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EXISTENCE OF POSITIVE SOLUTIONS FOR SEMI-POSITONE SYSTEMS

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Dedicated to Professor Claude Lobry on the occasion of his 60-th birthday

Abstract. This paper studies semi-positone systems of equations by using phase plane analysis and fixed point theory.

1. Introduction

This paper concerns the existence of positive solutions for the system

$$
\begin{gathered}
-u'' = \lambda f(u,v) \\
-v'' = \mu\, g(u,v) \\
u(0) = u(1) = 0 = v(0) = v(1).
\end{gathered} \tag{1.1}
$$

where $\lambda, \mu > 0$ are positive parameters and the nonlinear functions $f,\ g : [0,\infty) \to \mathbb{R}$ are continuous and satisfy the assumptions:

(A1) For v fixed,

$$
\begin{gathered}
f(s,v) < 0, \quad \text{for } 0 < s < s_0 \quad \text{or} \quad s > s_1 \\
f(s,v) > 0, \quad \text{for } s_0 < s < s_1, \\
\int_0^{s_1} f(s,v)ds > 0\,;
\end{gathered}
$$

(A2) For u fixed, g satisfies hypotheses similar to those of f in (A1);

(A3) (i) f (resp. g) is monotone increasing in v (resp. in u) or f (resp. g) is monotone decreasing in v (resp. in u) or

(ii) $\int_{s_0}^{s_1} \min_{0\le v\le s_1} f(u,v)du > -\int_0^{s_0} \min_{0\le v\le s_1} f(u,v)du$ (resp. g).

We point out that assumptions (A1) and (A2) on the nonlinearities f and g are natural and necessary for existence of positive solutions when $f(.,v)$ and $g(u,.)$ are negative at zero, as shown by the results of [9] and [6] in the case of a single equation.

Systems of type (1.1) are referred to as semi-positone systems. Such problems arise in many situations, for instance, in steady state problems of population models

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with nonlinear sources. In section 4 we provide an application to a model leading to a system of type (1.1). We refer the reader to [5], where Castro and Shivaji initially called such problems (with $f(0) < 0$) *non-positone problems*, in contrast with the terminology *positone problems*, coined by Cohen and Keller in [7], when the nonlinearity f was *positive* and *monotone*. In fact, P. L. Lions provided in [12] an extensive review on positive solutions of scalar positone equations and already remarked that the case when the nonlinearity is negative near the origin was mathematically challenging and important for applications.

Semipositone problems with scalar equations have motivated many researchers in the last fifteen years. We refer the reader to the survey paper by Castro-Maya-Shivaji [4] and references therein for a review. On the other hand, semi-positone systems have been studied in [1], [11] and [3], where the authors used the method of sub- and super-solutions. However, in their results the nonlinear functions f and g are usually monotonic in both variables u and v. In [3], the construction of sub- and super-solutions was based on the structure $f = f_1(u,v) - h_1(x)$ and $g = g_1(u,v) - h_2(x)$ of the nonlinear functions f and g, with the functions f_1 and g_1 having nonnegative partial derivative in the variables u and v. Also, in [9] and [6], semi-positone problems for a scalar equation have been studied under general assumptions on the nonlinearity f. As mentioned earlier, it was shown (among other things) that if $f(0) < 0$, then the condition $\int_0^{s_1} f(s)ds > 0$ is a necessary condition for existence of positive solutions (sufficiency was also shown in [6]). More recently, the authors in [8] studied semi-positone problems in the ODE case under very weak conditions on f by using phase plane analysis for conservative systems.

In the present paper, we prove existence of positive solutions for semi-positone systems under general conditions on the nonlinearities. We prove that the condition given in [9] and [6] is a sufficient condition in the case of systems, and we extend one of the results of [8] to systems of equations. Our approach combines the theory of conservative systems and tools of functional analysis, like fixed point theory.

For the rest of this article, we let $C_0[0,1]$ denote the space of continuous functions v on $[0,1]$ satisfying the boundary conditions $v(0) = v(1) = 0$, and endowed with the sup norm. Also, we let $\mathcal{K}$ denote the closed, convex cone of nonnegative functions in $C_0[0,1]$, namely, $\mathcal{K} = \{v \in C_0[0,1 : v \geq 0\}$. In addition, for a given v in $\mathcal{K}$, we define the continuous function

$$f_v(s,u) = f(u, v(s)), \quad \text{for } 0 \leq s \leq 1,\ u \geq 0,$$

and consider the auxiliary problem

$$\begin{gathered} -u'' = \lambda f_v(s,u) \\ u(0) = u(1) = 0 \end{gathered} \tag{1.2}$$

2. Existence result for (1.2)

The proof of existence of positive solutions for (1.2) is based on the method of sub and super-solutions. From the function f_v we construct the functions F_v and G_v below, where we shall omit the subscript v in the latter functions for simplicity of notation:

$$F(u) = \min_{0 \leq s \leq 1} f_v(s,u) \quad \text{and} \quad G(u) = \max_{0 \leq s \leq 1} f_v(s,u).$$

Since f is continuous in (u,v) and v is in $\mathcal{K}$, the functions F and G are well-defined, continuous, and satisfy the following properties:

(i)
$$\begin{gathered}F(s), G(s) < 0, \quad \text{for } 0 \le s < s_0, \text{ or } s > s_1 \\ F(s), G(s) > 0, \quad \text{for } s_0 < s < s_1 \\ \int_0^{s_1} F(s)ds > 0, \quad \int_0^{s_1} G(s)ds > 0\end{gathered}$$

(ii) $F(u) \le f_v(s,u) \le G(u)$, for $0 \le s \le 1$, and $u \ge 0$.

Note that solutions of the problems
$$\begin{gathered}-u'' = \lambda F(u) \\ u(0) = u(1) = 0\end{gathered} \tag{2.1}$$
and
$$\begin{gathered}-u'' = \lambda G(u) \\ u(0) = u(1) = 0\end{gathered} \tag{2.2}$$
are, respectively, sub-solutions and super-solutions of (1.2).

Existence of a sub-solution for (1.2)**.** We are interested in the existence of a positive solution for problem (2.1). We have the following result.

Lemma 2.1. *Assume that f satisfies assumptions* (A1) *and* (A3)*. Then, there exists $\Lambda_1 > 0$ such that the problem* (2.1) *has a positive solution if and only if $\lambda \ge \Lambda_1$.*

Proof. We prove this lemma by using phase-plane analysis on the conservative system associated to (2.1). Let us introduce the parameter $L = \sqrt{\lambda}$, and the new function $w(Lt) = u(t)$. Then problem (2.1) becomes
$$\begin{gathered}-w'' = F(w) \\ w(0) = w(L) = 0\,.\end{gathered}$$
Next, by defining $u_1 = w$ and $u_2 = u_1'$, the above problem is equivalent to the system
$$\begin{gathered}u_1' = u_2 \\ u_2' = -F(u_1) \\ u_1(0) = u_1(L) = 0\,.\end{gathered} \tag{2.3}$$
Therefore, we study the solutions of the conservative system (2.3). It is known that the energy is constant along the trajectories of (2.3) in the phase plane (u_1, u_2); i. e., we have
$$E = \frac{u_2^2}{2} + U(u_1) = \text{constant}\,, \tag{2.4}$$
where U is the potential energy function given by
$$U(x) = \int_0^x F(\xi)d\xi\,.$$
We represent in Figure 1 and 2, respectively, the potential energy function U and the phase-portrait of the system (2.3).

From equation (2.4), we define the time map $T:\,[0,k_0) \to (0,\infty)$,
$$T(k) = \int_0^{u(k)} \frac{d\xi}{\sqrt{2(E(k) - U(\xi))}}$$

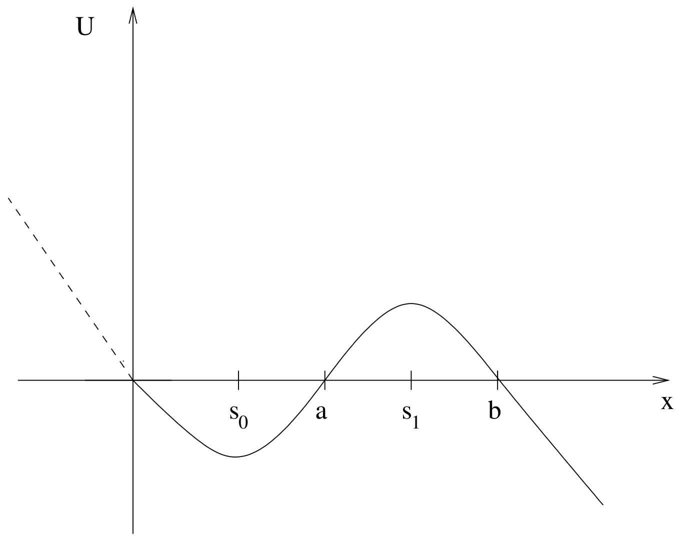

FIGURE 1. Potential energy function U

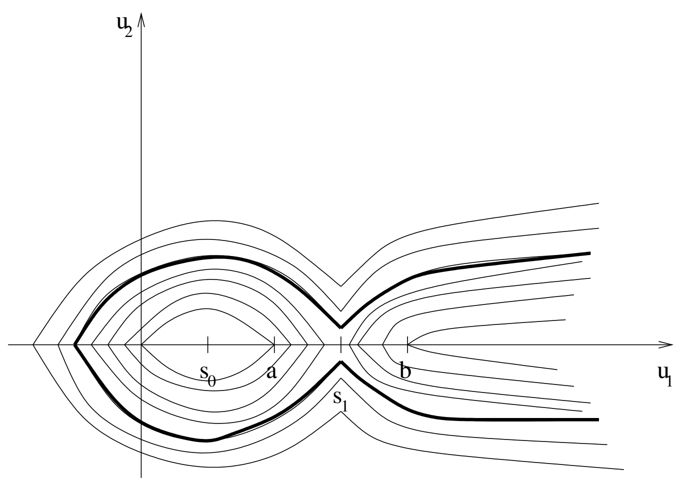

FIGURE 2. Phase-portrait of system (2.3)

for initial conditions $u_1(0) = 0$ and $u_2(0) = k$, where $u(k)$ is the distance reached by $u_1(t)$, from the origin, the first time when $u_2(t) = 0$ in the u_1u_2-plane. Therefore, problem (2.3) has a positive solution if and only if there is $k \geq 0$ such that $L = 2T(k)$.

Let a be the first zero of U in $(0, s_1)$ (see figure 1), $M = U(s_1) > 0$, $m = U(s_0)$, and $k_0 = \sqrt{2M}$. Then we have $a \leq u(k)$ for all $0 \leq k < k_0$ and $\lim_{k\to k_0^-} u(k) = \infty$. Also note that U is increasing in (s_0, s_1) and $E = U(u(k)) = \frac{k^2}{2}$ for all $k \geq 0$.

Therefore, for all $0 \le k < k_0$, we have $E - U(s) \le \frac{k_0^2}{2} - U(s_0) = \frac{k_0^2}{2} - m$ and

$$T(k) = \int_0^{u(k)} \frac{d\xi}{\sqrt{2(E - U(\xi))}} \ge \int_0^{u(k)} \frac{d\xi}{\sqrt{2(\frac{k_0^2}{2} - m)}} = \frac{u(k)}{\sqrt{k_0^2 - 2m}}$$

for all $0 \le k < k_0$. Hence, $\lim_{k \to k_0} T(k) = \infty$ and $T(k)$ has the form displayed in figure 3.

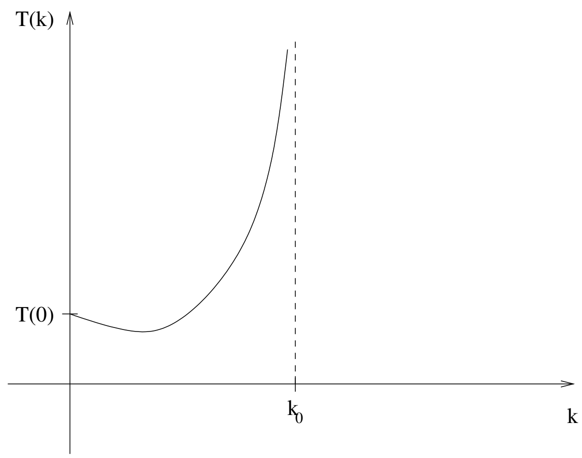

FIGURE 3. Time map $T(k)$

Now, if we denote by $T_1 = \min_{0 \le k < k_0^-} T(k)$ the minimum of the time map T on $[0, k_0)$, then the system (2.3) has a positive solution if and only if $L \ge 2T_1$. So, letting $\Lambda_1 = 4T_1^2$, problem (2.1) has a positive solution if and only if $\lambda \ge \Lambda_1$. □

Existence of a super-solution for (1.2)**.** We have a similar result on existence of a positive solution for (2.2). Recall that solutions of (2.2) are super-solutions of (1.2).

Lemma 2.2. *Assume that f satisfies assumptions (A1) and (A3). Then, there exists $\Lambda_2 > 0$ such that* (2.2) *has a positive solution if and only if $\lambda \ge \Lambda_2$.*

Proof. The proof is similar to that of Lemma 2.1. We recall the problem (2.2) is

$$\begin{gathered} -u'' = \lambda G(u) \\ u(0) = u(1) = 0 \end{gathered}$$

Similarly, the potential energy function V and the time map S are defined by

$$V(x) = \int_0^x G(\xi) d\xi, \quad S(k) = \int_0^{u(k)} \frac{d\xi}{\sqrt{2(E(k) - V(\xi))}}$$

and the associated conservative system is

$$\begin{gathered} x_1' = x_2 \\ x_2' = -G(x_1) \\ x_1(0) = x_2(L) = 0\,. \end{gathered} \tag{2.5}$$

The time map $S(k)$ has the same shape of the time map $T(k)$ in the proof of Lemma 2.1. Also, by using arguments similar to those in Lemma 2.1, we conclude that (2.2) has a positive solution if and only if $\lambda \geq \Lambda_2$, where $\Lambda_2 = 4(\min_{0\leq k<k_1} S(k))^2$ and $k_1 = \sqrt{2V(s_1)}$. □

Existence of a positive solution for (1.2). Before giving the main result in this section, we prove, in the next lemma, existence of a particular ordered pair (w_1, w_2) formed by a positive solution of (2.1) and a positive solution of (2.2).

Lemma 2.3. *Assume that f satisfies assumptions* (A1) *and* (A3). *Then, there exists $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$, there exists a positive solution w_1 of* (2.1) *and a positive solution w_2 of* (2.2) *with*

$$w_1 \leq w_2 \quad \textit{everywhere in } [0,1].$$

Proof. Let $\lambda_0 = \max(\Lambda_1,\ \Lambda_2)$, where Λ_1 and Λ_2 are given in Lemmas 2.1 and 2.2, and let us consider the potential energy functions U and V corresponding for the systems (2.3) and (2.5). These can be represented together in the same figure as follows (see figure 4).

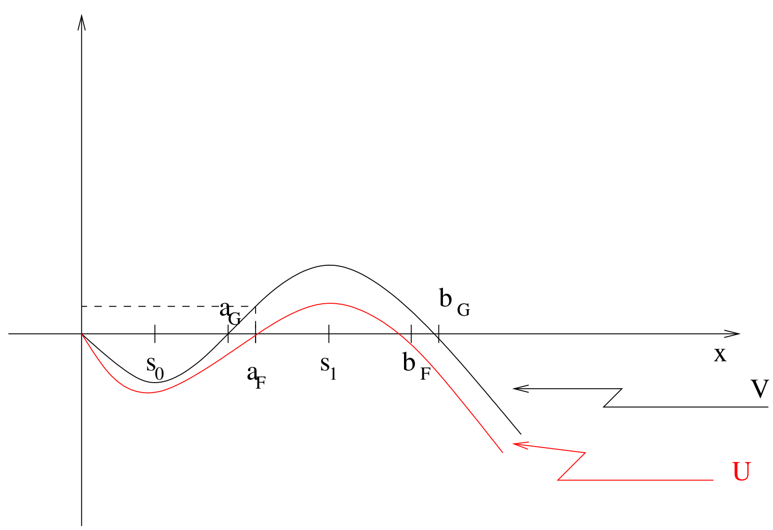

FIGURE 4. Potential energy functions for (2.3) and (2.5)

Let a_F and a_G be, respectively, the first zero of U and V in $(0, s_1)$ (cf. figure 4). Then

$$T(0) = \int_0^{a_F} \frac{d\xi}{\sqrt{-2U(\xi)}}, \quad S(0) = \int_0^{a_G} \frac{d\xi}{\sqrt{-2V(\xi)}}, \quad \text{and} \quad a_G \leq a_F < s_1.$$

Since $U(a_F) = 0$ and $V(a_F) > 0$, we define $\bar{k} = \sqrt{2V(a_F)} > 0$, and denote by (u_1, u_2) the solution of (2.3) corresponding to $T(0)$ and by (x_1, x_2) the solution of (2.5) corresponding to $S(\bar{k})$. Clearly, $T(0) = S(\bar{k})$ and $u_1(T(0)) = a_F = x_1(S(\bar{k}))$. The solutions (u_1, u_2) and (x_1, x_2) are represented together in figure 5.

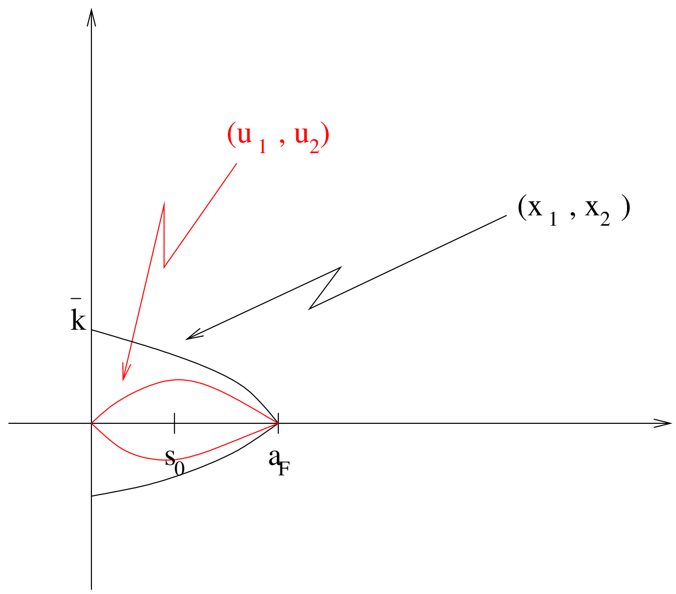

Figure 5. Special solutions of (2.3) and (2.5)

Now, we claim that $u_1(t) \le x_1(t)$ for all $0 \le t \le L$. The proof of this claim will be done by integrating backward the systems (2.3) and (2.5). Indeed, we first introduce the functions $\varphi_1(t) = u_1(\frac{L}{2} - t)$ and $\varphi_2(t) = u_2(\frac{L}{2} - t)$, so that we have

$$\begin{gathered} \varphi_1' = -\varphi_2 \\ \varphi_2' = F(\varphi_1) \\ \varphi_1(0) = a_F, \quad \varphi_2(0) = 0\,. \end{gathered} \tag{2.6}$$

Similarly, by introducing $\psi_1(t) = x_1(\frac{L}{2} - t)$ and $\psi_2(t) = x_2(\frac{L}{2} - t)$, we have

$$\begin{gathered} \psi_1' = -\psi_2 \\ \psi_2' = F(\psi_1) \\ \psi_1(0) = a_F, \quad \psi_2(0) = 0\,. \end{gathered} \tag{2.7}$$

Now, assume that there exists $t_0 \in (0, \frac{L}{2})$ such that $\varphi_1(t_0) = \psi_1(t_0)$ and $\varphi_2(t_0) = \psi_2(t_0)$ (equivalently, $u_1(\frac{L}{2} - t_0) = x_1(\frac{L}{2} - t_0)$ and $u_2(\frac{L}{2} - t_0) = x_2(\frac{L}{2} - t_0)$, or in other words the solution's curves in the phase plane touch each other). Then, we have

$$\varphi_2'(t_0) = F(\varphi_1(t_0)) \le G(\varphi_1(t_0)) = G(\varphi_1(t_0)) = \psi_2'(t_0).$$

Therefore, as t increases from t_0, $\varphi_1'(t)$ will remain greater than $\psi_1'(t)$, that is $u_1'(\frac{L}{2} - t)$ remains less than $x_1'(\frac{L}{2} - t)$ for all t in $[t_0, t_0 + \varepsilon]$ and for small $\varepsilon > 0$. Hence, $u_1(\frac{L}{2} - t) \le x_1(\frac{L}{2} - t)$, for all $t \in [t_0, t_0 + \varepsilon]$. And repeating this reasoning from $t_0 + \varepsilon$, we finally get $u_1(t) \le x_1(t)$ for all $0 \le t \le \frac{L}{2}$. By symmetry of the periodic solutions (u_1, u_2) and (x_1, x_2) about the horizontal axis in the phase plane, we have $u_1(t) \le x_1(t)$ for all $\frac{L}{2} \le t \le L$. To complete the proof, we let $w_1(t) = u_1(Lt)$ and $w_2(t) = x_1(Lt)$. □

Based on the above lemmas, we state and prove the main result of this section.

Theorem 2.4. *Assume that f satisfies assumptions* (A1) *and* (A3). *Then, for any $\lambda \geq \lambda_0$, problem* (1.2) *has a positive solution u satisfying*

$$0 \leq w_1 \leq u \leq w_2 \leq a_F$$

where λ_0, w_1 and w_2 are given in Lemma 2.3. *Moreover, u is uniquely defined by the conditions $u(\frac{1}{2}) = a_F$ and $u'(\frac{1}{2}) = 0$. We denote the mapping $v \mapsto u$ by $\mathcal{F}_1$.*

Proof. The existence of u is guaranteed by the existence of the particular sub- and super-solutions w_1 and w_2 given in Lemma 2.3. Therefore, we have to prove that u is uniquely defined by the conditions stated above.

Indeed, since $w_1 \leq u \leq w_2$ and $w_1(\frac{1}{2}) = w_2(\frac{1}{2}) = a_F$, it follows that u satisfies the Cauchy Problem

$$\begin{gathered} -u'' = F(u) \\ u(\frac{1}{2}) = a_F, \quad u'(\frac{1}{2}) = 0 \end{gathered} \tag{2.8}$$

Therefore, by integrating equation (2.8) forward and backward, we see that u is uniquely defined on both intervals $[\frac{1}{2}, 1]$ and $[0, \frac{1}{2}]$. This complete the proof. □

3. Existence of a positive solution for (1.1)

Let us consider the solution u of (1.2) given in Theorem 2.4. Observing that u belongs to $\mathcal{K}$, we consider (by symmetry) the problem

$$\begin{gathered} -v'' = \mu g_u(s, v) \\ v(0) = v(1) = 0 \end{gathered} \tag{3.1}$$

where $g_u(s, v) = g(u(s), v)$ for all $0 \leq s \leq 1$ and $v \geq 0$. Then, similarly to problem (1.2), we obtain the following result on existence of a positive solution for (3.1).

Corollary 3.1. *Assume that g satisfies assumptions* (A2)–(A3). *Then, there exists $\mu_0 > 0$ such that, for any $\mu \geq \mu_0$, problem* (3.1) *has a positive solution v uniquely defined by $v(\frac{1}{2}) = a_G$ and $v'(\frac{1}{2}) = 0$ and such that*

$$0 \leq v \leq s_1.$$

We denote the mapping $u \mapsto v$ by $\mathcal{F}_2$.

We shall use Schauder's Fixed Point Theorem [13, p 126] to prove existence of a positive solution for (1.1). For that, we define the mapping

$$\mathcal{F} = \mathcal{F}_2 \circ \mathcal{F}_1 : \mathcal{K} \to \mathcal{K}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by Theorem 2.4 and Corollary 3.1, respectively. Then, we have the following result.

Theorem 3.2. *Assume assumptions* (A1)–(A3). *Then, there exist $\lambda_0 > 0$, $\mu_0 > 0$ such that for any pair (λ, μ) with $\lambda \geq \lambda_0$, $\mu \geq \mu_0$, problem* (1.1) *has a positive solution (u, v). Furthermore, v is a fixed point of the mapping $\mathcal{F}$.*

Proof. Let $\mathcal{K}_0$ be the bounded, closed, convex subset of $C_0[0, 1]$ consisting of those functions $v \in \mathcal{K}$ satisfying $0 \leq v \leq s_1$; i. e.,

$$\mathcal{K}_0 = \{v \in \mathcal{K} : 0 \leq v \leq s_1\}.$$

On the other hand, by Theorem 2.4 and Corollary 3.1, we note that $\mathcal{F} : \mathcal{K} \to \mathcal{K}$ is well-defined and continuous (by continuous dependence of solutions of differential equations on parameters) and, furthermore, $\mathcal{F}$ maps $\mathcal{K}_0$ into $\mathcal{K}_0$. Also, by standard regularity, we have that $\mathcal{F}(v) \in C^2[0,1]$, so that $\mathcal{F}$ is completely continuous. Therefore, by Schauder's Fixed Point Theorem, $\mathcal{F}$ has a fixed point $v \in \mathcal{K}_0$. And, if we denote by u the solution of problem (1.2) given in Theorem 2.4 with v being the fixed point of $\mathcal{F}$, then $u \in \mathcal{K}_0$ and the pair (u,v) is a positive solution of (1.1). $\square$

4. An example

We shall consider the following reaction-diffusion system that describes the dynamics of two competing species in an environment Ω that we assume to be one-dimensional. Let $u(t,x)$ and $v(t,x)$ represent the concentrations or population densities of the two species. With the standard non-dimensionalization process, we may assume the diffusion coefficient equal to 1, and $\Omega = (0,1)$. The boundary of the environment is considered to be hostile to the species (so, one has zero Dirichlet boundary conditions) and we assume that the reproduction or growth of the species follows the law $\rho_1(u,v) = \lambda\frac{mu}{K+u+\frac{u^2}{\alpha}}(v+1)$ for the species u and $\rho_2(u,v) = \mu\frac{mv}{K+v+\frac{v^2}{\alpha}}(u+1)$ for the species v. Then, the dynamics of the two species is described by

$$\begin{gathered}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda(\frac{mu}{K+u+\frac{u^2}{\alpha}} - \alpha)(v+1) \\
\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \mu(\frac{mv}{K+v+\frac{v^2}{\alpha}} - \alpha)(u+1) \\
u(0) = u(1) = 0, \quad v(0) = v(1) = 0\,.
\end{gathered} \tag{4.1}$$

In the above system the the constants K, α, λ and μ are positive and the terms $-\lambda(v+1)$ and $-\mu(u+1)$ represent a mortality or harvesting of the species occurring in the interior of the environment. The steady state problem associated with (4.1) is given by

$$\begin{gathered}
-u'' = \lambda f(u,v) \\
-v'' = \mu g(u,v) \\
u(0) = u(1) = 0 = v(0) = v(1)\,,
\end{gathered} \tag{4.2}$$

where

$$\begin{gathered}
f(u,v) = \Big(\frac{mu}{K+u+\frac{u^2}{\alpha}} - \alpha\Big)(v+1)\,, \\
g(u,v) = \Big(\frac{mv}{K+v+\frac{v^2}{\alpha}} - \alpha\Big)(u+1)
\end{gathered}$$

In what follows, we show that our assumptions (A1)–(A3) are satisfied for this system. We only check the assumptions for the function f since the situation for g is similar.

(A1) is satisfied: Let $\alpha_0 := m + 2K - 2\sqrt{K(m+K)}$. Then, for any $0 < \alpha < \alpha_0$, we have $(m-\alpha)^2 > 4\alpha K$ and the function f satisfies $f(s_0,v) = f(s_1,v) = 0$ for any $v \geq 0$, where

$$s_0 = \frac{m-\alpha-\sqrt{(m-\alpha)^2-4\alpha K}}{2}, \quad s_1 = \frac{m-\alpha+\sqrt{(m-\alpha)^2-4\alpha K}}{2}.$$

In addition, $f(s,v) < 0$ for $0 < s < s_0$ or $s > s_1$, and $f(s,v) > 0$ for $s_0 < s < s_1$. On the other hand, for any $v \geq 0$ fixed, we have

$$\int_0^{s_1} f(u,v)du = (v+1)\Big[m\int_0^{s_1} \frac{u}{K+u+\frac{u^2}{\alpha}}du - \alpha s_1\Big] := (v+1)\,\varphi(\alpha)\,,$$

where

$$\varphi(\alpha) := m\int_0^{s_1} \frac{u}{K+u+\frac{u^2}{\alpha}}du \,-\, \alpha s_1$$

is such that

$$\varphi'(\alpha) = m\int_0^{s_1} \frac{u^3}{(\alpha(K+u)+u^2)^2}\,du \,-\, s_1\,.$$

From these, it clearly follows that $\lim_{\alpha\to 0^+}\varphi(\alpha) = 0$ and $\lim_{\alpha\to 0^+}\varphi'(\alpha) = +\infty$. Then, there exists $\alpha_1 > 0$ such that $\varphi(\alpha) > 0$ for all $\alpha \leq \alpha_1$, hence $\int_0^{s_1} f(u,v)du > 0$ for all $0 < \alpha \leq \alpha_1$. This shows that assumption (A1) is satisfied. Similarly, we see that assumption (A2) is satisfied.
(A3) part (ii) is satisfied: Let us consider the integral

$$\begin{aligned} I &= \int_0^{s_0} \min_{0\leq v\leq s_1} f(u,v)du + \int_{s_0}^{s_1} \min_{0\leq v\leq s_1} f(u,v)du \\ &= \int_0^{s_0} \min_{0\leq v\leq s_1} \Big(\frac{mu}{K+u+\frac{u^2}{\alpha}} - \alpha\Big)(v+1)du \\ &\quad + \int_{s_0}^{s_1} \min_{0\leq v\leq s_1} \Big(\frac{mu}{K+u+\frac{u^2}{\alpha}} - \alpha\Big)(v+1)du\,. \end{aligned}$$

Since

$$\frac{mu}{K+u+\frac{u^2}{\alpha}} - \alpha \leq 0 \quad \text{for all } u \in [0,s_0]$$

and

$$\frac{mu}{K+u+\frac{u^2}{\alpha}} - \alpha \geq 0 \quad \text{for all } u \in [s_0,s_1]\,,$$

we have

$$\begin{aligned} I &= \int_0^{s_0} \Big(\frac{mu}{K+u+\frac{u^2}{\alpha}} - \alpha\Big)(s_1+1)du + \int_{s_0}^{s_1} \Big(\frac{mu}{K+u+\frac{u^2}{\alpha}} - \alpha\Big)du \\ &= s_1\int_0^{s_0} \Big(\frac{mu}{K+u+\frac{u^2}{\alpha}} - \alpha\Big)du + \int_0^{s_1} \Big(\frac{mu}{K+u+\frac{u^2}{\alpha}} - \alpha\Big)du\,. \end{aligned}$$

Therefore, as in the case of $\varphi(\alpha)$ above, there exists $\alpha_2 > 0$ such that $I > 0$ for $0 < \alpha < \alpha_2$.

Conclusion. For any α satisfying $0 < \alpha \leq \min(\alpha_0,\alpha_1,\alpha_2)$, the functions f and g satisfy the assumptions (A1)–(A3).

Remark 4.1. The existence result given in this paper is valid for a system of n equations (with $n > 2$), the proof being similar to the one presented here. Moreover, this idea can be extended to handle systems of semi-positone PDEs.

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