $c_0 = \frac{1}{2}\min\{\frac{1}{3}, \frac{1}{a+1}(\frac{1}{2} - \frac{1}{a+2}(\frac{1}{2})^{a+2})\}$, and then $\alpha(1-t)^a[1 + y^b + y^{-d}] \leq \alpha(1-t)^a[1 + y^b + {c_0}^{-d}]$ for all $y \in [c_0, +\infty)$. Let $I(z) = \int_0^z \frac{1}{1+r^e+r^{-a}}dr$. Thus there exists an α_0 such that

$$\frac{I(\frac{1}{3})}{\alpha\frac{1}{a+1}\sup_{c_0 \leq r \leq 1} h(r)} > 1, \quad \forall \alpha \in (0, \alpha_0]$$

and then

$$\sup_{c_0 \leq c < +\infty} \frac{c}{3I^{-1}(\sup_{c_0 \leq r \leq c} h(r) \int_0^1 a(s)\Phi(s)ds)} > 1.$$

Hence, the conditions (P1), (P2) and (P3) hold. Thus Theorem 5.1 guarantees that (5.2)-(5.2) has at least two positive solutions.

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