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EXISTENCE OF WEAK SOLUTIONS FOR A NONUNIFORMLY ELLIPTIC NONLINEAR SYSTEM IN $\mathbb{R}^N$

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Abstract. We study the nonuniformly elliptic, nonlinear system

$$\begin{aligned} -\operatorname{div}(h_1(x)\nabla u) + a(x)u &= f(x,u,v) \quad \text{in } \mathbb{R}^N, \\ -\operatorname{div}(h_2(x)\nabla v) + b(x)v &= g(x,u,v) \quad \text{in } \mathbb{R}^N. \end{aligned}$$

Under growth and regularity conditions on the nonlinearities f and g, we obtain weak solutions in a subspace of the Sobolev space $H^1(\mathbb{R}^N,\mathbb{R}^2)$ by applying a variant of the Mountain Pass Theorem.

1. Introduction

We study the nonuniformly elliptic, nonlinear system

$$\begin{aligned} -\operatorname{div}(h_1(x)\nabla u) + a(x)u &= f(x,u,v) \quad \text{in } \mathbb{R}^N, \\ -\operatorname{div}(h_2(x)\nabla v) + b(x)v &= g(x,u,v) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{1.1}$$

where $N \ge 3$, $h_i \in L^1_{\rm loc}(\mathbb{R}^N)$, $h_i(x) \ge 1$ $i = 1,2$; $a, b \in C(\mathbb{R}^N)$. We assume that there exist $a_0, b_0 > 0$ such that

$$\begin{aligned} a(x) \ge a_0, \quad b(x) \ge b_0, \quad \forall x \in \mathbb{R}^N, \\ a(x) \to \infty, \quad b(x) \to \infty \quad \text{as } |x| \to \infty. \end{aligned} \tag{1.2}$$

System (1.1), with $h_1(x) = h_2(x) = 1$, has been studied by Costa [7]. There, under appropriate growth and regularity conditions on the functions $f(x,u,v)$ and $g(x,u,v)$, the weak solutions are exactly the critical points of a functional defined on a Hilbert space of functions u, v in $H^1(\mathbb{R}^N)$. In the scalar case, the problem

$$-\operatorname{div}(|x|^\alpha \nabla u) + b(x)u = f(x,u) \quad \text{in } \mathbb{R}^N,$$

with $N \ge 3$ and $\alpha \in (0,2)$, has been studied by Mihăilescu and Rădulescu [11]. In this situation, the authors overcome the lack of compactness of the problem by using the the Caffarelli-Kohn-Nirenberg inequality.

In this paper, under condition (1.2), we consider (1.1) which may be a nonuniformly elliptic system. We shall reduce (1.1) to a uniformly elliptic system by using appropriate weighted Sobolev spaces. Then applying a variant of the Mountain

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pass theorem in [8], we prove the existence of weak solutions of system (1.1) in a subspace of $H^1(\mathbb{R}^N,\mathbb{R}^2)$.

To prove our main results, we introduce the following some hypotheses:

(H1) There exists a function $F(x,w)\in C^1(\mathbb{R}^N\times\mathbb{R}^2,\mathbb{R})$ such that $\frac{\partial F}{\partial u}=f(x,w)$, $\frac{\partial F}{\partial v}=g(x,w)$, for all $x\in\mathbb{R}^N$, $w=(u,v)\in\mathbb{R}^2$.

(H2) $f(x,w),g(x,w)\in C^1(\mathbb{R}^N\times\mathbb{R}^2,\mathbb{R})$, $f(x,0,0)=g(x,0,0)=0$ for all $x\in\mathbb{R}^N$, there exists a positive constant τ_0 such that
$$|\nabla f(x,w)|+|\nabla g(x,w)|\le\tau_0|w|^{p-1}$$
for all $x\in\mathbb{R}^N$, $w=(u,v)\in\mathbb{R}^2$.

(H3) There exists a constant $\mu>2$ such that
$$0<\mu F(x,w)\le w\nabla F(x,w)$$
for all $x\in\mathbb{R}^N$ and $w\in\mathbb{R}^2\backslash\{(0,0)\}$.

Let $H^1(\mathbb{R}^N,\mathbb{R}^2)$ be the usual Sobolev space under the norm
$$\|w\|^2=\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2+|u|^2+|v|^2)dx,\quad w=(u,v)\in H^1(\mathbb{R}^N,\mathbb{R}^2)\,.$$
Consider the subspace
$$E=\{(u,v)\in H^1(\mathbb{R}^N,\mathbb{R}^2):\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2+a(x)|u|^2+b(x)|v|^2)dx<\infty\}.$$
Then E is a Hilbert space with the norm
$$\|w\|_E^2=\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2+a(x)|u|^2+b(x)|v|^2)dx.$$
By (1.2) it is clear that
$$\|w\|_E\ge m_0\|w\|_{H^1(\mathbb{R}^N,\mathbb{R}^2)},\quad\forall w\in E,m_0>0,$$
and the embeddings $E\hookrightarrow H^1(\mathbb{R}^N,\mathbb{R}^2)\hookrightarrow L^q(\mathbb{R}^N,\mathbb{R}^2)$, $2\le q\le 2^*$ are continuous. Moreover, the embedding $E\hookrightarrow L^2(\mathbb{R}^N,\mathbb{R}^2)$ is compact (see [7]). We now introduce the space
$$H=\{(u,v)\in E:\int_{\mathbb{R}^N}(h_1(x)|\nabla u|^2+h_2(x)|\nabla v|^2+a(x)|u|^2+b(x)|v|^2)dx<\infty\}$$
endowed with the norm
$$\|w\|_H^2=\int_{\mathbb{R}^N}(h_1(x)|\nabla u|^2+h_2(x)|\nabla v|^2+a(x)|u|^2+b(x)|v|^2)dx.$$

Remark 1.1. Since $h_1(x)\ge 1$, $h_2(x)\ge 1$ for all $x\in\mathbb{R}^N$ we have $\|w\|_E\le\|w\|_H$ with $\forall w\in H$ and $C_0^\infty(\mathbb{R}^N,\mathbb{R}^2)\subset H$.

Proposition 1.2. *The set H is a Hilbert space with the inner product*
$$\langle w_1,w_2\rangle=\int_{\mathbb{R}^N}(h_1(x)\nabla u_1\nabla u_2+h_2(x)\nabla v_1\nabla v_2+a(x)u_1u_2+b(x)v_1v_2)dx$$
for all $w_1=(u_1,v_1)$, $w_2=(u_2,v_2)\in H$.

Proof. It suffices to check that any Cauchy sequences $\{w_m\}$ in H converges to $w \in H$. Indeed, let $\{w_m\} = \{(u_m, v_m)\}$ be a Cauchy sequence in H. Then

$$\lim_{m,k\to\infty} \int_{\mathbb{R}^N} \left(h_1(x)|\nabla u_m - \nabla u_k|^2 + h_2(x)|\nabla v_m - \nabla v_k|^2\right) dx \\ + \lim_{m,k\to\infty} \int_{\mathbb{R}^N} \left(a(x)|u_m - u_k|^2 + b(x)|v_m - v_k|^2\right) dx = 0$$

and $\{\|w_m\|_H\}$ is bounded.

Moreover, by Remark 1.1, $\{w_m\}$ is also a Cauchy sequence in E. Hence the sequence $\{w_m\}$ converges to $w = (u, v) \in E$; i.e.,

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} \left(|\nabla u_m - \nabla u|^2 + |\nabla v_m - \nabla v|^2\right) dx \\ + \lim_{m\to\infty} \int_{\mathbb{R}^N} \left(a(x)|u_m - u|^2 + b(x)|v_m - v|^2\right) dx = 0.$$

It follows that $\{\nabla w_m = (\nabla u_m, \nabla v_m)\}$ converges to $\nabla w = (\nabla u, \nabla v)$ and $\{w_m\}$ converges to w in $L^2(\mathbb{R}^N, \mathbb{R}^2)$. Therefore $\{\nabla w_m(x)\}$ converges to $\{\nabla w(x)\}$ and $\{w_m(x)\}$ converges to $w(x)$ for almost everywhere $x \in \mathbb{R}^N$. Applying Fatou's lemma we get

$$\int_{\mathbb{R}^N} (h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2 + a(x)|u|^2 + b(x)|v|^2)dx \\ \le \liminf_{m\to\infty} \int_{\mathbb{R}^N} (h_1(x)|\nabla u_m|^2 + h_2(x)|\nabla v_m|^2 + a(x)|u_m|^2 + b(x)|v_m|^2)dx < \infty.$$

Hence $w = (u, v) \in H$. Applying again Fatou's lemma

$$\begin{aligned} 0 &\le \lim_{m\to\infty} \int_{\mathbb{R}^N} \left(h_1(x)|\nabla u_m - \nabla u|^2 + h_2(x)|\nabla v_m - \nabla v|^2\right) dx \\ &\quad + \lim_{m\to\infty} \int_{\mathbb{R}^N} \left(a(x)|u_m - u|^2 + b(x)|v_m - v|^2\right) dx \\ &\le \lim_{m\to\infty} \Big[\liminf_{k\to\infty} \int_{\mathbb{R}^N} \left(h_1(x)|\nabla u_m - \nabla u_k|^2 + h_2(x)|\nabla v_m - \nabla v_k|^2\right) dx\Big] \\ &\quad + \lim_{m\to\infty} \Big[\liminf_{k\to\infty} \int_{\mathbb{R}^N} \left(a(x)|u_m - u_k|^2 + b(x)|v_m - v_k|^2\right) dx\Big] = 0. \end{aligned}$$

We conclude that $\{w_m\}$ converges to $w = (u, v)$ in H. □

Definition 1.3. We say that $w = (u, v) \in H$ is a weak solution of system (1.1) if

$$\int_{\mathbb{R}^N} (h_1(x)\nabla u\nabla\varphi + h_2(x)\nabla v\nabla\psi + a(x)u\varphi + b(x)v\psi)dx \\ - \int_{\mathbb{R}^N} (f(x, u, v)\varphi + g(x, u, v)\psi)dx = 0$$

for all $\Phi = (\varphi, \psi) \in H$.

Our main result is stated as follows.

Theorem 1.4. *Assuming* (1.2) *and* (H1)–(H3) *are satisfied, the system* (1.1) *has at least one non-trivial weak solution in* H.

This theorem will be proved by using variational techniques based on a variant of the Mountain pass theorem in [8]. Let us define the functional $J : H \to \mathbb{R}$ given by

$$\begin{aligned} J(w) &= \frac{1}{2}\int_{\mathbb{R}^N}(h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2 + a(x)|u|^2 + b(x)|v|^2)dx \\ &\quad - \int_{\mathbb{R}^N} F(x,u,v)dx \\ &= T(w) - P(w) \quad \text{for } w = (u,v) \in H, \end{aligned} \tag{1.3}$$

where

$$T(w) = \frac{1}{2}\int_{\mathbb{R}^N}(h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2 + a(x)|u|^2 + b(x)|v|^2)dx, \tag{1.4}$$

$$P(w) = \int_{\mathbb{R}^N} F(x,u,v)dx. \tag{1.5}$$

2. Existence of weak solutions

In general, due to $h(x) \in L^1_{\rm loc}(\mathbb{R}^N)$, the functional J may be not belong to $C^1(H)$ (in this work, we do not completely care whether the functional J belongs to $C^1(H)$ or not). This means that we cannot apply directly the Mountain pass theorem by Ambrosetti-Rabinowitz (see [4]). In the situation, we recall the following concept of weakly continuous differentiability. Our approach is based on a weak version of the Mountain pass theorem by Duc (see [8]).

Definition 2.1. Let J be a functional from a Banach space Y into $\mathbb{R}$. We say that J is weakly continuously differentiable on Y if and only if the following conditions are satisfied

(i) J is continuous on Y.
(ii) For any $u \in Y$, there exists a linear map $DJ(u)$ from Y into $\mathbb{R}$ such that

$$\lim_{t\to 0}\frac{J(u+tv) - J(u)}{t} = \langle DJ(u), v\rangle, \quad \forall v \in Y.$$

(iii) For any $v \in Y$, the map $u \mapsto \langle DJ(u), v\rangle$ is continuous on Y.

We denote by $C^1_w(Y)$ the set of weakly continuously differentiable functionals on Y. It is clear that $C^1(Y) \subset C^1_w(Y)$, where $C^1(Y)$ is the set of all continuously Frechet differentiable functionals on Y. The following proposition concerns the smoothness of the functional J.

Proposition 2.2. *Under the assumptions of Theorem 1.4, the functional $J(w), w \in H$ given by* (1.3) *is weakly continuously differentiable on H and*

$$\begin{aligned} \langle DJ(w), \Phi\rangle &= \int_{\mathbb{R}^N}(h_1(x)\nabla u\nabla\varphi + h_2(x)\nabla v\nabla\psi + a(x)u\varphi + b(x)v\psi)dx \\ &\quad - \int_{\mathbb{R}^N}(f(x,u,v)\varphi + g(x,u,v)\psi)dx \end{aligned} \tag{2.1}$$

for all $w = (u,v)$, $\Phi = (\varphi,\psi) \in H$.

Proof. By conditions (H1)–(H3) and the embedding $H \hookrightarrow E$ is continuous, it can be shown (cf. [5, Theorem A.VI]) that the functional P is well-defined and of class $C^1(H)$. Moreover, we have

$$\langle DP(w), \Phi\rangle = \int_{\mathbb{R}^N} (f(x,u,v)\varphi + g(x,u,v)\psi)dx$$

for all $w = (u, v)$, $\Phi = (\varphi, \psi) \in H$.

Next, we prove that T is continuous on H. Let $\{w_m\}$ be a sequence converging to w in H, where $w_m = (u_m, v_m)$, $m = 1, 2, \dots$, $w = (u, v)$. Then

$$\begin{aligned}&\lim_{m\to\infty}\int_{\mathbb{R}^N}[h_1(x)|\nabla u_m - \nabla u|^2 + h_2(x)|\nabla v_m - \nabla v|^2]dx\\ &+ \lim_{m\to\infty}\int_{\mathbb{R}^N}[a(x)|u_m - u|^2 + b(x)|v_m - v|^2]dx = 0\end{aligned}$$

and $\{\|w_m\|_H\}$ is bounded. Observe further that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N} h_1(x)|\nabla u_m|^2dx - \int_{\mathbb{R}^N} h_1(x)|\nabla u|^2dx\Big|\\
&= \Big|\int_{\mathbb{R}^N} h_1(x)(|\nabla u_m|^2 - |\nabla u|^2)dx\Big|\\
&\le \int_{\mathbb{R}^N} h_1(x)||\nabla u_m| - |\nabla u||(|\nabla u_m| + |\nabla u|)dx\\
&\le \int_{\mathbb{R}^N} h_1(x)|\nabla u_m - \nabla u||\nabla u_m|dx + \int_{\mathbb{R}^N} h_1(x)|\nabla u_m - \nabla u||\nabla u|dx\\
&\le \Big(\int_{\mathbb{R}^N} h_1(x)|\nabla u_m - \nabla u|^2dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N} h_1(x)|\nabla u_m|^2dx\Big)^{1/2}\\
&\quad + \Big(\int_{\mathbb{R}^N} h_1(x)|\nabla u_m - \nabla u|^2dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N} h_1(x)|\nabla u|^2dx\Big)^{1/2}\\
&\le (\|w_m\|_H + \|w\|_H)\|w_m - w\|_H.\end{aligned}$$

Similarly, we obtain

$$\begin{gathered}\Big|\int_{\mathbb{R}^N} h_2(x)|\nabla v_m|^2dx - \int_{\mathbb{R}^N} h_2(x)|\nabla v|^2dx\Big| \le (\|w_m\|_H + \|w\|_H)\|w_m - w\|_H,\\
\Big|\int_{\mathbb{R}^N} a(x)|u_m|^2dx - \int_{\mathbb{R}^N} a(x)|u|^2dx\Big| \le (\|w_m\|_H + \|w\|_H)\|w_m - w\|_H,\\
\Big|\int_{\mathbb{R}^N} b(x)|v_m|^2dx - \int_{\mathbb{R}^N} b(x)|v|^2dx\Big| \le (\|w_m\|_H + \|w\|_H)\|w_m - w\|_H.\end{gathered}$$

From the above inequalities, we obtain

$$|T(w_m) - T(w)| \le 4(\|w_m\|_H + \|w\|_H)\|w_m - w\|_H \to 0 \quad \text{as } m \to \infty.$$

Thus T is continuous on H. Next we prove that for all $w = (u, v)$, $\Phi = (\varphi, \psi) \in H$,

$$\langle DJ(w), \Phi\rangle = \int_{\mathbb{R}^N} (h_1(x)\nabla u\nabla\varphi + h_2(x)\nabla v\nabla\psi + a(x)u\varphi + b(x)v\psi)dx.$$

Indeed, for any $w = (u, v)$, $\Phi = (\varphi, \psi) \in H$, any $t \in (-1, 1)\backslash\{0\}$ and $x \in \mathbb{R}^N$ we have

$$\begin{aligned}\Big|\frac{h_1(x)|\nabla u + t\nabla\varphi|^2 - h_1(x)|\nabla u|^2}{t}\Big| &= \Big|2\int_0^1 h_1(x)(\nabla u + st\nabla\varphi)\nabla\varphi ds\Big| \\ &\le 2h_1(x)(|\nabla u| + |\nabla\varphi|)|\nabla\varphi| \\ &\le h_1(x)|\nabla u|^2 + 3h_1(x)|\nabla\varphi|^2.\end{aligned}$$

Since $h_1(x)|\nabla u|^2$, $h_1(x)|\nabla\varphi|^2 \in L^1(\mathbb{R}^N)$, $g(x) = h_1(x)|\nabla u|^2 + 3h_1(x)|\nabla\varphi|^2 \in L^1(\mathbb{R}^N)$. Applying Lebesgue's Dominated convergence theorem we get

$$\lim_{t\to 0}\int_{\mathbb{R}^N}\frac{h_1(x)|\nabla u + t\nabla\varphi|^2 - h_1(x)|\nabla u|^2}{t}dx = 2\int_{\mathbb{R}^N} h_1(x)\nabla u\nabla\varphi dx. \tag{2.2}$$

Similarly, we have

$$\lim_{t\to 0}\int_{\mathbb{R}^N}\frac{h_2(x)|\nabla v + t\nabla\psi|^2 - h_2(x)|\nabla v|^2}{t}dx = 2\int_{\mathbb{R}^N} h_2(x)\nabla v\nabla\psi dx, \tag{2.3}$$

$$\lim_{t\to 0}\int_{\mathbb{R}^N}\frac{a(x)|u + t\varphi|^2 - a(x)|u|^2}{t}dx = 2\int_{\mathbb{R}^N} a(x)u\varphi dx, \tag{2.4}$$

$$\lim_{t\to 0}\int_{\mathbb{R}^N}\frac{b(x)|v + t\psi|^2 - b(x)|v|^2}{t}dx = 2\int_{\mathbb{R}^N} b(x)v\psi\, dx. \tag{2.5}$$

Combining (2.2)-(2.5), we deduce that

$$\begin{aligned}\langle DT(w), \Phi\rangle &= \lim_{t\to 0}\frac{T(w + t\Phi) - T(w)}{t} \\ &= \int_{\mathbb{R}^N}(h_1(x)\nabla u\nabla\varphi + h_2(x)\nabla v\nabla\psi + a(x)u\varphi + b(x)v\psi)\, dx.\end{aligned}$$

Thus T is weakly differentiable on H.

Let $\Phi = (\varphi, \psi) \in H$ be fixed. We now prove that the map $w \mapsto \langle DT(w), \Phi\rangle$ is continuous on H. Let $\{w_m\}$ be a sequence converging to w in H. We have

$$\begin{aligned}&\big|\langle DT(w_m), \Phi\rangle - \langle DT(w), \Phi\rangle\big| \\ &\le \int_{\mathbb{R}^N} h_1(x)|\nabla u_m - \nabla u||\nabla\varphi|dx + \int_{\mathbb{R}^N} h_2(x)|\nabla v_m - \nabla v||\nabla\psi|dx \\ &\quad + \int_{\mathbb{R}^N} a(x)|u_m - u||\varphi|dx + \int_{\mathbb{R}^N} b(x)|v_m - v||\psi|dx.\end{aligned}$$

It follows by applying Cauchy's inequality that

$$|\langle DT(w_m), \Phi\rangle - \langle DT(w), \Phi\rangle| \le 4\|\Phi\|_H\|w_m - w\|_H \to 0 \quad \text{as } m \to \infty. \tag{2.6}$$

Thus the map $w \mapsto \langle DT(w), \Phi\rangle$ is continuous on H and we conclude that functional T is weakly continuously differentiable on H. Finally, J is weakly continuously differentiable on H. □

Remark 2.3. From Proposition 2.2 we observe that the weak solutions of system (1.1) correspond to the critical points of the functional $J(w), w \in H$ given by (1.3). Thus our idea is to apply a variant of the Mountain pass theorem in [8] for obtaining non-trivial critical points of J and thus they are also the non-trivial weak solutions of system (1.1).

Proposition 2.4. *The functional $J(w), w \in H$ given by* (1.3) *satisfies the Palais-Smale condition.*

Proof. Let $\{w_m = (u_m, v_m)\}$ be a sequence in H such that

$$\lim_{m\to\infty} J(w_m) = c, \quad \lim_{m\to\infty} \|DJ(w_m)\|_{H^*} = 0.$$

First, we prove that $\{w_m\}$ is bounded in H. We assume by contradiction that $\{w_m\}$ is not bounded in H. Then there exists a subsequence $\{w_{m_j}\}$ of $\{w_m\}$ such that $\|w_{m_j}\|_H \to \infty$ as $j \to \infty$. By assumption (H3) it follows that

$$\begin{aligned}
&J(w_{m_j}) - \frac{1}{\mu}\langle DJ(w_{m_j}), w_{m_j}\rangle \\
&= \big(\frac{1}{2} - \frac{1}{\mu}\big)\|w_{m_j}\|_H^2 + \big(\frac{1}{\mu}\langle DP(w_{m_j}), w_{m_j}\rangle - P(w_{m_j})\big) \\
&\geq \gamma_0 \|w_{m_j}\|_H^2,
\end{aligned}$$

where $\gamma_0 = \frac{1}{2} - \frac{1}{\mu}$. This yields

$$\begin{aligned}
J(w_{m_j}) &\geq \gamma_0 \|w_{m_j}\|_H^2 + \frac{1}{\mu}\langle DJ(w_{m_j}), w_{m_j}\rangle \\
&\geq \gamma_0 \|w_{m_j}\|_H^2 - \frac{1}{\mu}\|DJ(w_{m_j})\|_{H^*}.\|w_{m_j}\|_H \\
&= \|w_{m_j}\|_H \big(\gamma_0 \|w_{m_j}\|_H - \frac{1}{\mu}\|DJ(w_{m_j})\|_{H^*}\big).
\end{aligned} \tag{2.7}$$

Letting $j \to \infty$, since $\|w_{m_j}\|_H \to \infty$, $\|DJ(w_{m_j})\|_{H^*} \to 0$ we deduce that $J(w_{m_j}) \to \infty$, which is a contradiction. Hence $\{w_m\}$ is bounded in H.

Since H is a Hilbert space and $\{w_m\}$ is bounded in H, there exists a subsequence $\{w_{m_k}\}$ of $\{w_m\}$ weakly converging to w in H. Moreover, since the embedding $H \hookrightarrow E$ is continuous, $\{w_{m_k}\}$ is weakly convergent to w in E. We shall prove that

$$T(w) \leq \liminf_{k\to\infty} T(w_{m_k}). \tag{2.8}$$

Since the embedding $E \hookrightarrow L^2(\mathbb{R}^N, \mathbb{R}^2)$ is compact, $\{w_{m_k}\}$ converges strongly to w in $L^2(\mathbb{R}^N, \mathbb{R}^2)$. Therefore, for all $\Omega \subset\subset \mathbb{R}^N$, $\{w_{m_k}\}$ converges strongly to w in $L^1(\Omega, \mathbb{R}^2)$. Besides, for any $\Phi = (\varphi, \psi) \in E$ we have

$$\begin{aligned}
&\big|\int_\Omega (a(x)(u_{m_k} - u)\varphi + b(x)(v_{m_k} - v)\psi)\,dx\big| \\
&\leq \max\Big(\sup_\Omega a(x), \sup_\Omega b(x)\Big)\Big(\int_\Omega |u_{m_k} - u||\varphi|dx + \int_\Omega |v_{m_k} - v||\psi|dx\Big).
\end{aligned}$$

Applying Cauchy inequality we obtain

$$\begin{aligned}
&\big|\int_\Omega (a(x)(u_{m_k} - u)\varphi + b(x)(v_{m_k} - v)\psi)\,dx\big| \\
&\leq \gamma_1 \|\Phi\|_{L^2(\mathbb{R}^N,\mathbb{R}^2)} \|w_{m_k} - w\|_{L^2(\mathbb{R}^N,\mathbb{R}^2)},
\end{aligned}$$

where $\gamma_1 = \max(\sup_\Omega a(x), \sup_\Omega b(x)) > 0$. Letting $k \to \infty$ we get

$$\lim_{k\to\infty} \int_\Omega (a(x)(u_{m_k} - u)\varphi + b(x)(v_{m_k} - v)\psi)\,dx = 0. \tag{2.9}$$

On the other hand, since w_{m_k} converges weakly to w in E; i.e.,

$$\lim_{k\to\infty}\int_{\mathbb{R}^N}[(\nabla u_{m_k}-\nabla u)\nabla\varphi+(\nabla v_{m_k}-\nabla v)\nabla\psi]\,dx + \lim_{k\to\infty}\int_{\mathbb{R}^N}[a(x)(u_{m_k}-u)\varphi+b(x)(v_{m_k}-v)\psi]\,dx=0$$

for all $\Phi=(\varphi,\psi)\in E$, by (2.9) and $C_0^\infty(\mathbb{R}^N,\mathbb{R}^2)\subset H\subset E$ we infer that

$$\lim_{k\to\infty}\int_{\Omega}[(\nabla u_{m_k}-\nabla u)\nabla\varphi+(\nabla v_{m_k}-\nabla v)\nabla\psi]\,dx=0,$$

for all $\Omega\subset\subset\mathbb{R}^N$. This implies that $\{\nabla w_{m_k}\}$ converges weakly to ∇w in $L^1(\Omega,\mathbb{R}^2)$. Applying [15, Theorem 1.6], we obtain

$$T(w)\le\liminf_{k\to\infty}T(w_{m_k}).$$

Thus (2.8) is proved. We now prove that

$$\lim_{k\to\infty}\langle DP(w_{m_k}),w_{m_k}-w\rangle=\lim_{k\to\infty}\int_{\mathbb{R}^N}\nabla F(x,w_{m_k}).(w_{m_k}-w)dx=0. \tag{2.10}$$

Indeed, by (H2), we have

$$\begin{aligned}
&|\nabla F(x,w_{m_k})(w_{m_k}-w)|\\
&=|f(x,w_{m_k})(u_{m_k}-u)+g(x,w_{m_k})(v_{m_k}-v)|\\
&\le|\nabla f(x,\theta_1 w_{m_k})||w_{m_k}||u_{m_k}-u|+|\nabla g(x,\theta_2 w_{m_k})||w_{m_k}||v_{m_k}-v|\\
&\le A_1|w_{m_k}|^p|u_{m_k}-u|+A_2|w_{m_k}|^p|v_{m_k}-v|\\
&\le A_3|w_{m_k}|^p|w_{m_k}-w|,\quad 0<\theta_1,\theta_2<1
\end{aligned}$$

where A_i $(i=1,2,3)$ are positive constants.

Set $2^*=\frac{2N}{N-2}$, $p_1=\frac{2^*}{p-1}$, $p_2=p_3=\frac{2p_1}{p_1-1}$. We have $p_1>1$, $2<p_2,p_3<2^*$ and $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$. Therefore,

$$\begin{aligned}
\lim_{k\to\infty}\int_{\mathbb{R}^N}\nabla F(x,w_{m_k}).(w_{m_k}-w)dx&\le A_3\int_{\mathbb{R}^N}|w_{m_k}|^{p-1}|w_{m_k}-w||w_{m_k}|dx\\
&\le A_3\|w_{m_k}\|_{L^{2^*}}^{p-1}\|w_{m_k}-w\|_{L^{p_2}}\|w_{m_k}\|_{L^{p_3}}.
\end{aligned}$$

On the other hand, using the continuous embeddings $H\hookrightarrow E\hookrightarrow L^q(\mathbb{R}^N)$, $2\le q\le 2^*$ together with the interpolation inequality (where $\frac{1}{p_2}=\frac{\delta}{2}+\frac{1-\delta}{2^*}$), it follows that

$$\|w_{m_k}-w\|_{L^{p_2}(\mathbb{R}^N)}\le\|w_{m_k}-w\|_{L^2(\mathbb{R}^N)}^{\delta}.\|w_{m_k}-w\|_{L^{2^*}}^{1-\delta}.$$

Since the embedding $E\hookrightarrow L^2(\mathbb{R}^N)$ is compact we have $\|w_{m_k}-w\|_{L^2(\mathbb{R}^N)}\to 0$ as $k\to\infty$. Hence $\|w_{m_k}-w\|_{L^{p_2}(\mathbb{R}^N)}\to 0$ as $k\to\infty$ and (2.10) is proved.

On the other hand, by (2.10) and (2.1) it follows

$$\lim_{k\to\infty}\langle DT(w_{m_k}),w_{m_k}-w\rangle=0.$$

Hence, by the convex property of the functional T we deduce that

$$\begin{aligned}
T(w)-\lim_{k\to\infty}\sup T(w_{m_k})&=\lim_{k\to\infty}\inf\,[T(w)-T(w_{m_k})] &(2.11)\\
&\ge\lim_{k\to\infty}\langle DT(w_{m_k}),w-w_{m_k}\rangle=0. &(2.12)
\end{aligned}$$

Relations (2.8) and (2.11) imply

$$T(w) = \lim_{k\to\infty} T(w_{m_k}). \tag{2.13}$$

Finally, we prove that $\{w_{m_k}\}$ converges strongly to w in H. Indeed, we assume by contradiction that $\{w_{m_k}\}$ is not strongly convergent to w in H. Then there exist a constant $\epsilon_0 > 0$ and a subsequence $\{w_{m_{k_j}}\}$ of $\{w_{m_k}\}$ such that $\|w_{m_{k_j}} - w\|_H \geq \epsilon_0 > 0$ for any $j = 1, 2, \dots$. Hence

$$\frac{1}{2}T(w_{m_{k_j}}) + \frac{1}{2}T(w) - T\Big(\frac{w_{m_{k_j}} + w}{2}\Big) = \frac{1}{4}\|w_{m_{k_j}} - w\|_H^2 \geq \frac{1}{4}\epsilon_0^2. \tag{2.14}$$

With the same arguments as in the proof of (2.8), and remark that the sequence $\{\frac{w_{m_{k_j}}+w}{2}\}$ converges weakly to w in E, we have

$$T(w) \leq \liminf_{j\to\infty} T\Big(\frac{w_{m_{k_j}} + w}{2}\Big). \tag{2.15}$$

Hence letting $j \to \infty$, from (2.13) and (2.14) we infer that

$$T(w) - \liminf_{j\to\infty} T\Big(\frac{w_{m_{k_j}} + w}{2}\Big) \geq \frac{1}{4}\epsilon_0^2. \tag{2.16}$$

Relations (2.15) and (2.16) imply $0 \geq \frac{1}{4}\epsilon_0^2 > 0$, which is a contradiction. Therefore, we conclude that $\{w_{m_k}\}$ converges strongly to w in H and J satisfies the Palais - Smale condition on H. $\square$

To apply the Mountain pass theorem we shall prove the following proposition which shows that the functional J has the Mountain pass geometry.

Proposition 2.5. *(i) There exist $\alpha > 0$ and $r > 0$ such that $J(w) \geq \alpha$, for all $w \in H$ with $\|w\|_H = r$.*

(ii) There exists $w_0 \in H$ such that $\|w_0\|_H > r$ and $J(w_0) < 0$.

Proof. (i) From (H3), it is easy to see that

$$F(x,z) \geq \min_{|s|=1} F(x,s).|z|^\mu > 0 \quad \forall x \in \mathbb{R}^N \text{ and } |z| \geq 1, z \in \mathbb{R}^2, \tag{2.17}$$

$$0 < F(x,z) \leq \max_{|s|=1} F(x,s).|z|^\mu \quad \forall x \in \mathbb{R}^N \text{ and } 0 < |z| \leq 1, \tag{2.18}$$

where $\max_{|s|=1} F(x,s) \leq C$ in view of (H2). It follows from (2.18) that

$$\lim_{|z|\to 0} \frac{F(x,z)}{|z|^2} = 0 \quad \text{uniformly for } x \in \mathbb{R}^N. \tag{2.19}$$

By using the embeddings $H \hookrightarrow E \hookrightarrow L^2(\mathbb{R}^N, \mathbb{R}^2)$, with simple calculations we infer from (2.19) that $\inf_{\|w\|_H = r} J(w) = \alpha > 0$ for $r > 0$ small enough. This implies (i).

(ii) By (2.17), for each compact set $\Omega \subset \mathbb{R}^N$ there exists $\overline{c} = \overline{c}(\Omega)$ such that

$$F(x,z) \geq \overline{c}|z|^\mu \quad \text{for all } x \in \Omega, |z| \geq 1. \tag{2.20}$$

Let $0 \neq \Phi = (\varphi, \psi) \in C^1(\mathbb{R}^N, \mathbb{R}^2)$ having compact support, for $t > 0$ large enough, from (2.20) we have

$$J(t\Phi) = \frac{1}{2}t^2\|\Phi\|_H^2 - \int_{\mathbb{R}^N} F(x, t\Phi)dx \leq \frac{1}{2}t^2\|\Phi\|_H^2 - t^\mu\overline{c}\int_\Omega |\Phi|^\mu dx, \tag{2.21}$$

where $\overline{c} = \overline{c}(\Omega)$, $\Omega = (\operatorname{supp}\varphi \cup \operatorname{supp}\psi)$. Then (2.21) and $\mu > 2$ imply (ii). $\square$

Proof of Theorem 1.4. It is clear that $J(0) = 0$. Furthermore, the acceptable set

$$G = \{\gamma \in C([0,1], H) : \gamma(0) = 0, \gamma(1) = w_0\},$$

where w_0 is given in Proposition 2.5, is not empty (it is easy to see that the function $\gamma(t) = t\omega_0 \in G$). By Proposition 1.2 and Propositions 2.2-2.5, all assumptions of the Mountain pass theorem introduced in [8] are satisfied. Therefore there exists $\hat{w} \in H$ such that

$$0 < \alpha \le J(\hat{w}) = \inf\{\max J(\gamma([0,1])) : \gamma \in G\}$$

and $\langle DJ(\hat{w}), \Phi\rangle = 0$ for all $\Phi \in H$; i.e., $\hat{w}$ is a weak solution of system (1.1). The solution $\hat{w}$ is a non-trivial solution by $J(\hat{w}) \ge \alpha > 0$. The proof is complete. $\square$

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