Electronic Journal of Differential Equations, Vol. 2008(2008), No. 123, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE OF COUNTABLY MANY POSITIVE SOLUTIONS FOR nTH-ORDER m-POINT BOUNDARY-VALUE PROBLEMS ON TIME SCALES

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ABSTRACT. In this paper, we study the existence of positive solutions for the nonlinear n-th order with m-point singular boundary-value problem. By using the fixed point index theory and a new fixed point theorem in cones, the existence of countably many positive solutions for a nonlinear singular boundary value problem are obtained.

1. INTRODUCTION

In this paper, by introducing a new operator, improving and generating a p-Laplace operator for some $p > 1$, we study the existence of countably many positive solutions for n-th order with m-point nonlinear boundary-value problems

$$(\varphi(u^{\Delta^{n-1}})(t))^{\nabla} + a(t)f(u(t), u^{\Delta}(t), \dots, u^{\Delta^{n-2}}(t)) = 0, \quad 0 < t < T, \tag{1.1}$$

subject to the boundary conditions

$$\begin{gathered} u^{\Delta^i}(0) = 0, \quad i = 0, 1, \dots, n-3, \\ u^{\Delta^{n-2}}(0) = \sum_{i=1}^{m-2} \alpha_i u^{\Delta^{n-2}}(\xi_i), \quad u^{\Delta^{n-1}}(T) = 0, \end{gathered} \tag{1.2}$$

where $\varphi : R \to R$ is the increasing homeomorphism and positive homomorphism and $\varphi(0) = 0$. $\xi_i \in [0, T]_{\mathbf{T}}$ with $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < T$ and α_i satisfy $\alpha_i \in [0, T]_{\mathbf{T}}$, $0 < \sum_{i=1}^{m-2} \alpha_i < 1$. $a(t) : [0, T]_{\mathbf{T}} \to [0, +\infty)$ and has countably many singularities in $[0, T]_{\mathbf{T}}$.

A projection $\varphi : R \to R$ is called an increasing homeomorphism and positive homomorphism, if the following conditions are satisfied:

(1) if $x \le y$, then $\varphi(x) \le \varphi(y)$, for all $x, y \in R$;
(2) φ is a continuous bijection and its inverse mapping is also continuous;
(3) $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in [0, +\infty)$.

In the above definition, we can replace condition (3) by the following stronger condition:

2000 *Mathematics Subject Classification.* 34B18.
Key words and phrases. Time scales; positive solutions; singular boundary-value; fixed-point index theory.

Submitted February 20, 2008. Published September 4, 2008.

(4) $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in \mathbb{R}$, where $\mathbb{R} = (-\infty, +\infty)$.

Remmark 1.1. If conditions (1), (2) and (4) hold, then φ is homogenous generating a p-Laplace operator; i.e., $\varphi(x) = |x|^{p-2}x$, for some $p > 1$.

Moreover, throughout this paper the following conditions hold:

(C1) $f : [0, +\infty) \to [0, +\infty)$ is continuous;

(C2) $a : [0, T]_{\mathbf{T}} \to [0, +\infty)$ and has countably many singularities in $[0, T]_{\mathbf{T}}$, i.e., there exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $0 < t_{i+1} < t_i < \frac{T}{2}$, $\lim_{i\to\infty} t_i = t_0 < \frac{T}{2}$, and $t_0 \in [0, T]_{\mathbf{T}}$. $\lim_{t\to t_i} a(t) = \infty$, $i = 1, 2, \dots$, and $a(t)$ does not vanish identically on any subinterval of $[0, T]_{\mathbf{T}}$. Moreover

$$0 < \int_0^T a(s)\nabla s < +\infty.$$

Recently, there is much attention paid to the existence of positive solutions for three-point boundary-value problems on time scales, see [2, 4, 5, 7, 8, 13, 16, 17] and references therein. However, there are not many results concerning the increasing homeomorphism and positive homomorphism operator on time scales.

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. We make the blanket assumption that $0, T$ are points in $\mathbb{T}$. By an interval $(0, T)$, we always mean the intersection of the real interval $(0, T)$ with the given time scale; that is $(0, T) \cap \mathbb{T}$.

Anderson [2] discussed the dynamic equation on time scales:

$$u^{\Delta\nabla}(t) + a(t)f(u(t)) = 0, \quad t \in (0, T), \tag{1.3}$$

$$u(0) = 0, \quad \alpha u(\eta) = u(T). \tag{1.4}$$

He obtained some results for the existence of one positive solution of the problem (1.3) and (1.4) based on the limits $f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}$ and $f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}$. He also obtained the existence of at least three positive solutions.

Kaufmann [8] studied the problem (1.3) and (1.4) and obtained existence results of finitely many positive solutions and countably many positive solutions.

Zhou and Su [17] studied the quasi-linear equation with p-Laplacian operator:

$$(\phi_p(u^{(n-1)}))' + g(t)f(u(t), u'(t), \dots, u^{(n-2)}(t)) = 0, \quad 0 < t < T, \tag{1.5}$$

$$\begin{gathered} u^{(i)}(0) = 0 \quad 0 \le i \le n-3, \\ u^{(n-2)}(0) - B_0(u^{(n-1)}(\xi)) = 0 \quad n \ge 3, \\ u^{(n-2)}(1) + B_1(u^{(n-1)}(\eta)) = 0 \quad n \ge 3. \end{gathered} \tag{1.6}$$

They obtained the existence of one solution, and of multiple solutions by using the fixed-point index theory.

Liu and Zhang [12] considered the existence of positive solutions of the following quasi-linear differential equation

$$(\varphi(x'))' + a(t)f(x(t)) = 0, \quad 0 < t < 1, \tag{1.7}$$

$$x(0) - \beta x'(0) = 0, \quad x(1) + \delta x'(1) = 0. \tag{1.8}$$

Where $\varphi : R \to R$ is an increasing homeomorphism and positive homomorphism and $\varphi(0) = 0$. They obtained the existence of one or two positive solutions of the problem (1.7) and (1.8) by using a fixed-point index theorem in cones.

But whether or not we can obtain the countably many positive solutions of nth-order with m-point boundary value problem (1.1) and (1.2) still remain unknown.

So the goal of present paper is to improve and generate p-Laplacian operator and establish some criteria for the existence of countable many solutions.

The plan of the paper is as follows. In Section 2, for the convenience of the reader we give some definitions. In Section 3, we present some lemmas in order to prove our main results. Section 4 is developed in order to present and prove our main results. In Section 5 we present the example of the increasing homeomorphism and positive homomorphism operators.

2. Some definitions and fixed point theorems

For convenience, we list the following definitions which can be found in [1, 3, 4, 5, 7].

Definition 2.1. A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$. For $t < \sup \mathbb{T}$ and $r > \inf \mathbb{T}$, define the forward jump operator σ and backward jump operator ρ, respectively, by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\} \in \mathbb{T},$$
$$\rho(r) = \sup\{\tau \in \mathbb{T} : \tau < r\} \in \mathbb{T}.$$

for all $t, r \in \mathbb{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered; if $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbb{T}$ has a right scattered minimum m, define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise set $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, define $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$; otherwise set $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2.2. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, the delta derivative of f at the point t is defined to be the number $f^\Delta(t)$, (provided it exists), with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \le \epsilon|\sigma(t) - s|,$$

for all $s \in U$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, the nabla derivative of f at t is the number $f^\nabla(t)$, (provided it exists), with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \le \epsilon|\rho(t) - s|,$$

for all $s \in U$.

Definition 2.3. A function f is left-dense continuous (i.e. ld-continuous), if f is continuous at each left-dense point in $\mathbb{T}$ and its right-sided limit exists at each right-dense point in $\mathbb{T}$. It is well-known that if f is ld-continuous, then there is a function $F(t)$ such that $F^\nabla(t) = f(t)$. In this case, it is defined that

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

If $u^{\Delta\nabla}(t) \le 0$ on $[0, T]$, then we say u is concave on $[0, T]$.

Definition 2.4. Let $(E, \|.\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following are satisfied:

(a) if $y \in P$ and $\lambda \ge 0$, then $\lambda y \in P$;
(b) if $y \in P$ and $-y \in P$, then $y = 0$.

If $P \subset E$ is a cone, we denote the order induced by P on E by $\leq$, that is, $x \leq y$ if and only if $y - x \in P$.

Definition 2.5. Given a nonnegative continuous functional γ on a cone P of E, for each $d > 0$ we define the set

$$P(\gamma, d) = \{x \in P : \gamma(x) < d\}.$$

The following fixed point theorems are fundamental and important for the proofs of our main results.

Theorem 2.6 ([6]). *Let E be a Banach space and $P \subset E$ be a cone in E. Let $r > 0$ define $\Omega_r = \{x \in P : \|x\| < r\}$. Assume that $A : P \bigcap \overline{\Omega}_r \to P$ is completely continuous operator such that $Ax \neq x$ for $x \in \partial\Omega_r$.*

(i) *If $\|Ax\| < \|x\|$ for $x \in \partial\Omega_r$, then $i(A, \Omega_r,\ P) = 1$.*
(ii) *If $\|Ax\| > \|x\|$ for $x \in \partial\Omega_r$, then $i(A, \Omega_r,\ P) = 0$.*

Theorem 2.7 ([15]). *Let P be a cone in a Banach space E. Let α, β and γ be three increasing, nonnegative and continuous functionals on P, satisfying for some $c > 0$ and $M > 0$ such that*

$$\gamma(x) \leq \beta(x) \leq \alpha(x), \quad \|x\| \leq M\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. Suppose there exists a completely continuous operator $A : \overline{P(\gamma, c)} \to P$ and $0 < a < b < c$ such that

(i) *$\gamma(Ax) < c$, for all $x \in \partial P(\gamma, c)$;*
(ii) *$\beta(Ax) > b$, for all $x \in \partial P(\beta, b)$;*
(iii) *$P(\alpha, a) \neq \emptyset$, and $\alpha(Ax) < a$, for all $x \in \partial P(\alpha, a)$. Then A has at least three fixed points x_1, x_2, $x_3 \in \overline{P(\gamma, c)}$ such that*

$$0 \leq \alpha(x_1) < a < \alpha(x_2), \quad \beta(x_2) < b < \beta(x_3), \quad \gamma(x_3) < c.$$

3. Preliminaries and Lemmas

In the rest of this article, $\mathbf{T}$ is closed subset of $\mathbb{R}$ with $0 \in \mathbf{T}_\kappa$, $T \in \mathbf{T}^\kappa$. And

$$E = \big\{u \in C_{ld}^{n-2}[0, T] : u^{\Delta^i}(0) = 0,\ 0 \leq i \leq n-3\big\}.$$

Then E is a Banach space with the norm $\|u\| = \sup_{t \in [0,T]} |u^{\Delta^{n-2}}(t)|$. And let

$$P = \big\{u \in E : u^{\Delta^{n-2}}(t) \geq 0,\ u^{\Delta^{n-2}}(t) \text{is concave nondecreasing on } [0, T]\big\}.$$

Obviously, P is a cone in E. Set $P_r = \{u \in P : \|u\| \leq r\}$. We can easily get the following Lemmas.

Lemma 3.1. *Suppose condition* (C2) *holds. Then there exists a constant $\theta \in \max\{t \in T |\ 0 < t < \frac{T}{2}\}$ that satisfies*

$$0 < \int_\theta^{T-\theta} a(s) \nabla s < +\infty.$$

Furthermore, the function

$$H(t) = \int_t^{T-t_1} \varphi^{-1}\Big(\int_s^{T-t_1} a(\tau)\nabla\tau\Big)\Delta s + \frac{\sum_{i=1}^{m-2} \alpha_i \int_{t_1}^t \varphi^{-1}\big(\int_s^t a(\tau)\nabla\tau\big)\Delta s}{1 - \sum_{i=1}^{m-2} \alpha_i}$$

is continuous and positive on $[t_1, T-t_1]$. *Furthermore there exists a constant* $L > 0$ *such that*

$$L = \min_{t\in[t_1,T-t_1]} H(t) > 0.$$

Proof. At first, it is easily seen that $H(t)$ is continuous on $[t_1, T-t_1]$. Let

$$H_1(t) = \int_t^{T-t_1} \varphi^{-1}\Big(\int_s^{T-t_1} a(\tau)\nabla\tau\Big)\Delta s,$$

$$H_2(t) = \frac{\sum_{i=1}^{m-2}\alpha_i \int_{t_1}^{t}\varphi^{-1}\left(\int_s^t a(\tau)\nabla\tau\right)\Delta s}{1-\sum_{i=1}^{m-2}\alpha_i}.$$

Then from condition (C2), we know that $H_1(t)$ strictly monotone decreasing on $[t_1, T-t_1]$ and $H_1(T-t_1) = 0$. Similarly function $H_2(t)$ is strictly monotone increasing on $[t_1, T-t_1]$ and $H_2(t_1) = 0$. Since $H_1(t)$ and $H_2(t)$ are not equal to zero at the same time. So the function $H(t) = H_1(t) + H_2(t)$ is positive on $[t_1, T-t_1]$, which implies $L = \min_{t\in[t_1,T-t_1]} H(t) > 0$. □

Lemma 3.2. *If* $u \in P$. *Then*

$$u^{\Delta^{n-2}}(t) \geq \frac{\theta}{T}\|u\|, \quad t\in[\theta, T-\theta],$$

The proof of the above lemma is similar to the proof of in [7], so we omit it. Now, we define a mapping $F : P \to C_{ld}^{n-1}[0,T]$ by

$$(Fu)(t) = \int_0^t\int_0^{\zeta_1}\dots\int_0^{\zeta_{n-3}} w(\zeta_{n-2})\Delta\zeta_{n-2}\Delta\zeta_{n-3}\dots\Delta\zeta_1, \tag{3.1}$$

where

$$\begin{aligned} w(\zeta_{n-2}) = &\int_0^{\zeta_{n-2}} \varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau), u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\Big)\Delta s \\ &+ \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi^{-1}\big(\int_s^T a(\tau)f(u(\tau),u'(\tau),\dots,u^{(n-2)}(\tau))\nabla\tau\big)\Delta s}{1-\sum_{i=1}^{m-2}\alpha_i}. \end{aligned}$$

Then it is easy to see that

$$\begin{aligned} (Fu)^{\Delta^{n-2}}(t) = &\int_0^t \varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau), u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\Big)\Delta s \\ &+ \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau),u'(\tau),\dots,u^{(n-2)}(\tau))\nabla\tau\Big)\Delta s}{1-\sum_{i=1}^{m-2}\alpha_i} \\ \geq{}& 0, \quad 0\leq t\leq T. \end{aligned}$$

$$\begin{aligned} (Fu)^{\Delta^{n-1}}(t) &= \varphi^{-1}\Big(\int_t^T a(\tau)f(u(\tau), u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\Big) \\ &\geq 0, \quad 0\leq t\leq T. \end{aligned}$$

We also have

$$[\varphi((Fu)^{\Delta^{n-1}})(t)]^{\nabla} = -a(t)f(u(t), u^{\Delta}(t),\dots,u^{\Delta^{n-2}}(t)) \leq 0.$$

Together with φ is a increasing operator, we know $(Fu)^{\Delta^{n-2}}$ is a concave function. This shows $F(P) \subset P$.

Using the Arzela-Ascoli Theorem, we obtain the following lemma.

Lemma 3.3. *The operator $F : P \to P$ is completely continuous.*

Lemma 3.4. *Suppose that conditions* (C1), (C2) *hold. Then the solution $u(t) \in P$ of* (1.1), (1.2) *satisfies*

$$u(t) \le u^{\Delta}(t) \le \dots \le u^{\Delta^{n-3}}(t), \quad 0 \le t \le T,$$

and for $\theta \in (0, \frac{T}{2})$ in Lemma 3.1, we have

$$u^{\Delta^{n-3}}(t) \le \frac{T}{\theta} u^{\Delta^{n-2}}(t), \quad \theta \le t \le T - \theta.$$

The proof of the above lemma is similar to the proof of in [17, lemma 2.4].

4. Main results

For notational convenience, we define

$$\lambda_1 = \frac{1}{L}, \quad \lambda_2 = \frac{(1 - \sum_{i=1}^{m-2} \alpha_i)}{\int_0^T \varphi^{-1}\Big(\int_s^T a(\tau)\nabla\tau\Big)\Delta s}.$$

The main results of this paper are the following.

Theorem 4.1. *Suppose that conditions* (C1)-(C2) *hold. Let $\{\theta_k\}_{k=1}^{\infty}$ be such that $\theta_k \in (t_{k+1}, \ t_k)$ $(k = 1, 2, \dots)$. Let $\{r_k\}_{k=1}^{\infty}$ and $\{R_k\}_{k=1}^{\infty}$ be such that*

$$R_{k+1} < \frac{\theta_k}{T} r_k < r_k < m r_k < R_k, \quad m r_k \le M R_k, \quad k = 1, 2, \dots.$$

Furthermore for each natural number k we assume that f satisfy:

- (C3) *$f(v_1, v_2, \dots, v_{n-1}) \ge \varphi(m r_k)$ for all $0 \le v_1, v_2, \dots, v_{n-2} \le \frac{T}{\theta_k} r_k$, $\frac{\theta_k}{T} r_k \le v_{n-1} \le r_k$;*
- (C4) *$f(v_1, v_2, \dots, v_{n-1}) \le \varphi(M R_k)$ for all $0 \le v_1, v_2, \dots, v_{n-1} \le R_k$.*

Where $m \in (\lambda_1, \infty)$, $M \in (0, \lambda_2)$. Then the boundary-value problem (1.1), (1.2) *has infinitely many solutions $\{u_k\}_{k=1}^{\infty}$ such that*

$$r_k \le \|u_k\| \le R_k, \quad k = 1, 2, \dots.$$

Proof. Since $0 < t_0 < t_{k+1} < \theta_k < t_k < \frac{T}{2}$, $k = 1, 2, \dots$, for any $k \in \mathbb{N}$ and $u \in P$, by the Lemma 3.2 we have

$$u^{\Delta^{n-2}}(t) \ge \frac{\theta_k}{T}\|u\|, \quad t \in [\theta_k, T - \theta_k]. \tag{4.1}$$

Consider the sequences $\{\Omega_{1,k}\}_{k=1}^{\infty}$ and $\{\Omega_{2,k}\}_{k=1}^{\infty}$ of open subsets of E defined by

$$\Omega_{1,k} = \{u \in P : \|u\| < r_k\}, \quad k = 1, 2, \dots,$$
$$\Omega_{2,k} = \{u \in P : \|u\| < R_k\}, \quad k = 1, 2, \dots.$$

For a fixed k and $u \in \partial\Omega_{1,k}$. From (4.1) we have

$$r_k = \|u\| \ge u^{\Delta^{n-2}}(t) \ge \frac{\theta_k}{T}\|u\| = \frac{\theta_k}{T} r_k, \quad t \in [\theta_k, T - \theta_k].$$

Since $(t_1, T - t_1) \subset [\theta_k, T - \theta_k]$, in the following we consider three cases:
(i) If $\xi_1 \in [t_1, T - t_1]$. In this case, from (3.1), condition (C3) and Lemma 3.1, we have

$$\|Fu\| = (Fu)^{\Delta^{n-2}}(T)$$

$$
\begin{aligned}
&= \int_0^T \varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau),u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\Big)\Delta s \\
&\quad + \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau),u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\Big)\Delta s}{1-\sum_{i=1}^{m-2}\alpha_i} \\
&\ge \int_{\xi_1}^{T-t_1} \varphi^{-1}\Big(\int_s^{T-t_1} a(\tau)f(u(\tau),u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\Big)\Delta s \\
&\quad + \frac{\sum_{i=1}^{m-2}\alpha_i\int_{t_1}^{\xi_1}\varphi^{-1}\Big(\int_s^{\xi_1} a(\tau)f(u(\tau),u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\Big)\Delta s}{1-\sum_{i=1}^{m-2}\alpha_i} \\
&\ge (mr_k)\Big[\int_{\xi_1}^{T-t_1} \varphi^{-1}\Big(\int_s^{T-t_1} a(\tau)\nabla\tau\Big)\Delta s \\
&\quad + \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\int_{t_1}^{\xi_1}\varphi^{-1}\Big(\int_s^{\xi_1} a(\tau)\nabla\tau\Big)\Delta s\Big] \\
&= mr_kH(\xi_1) > mr_kL > r_k = \|u\|.
\end{aligned}
$$

(ii) If $\xi_1 \in [0, t_1]$. In this case, from (3.1), condition (C_3) and Lemma3.1, we have

$$
\begin{aligned}
\|Fu\| &\ge \int_0^T \varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau),u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\Big)\Delta s \\
&\ge \int_{t_1}^{T-t_1} \varphi^{-1}\Big(\int_s^{T-t_1} a(\tau)f(u(\tau),u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\Big)\Delta s \\
&\ge (mr_k)\Big[\int_{t_1}^{T-t_1} \varphi^{-1}\Big(\int_s^{T-t_1} a(\tau)\nabla\tau\Big)\Delta s\Big] \\
&= mr_kH(t_1) > mr_kL > r_k = \|u\|.
\end{aligned}
$$

(iii) If $\xi_1 \in [T-t_1, T]$. In this case, from (3.1), condition (C_3) and Lemma3.1, we have

$$
\begin{aligned}
\|Fu\| &\ge \frac{\sum_{i=1}^{m-2}\alpha_i\int_{t_1}^{T-t_1}\varphi^{-1}\big(\int_s^{T-t_1} a(\tau)f(u(\tau),u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\big)\Delta s}{1-\sum_{i=1}^{m-2}\alpha_i} \\
&\ge (mr_k)\Big[\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\int_{T-t_1}^{t_1}\varphi^{-1}\Big(\int_s^{T-t_1} a(\tau)\nabla\tau\Big)\Delta s\Big] \\
&= mr_kH(T-t_1) > mr_kL > r_k = \|u\|.
\end{aligned}
$$

Thus in all cases, an application of Theorem 2.6 implies

$$
i(F,\ \Omega_{1,k},\ P) = 0. \tag{4.2}
$$

On the another hand, let $u(t) \in \partial\Omega_{2,k}$, we have $u^{\Delta^{n-2}}(t) \le \|u\| = R_k$, by (C_4) we have

$$
\begin{aligned}
\|Fu\| &= (Fu)^{(n-2)}(T) \\
&= \int_0^T \varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau),u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\Big)\Delta s \\
&\quad + \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau),u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\Big)\Delta s}{1-\sum_{i=1}^{m-2}\alpha_i}
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_0^T \varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau),u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\Big)\Delta s\\
&\quad+\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_{m-2}}\varphi^{-1}\big(\int_s^T a(\tau)f(u(\tau),u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\big)\Delta s}{1-\sum_{i=1}^{m-2}\alpha_i}\\
&\le MR_k\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\int_0^T \varphi^{-1}\Big(\int_s^T a(\tau)\nabla\tau\Big)\Delta s\Big]\\
&= R_k = \|u\|.
\end{aligned}
$$

Thus Theorem 2.6 implies

$$
i(T,\ \Omega_{2,k},\ P)=1. \tag{4.3}
$$

Hence since $r_k < R_k$ for $k\in\mathbb{N}$, (4.2) and (4.3), it follows from additivity of the fixed-point index that

$$
i(T,\ \Omega_{2,k}\backslash\overline{\Omega}_{1,k},\ P)=1 \quad \text{for } k\in\mathbb{N}.
$$

Thus F has a fixed point in $\Omega_{2,k}\backslash\overline{\Omega}_{1,k}$ such that $r_k\le\|u_k\|\le R_k$. Since $k\in\mathbb{N}$ was arbitrary, the proof is complete. □

To use Theorem 2.7, let $\theta_k < r_k < 1-\theta_k$ and θ_k of Theorem 4.1, we define the nonnegative, increasing, continuous functionals

$$
\begin{gathered}
\gamma_k(u)=\max_{\theta_k\le t\le r_k} u^{\Delta^{n-2}}(t)=u^{\Delta^{n-2}}(r_k),\\
\beta_k(u)=\min_{r_k\le t\le T-\theta_k} u^{\Delta^{n-2}}(t)=u^{\Delta^{n-2}}(r_k),\\
\alpha_k(u)=\max_{\theta_k\le t\le T-\theta_k} u^{\Delta^{n-2}}(t)=u^{\Delta^{n-2}}(T-\theta_k).
\end{gathered}
$$

It is obvious that for each $u\in P$,

$$
\gamma_k(u)\le\beta_k(u)\le\alpha_k(u).
$$

In addition, by Lemma 3.2, for each $u\in P$,

$$
\gamma_k(u)=u^{\Delta^{n-2}}(r_k)\ge\frac{\theta_k}{T}\|u\|.
$$

Thus

$$
\|u\|\le\frac{T}{\theta_k}\gamma_k(u) \quad \text{for all } u\in P.
$$

For convenience, we denote

$$
\begin{gathered}
\lambda=\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\int_0^T\varphi^{-1}\Big(\int_s^T a(\tau)\nabla\tau\Big)\Delta s\Big],\\
\eta_k=\int_{\theta_k}^{r_k}\varphi^{-1}\Big(\int_s^{T-\theta_k} a(\tau)\nabla\tau\Big)\Delta s.
\end{gathered}
$$

Theorem 4.2. *Suppose* (C1)-(C2) *hold. Let* $\{\theta_k\}_{k=1}^{\infty}$ *be such that* $\theta_k\in(t_{k+1},\ t_k)$ *(*$k=1,2,\dots$*). Let* $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ *and* $\{c_k\}_{k=1}^{\infty}$ *be such that*

$$
c_{k+1}<a_k<\frac{\theta_k}{T}b_k<b_k<c_k, \quad \text{and } \rho_k b_k<\eta_k c_k, \text{ for } k=1,2,\dots.
$$

Furthermore for each natural number k we assume that f satisfies:

(C5) $f(v_1,v_2,\dots,v_{n-1})<\varphi(\frac{c_k}{\lambda})$, *for all* $0\le v_1,v_2,\dots,v_{n-1}\le\frac{T}{\theta_k}c_k$;

(C6) $f(v_1, v_2, \dots, v_{n-1}) > \varphi(\frac{b_k}{\eta_k})$, *for all* $0 \le v_1, v_2, \dots, v_{n-2} \le \frac{T}{\theta_k} b_k$, $b_k \le v_{n-1}(t) \le \frac{T}{\theta_k} b_k$;

(C7) $f(v_1, v_2, \dots, v_{n-1}) < \varphi(\frac{a_k}{\lambda})$, *for all* $0 \le v_1, v_2, \dots, v_{n-1} \le \frac{T}{\theta_k} a_k$.

Then the boundary-value problem (1.1), (1.2) *has three infinite families of solutions* $\{u_{1k}\}_{k=1}^{\infty}$ $\{u_{2k}\}_{k=1}^{\infty}$ *and* $\{u_{3k}\}_{k=1}^{\infty}$ *satisfying*

$$0 \le \alpha_k(u_{1k}) < a_k < \alpha_k(u_{2k}), \quad \beta_k(u_{2k}) < b_k < \beta_k(u_{3k}), \quad \gamma(u_{3k}) < c_k,$$

for $n \in \mathbb{N}$.

Proof. We define the completely continuous operator F by 3.1. So it is easy to check that $F : \overline{P(\gamma_k, c_k)} \to P$, for $k \in \mathbb{N}$.

We now show that all the conditions of Theorem 2.7 are satisfied. To make use of property (i) of Theorem 2.7, we choose $u \in \partial P(\gamma_k, c_k)$. Then $\gamma_k(u) = \max_{\theta_k \le t \le r_k} u^{\Delta^{n-2}}(t) = u^{\Delta^{n-2}}(r_k) = c_k$, this implies that $0 \le u^{\Delta^{n-2}}(t) \le c_k$ for $[0, r_k]$. If we recall that $\|u\| \le \frac{T}{\theta_k}\gamma_k(u) = \frac{T}{\theta_k} c_k$. So we have

$$0 \le u^{\Delta^i}(t) \le \frac{T}{\theta_k} c_k, \quad 0 \le t \le T,\ i = 0, 1, \dots, n-1.$$

Then assumption (C5) implies

$$f(u(t), u^{\Delta}(t), \dots, u^{\Delta^{n-2}}(t)) < \varphi(\frac{c_k}{\lambda}), \quad 0 \le t \le T.$$

Therefore

$$\begin{aligned}
\gamma_k(Fu) &= \max_{\theta_k \le t \le r_k} (Fu)^{\Delta^{n-2}}(t) = (Fu)^{\Delta^{n-2}}(r_k) \\
&\le \int_0^T \varphi^{-1}\Big(\int_s^T a(\tau) f(u(\tau), u^{\Delta}(\tau), \dots, u^{\Delta^{n-2}}(\tau))\nabla\tau\Big)\Delta s \\
&\quad + \frac{\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_{m-2}} \varphi^{-1}\left(\int_s^T a(\tau) f(u(\tau), u^{\Delta}(\tau), \dots, u^{\Delta^{n-2}}(\tau))\nabla\tau\right)\Delta s}{1 - \sum_{i=1}^{m-2}\alpha_i} \\
&\le \frac{c_k}{\lambda}\frac{1}{1 - \sum_{i=1}^{m-2}\alpha_i}\Big[\int_0^T \varphi^{-1}\Big(\int_s^T a(\tau)\nabla\tau\Big)\Delta s\Big] \\
&= c_k.
\end{aligned}$$

Hence condition (i) is satisfied.

Secondly, we show that (ii) of Theorem 2.7 is fulfilled. For this we select $u \in \partial P(\beta_k, b_k)$. Then $\beta_k(u) = \min_{r_k \le t \le T - \theta_k} u^{\Delta^{n-2}}(t) = u^{\Delta^{n-2}}(r_k) = b_k$, this fact implies that $u^{\Delta^{n-2}}(t) \ge b_k$, for $r_k \le t \le T$. Noticing that $\|u\| \le \frac{T}{\theta_k}\gamma_k(u) \le \frac{T}{\theta_k}\beta_k(u) = \frac{T}{\theta_k} b_k$, we have

$$b_k \le u^{\Delta^{n-2}}(t) \le \frac{T}{\theta_k} b_k, \quad \text{for } r_k \le t \le T.$$

By (C6), we have

$$f(u(t), u^{\Delta}(t), \dots, u^{\Delta^{n-2}}(t)) > \varphi(\frac{b_k}{\eta_k}).$$

Therefore,

$$\beta_k(Fu) = \min_{r_k \le t \le T - \theta_k} (Fu)^{\Delta^{n-2}}(t) = (Fu)^{\Delta^{n-2}}(r_k)$$

$$
\begin{aligned}
&= \int_0^{r_k} \varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau),u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\Big)\Delta s\\
&\quad + \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau),u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\Big)\Delta s}{1-\sum_{i=1}^{m-2}\alpha_i}\\
&\ge \int_{\theta_k}^{r_k} \varphi^{-1}\Big(\int_s^{T-\theta_k} a(\tau)f(u(\tau),u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\Big)\Delta s\\
&= \frac{b_k}{\eta_k}\Big[\int_{\theta_k}^{r_k}\varphi^{-1}\Big(\int_s^{T-\theta_k} a(\tau)\nabla\tau\Big)\Delta s\Big]\\
&= b_k.
\end{aligned}
$$

Hence condition (ii) is satisfied.

Finally, we verify that (iii) of Theorem 2.7 is satisfied. Noting that $u^{\Delta^{n-2}}(t) \equiv \frac{a_k}{4}$, $0 \le t \le T$ is a member of $P(\alpha_k, a_k)$ and $\alpha_k(u) = \frac{a_k}{4} < a_k$. So $P(\alpha_k, a_k) \ne \emptyset$. Now let $u \in \partial P(\alpha_k, a_k)$. Then $\alpha_k(u) = \max_{\theta_k \le t \le T-\theta_k} u^{\Delta^{n-2}}(t) = u^{\Delta^{n-2}}(T-\theta_k) = a_k$. This implies that $0 \le u^{\Delta^{n-2}}(t) \le a_k, 0 \le t \le T-\theta_k$. Noticing that $\|u\| \le \frac{T}{\theta_k}\gamma_k(u) \le \frac{T}{\theta_k}\alpha_k(u) = \frac{T}{\theta_k}a_k$. Then we get

$$
0 \le u^{\Delta^i}(t) \le \frac{a_k}{r_k}, \quad 0 \le t \le T,\ i = 0,1,\dots,n-1.
$$

Then assumption (C7) implies

$$
f(u(t),u^{\Delta}(t),\dots,u^{\Delta^{n-2}}(t)) < \varphi\big(\frac{a_k}{\lambda}\big), \quad 0 \le t \le T.
$$

As before, we get

$$
\begin{aligned}
\alpha_k(Fu) &= \max_{\theta_k \le t \le T-\theta_k}(Fu)(t) = (Fu)^{\Delta^{n-2}}(T-\theta_k)\\
&\le \int_0^{T} \varphi^{-1}\Big(\int_s^T a(\tau)f(u(\tau),u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\Big)\Delta s\\
&\quad + \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_{m-2}}\varphi^{-1}\big(\int_s^T a(\tau)f(u(\tau),u^{\Delta}(\tau),\dots,u^{\Delta^{n-2}}(\tau))\nabla\tau\big)\Delta s}{1-\sum_{i=1}^{m-2}\alpha_i}\\
&\le \frac{a_k}{\lambda}\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\int_0^{T}\varphi^{-1}\Big(\int_s^{T} a(\tau)\nabla\tau\Big)\Delta s\Big]\\
&= a_k.
\end{aligned}
$$

Thus (iii) of Theorem 2.7 is satisfied. Since all hypotheses of Theorem 2.7 are satisfied, the assertion follows. □

Remmark 4.3. If we add the condition of $a(t)f(u(t),u^{\Delta}(t),\dots,u^{\Delta^{n-2}}(t)) \not\equiv 0$, $t \in [0,T]$, to Theorem 4.2 we can get three infinite families of positive solutions $\{u_{1k}\}_{k=1}^{\infty}$, $\{u_{2k}\}_{k=1}^{\infty}$, and $\{u_{3k}\}_{k=1}^{\infty}$ satisfying

$$
0 < \alpha_k(u_{1k}) < a_k < \alpha_k(u_{2k}), \quad \beta_k(u_{2k}) < b_k < \beta_k(u_{3k}), \quad \gamma(u_{3k}) < c_k,
$$

for $n \in \mathbb{N}$.

Remmark 4.4. The same conclusions of Theorem 4.1 and Theorem 4.2 hold when conditions (1), (2) and (4) are satisfied. Especially, for p-Laplacian operator $\varphi(x) = |x|^{p-2}x$, for some $p > 1$, our conclusions are also true and new.

5. Applications

There exists a function $a(t)$ satisfying condition (C2).

Example 5.1. Let $\mathbf{T} \equiv 1$ and

$$\delta = 2\big(\frac{\pi^2}{3} - \frac{9}{4}\big), \quad t^* = \frac{15}{32}, \quad t_i = t^* - \sum_{k=1}^{i} \frac{1}{2(k+1)^4}, \quad i = 1, 2, \dots .$$

Consider the function $a(t) : [0, 1] \to (0, \infty)$, $a(t) = \sum_{i=1}^{\infty} a_i(t)$, $t \in [0, 1]$, where

$$a_i(t) = \begin{cases} \frac{1}{(2i-1)(2i+1)(t_{i+1}+t_i)}, & 0 \le t < \frac{t_{i+1}+t_i}{2}, \\ \frac{1}{\delta(t_i - t)^{1/2}}, & \frac{t_{i+1}+t_i}{2} \le t < t_i, \\ \frac{1}{\delta(t-t_i)^{1/2}}, & t_i < t \le \frac{t_i+t_{i-1}}{2}, \\ 0, & \frac{t_i+t_{i-1}}{2} < t \le t_1. \\ \frac{1}{2(2i-1)(2i+1)(1-t_1)}, & t_1 \le t \le 1. \end{cases}$$

It is easy to check that $t_1 = \frac{7}{16} < \frac{1}{2}$, $t_i - t_{i+1} = \frac{1}{2(i+2)^4}$, $i = 1, 2, \dots$ (denote $\sum_{i=1}^{\infty} \frac{1}{i^4} = \frac{\pi^4}{90}$), and

$$t_0 = \lim_{i\to\infty} t_i = \frac{15}{32} - \sum_{k=1}^{\infty} \frac{1}{2(k+1)^4} = \frac{31}{32} - \frac{\pi^4}{180} > \frac{1}{5}$$

and because $\sum_{i=1}^{\infty} 1/i^2 = \pi^2/6$, we have

$$\begin{aligned} &\sum_{i=1}^{\infty} \int_0^1 a_i(t) \nabla t \\ &= \sum_{i=1}^{\infty} \frac{1}{(2i-1)(2i+1)} + \frac{1}{\delta} \sum_{i=1}^{\infty} \Big[\int_{\frac{t_{i+1}+t_i}{2}}^{t_i} \frac{1}{(t_i - t)^{1/2}} \nabla t + \int_{t_i}^{\frac{t_i+t_{i-1}}{2}} \frac{1}{(t-t_i)^{\frac{1}{2}}} \nabla t \Big] \\ &= \frac{1}{2} + \frac{\sqrt{2}}{\delta} \sum_{i=1}^{\infty} \Big[(t_i - t_{i+1})^{1/2} + (t_{i-1} - t_i)^{1/2} \Big] \\ &= \frac{1}{2} + \frac{1}{\delta} \sum_{i=1}^{\infty} \Big[\frac{1}{(i+2)^2} + \frac{1}{(i+1)^2} \Big] \\ &= \frac{1}{2} + \frac{1}{\delta} \big[\frac{\pi^2}{3} - \frac{9}{4} \big] = 1. \end{aligned}$$

Therefore,

$$\int_0^1 a(t) \nabla t = \sum_{i=1}^{\infty} \int_0^1 a_i(t) \nabla t = 1 < \infty.$$

Which implies that Condition (C2).

Example 5.2. As an example we mention the boundary-value problem

$$[\varphi(u^{\Delta^2})]^{\nabla} + a(t) f(u(t)) = 0, \quad t \in [0, 1]_{\mathbf{T}}, \tag{5.1}$$

$$\begin{gathered} u(0) = u^{\Delta}(0) = 0, \\ u^{\Delta^2}(0) = \frac{1}{4} u^{\Delta^2}(\frac{1}{4}) + \frac{1}{2} u^{\Delta^2}(\frac{1}{2}), \quad u^{\Delta^3}(1) = 0, \end{gathered} \tag{5.2}$$

where

$$\varphi(u)=\begin{cases}\frac{u^7}{1+u^2}, & u\le 0,\\ u^2, & u>0,\end{cases}$$

and

$$f(u(t))=\begin{cases}M^2R_1^2, & u\in(R_1,+\infty),\\ m^2r_k^2+\frac{M^2R_k^2-m^2r_k^2}{R_k-r_k}(u-r_k), & u\in[r_k,R_k],\\ m^2r_k^2, & u\in(\frac{\theta_k}{T}r_k,r_k),\\ M^2R_{k+1}^2+\frac{m^2r_k^2-M^2R_{k+1}^2}{\frac{\theta_k}{T}r_k-R_{k+1}}(u-R_{k+1}), & u\in(R_{k+1},\frac{\theta_k}{T}r_k],\\ 0, & u=0.\end{cases}$$

Since $H'(t)\le 0$, So it is easy to see by calculating that

$$L=\min_{[t_1,1-t_1]}H(t)=H(1-t_1)=\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\frac{1}{1-t_1}\big[\frac{2}{3}(1-2t_1)^{\frac{3}{2}}\big]=\frac{8}{9}.$$

Then

$$\lambda_1=\frac{1}{L}=\frac{9}{8},\quad \lambda_2=\frac{2}{5}.$$

Therefore, we take $m=10\in(\frac{9}{8},+\infty)$, $M=\frac{1}{5}\in(0,\frac{2}{5})$ and let

$$\theta_k=t^*-\frac{1}{2}\Big(\sum_{i=1}^{k+1}\frac{1}{2(i+1)^4}+\sum_{i=1}^{k}\frac{1}{2(i+1)^4}\Big)\in(0,\frac{15}{32})$$

For $R_k=\frac{1}{800^k}$ and $r_k=\frac{1}{300\times 800^k}$, $k=1,2,\dots$ we have

$$\frac{1}{800^{k+1}}<\frac{\theta_k}{300\times 800^k}<\frac{1}{300\times 800^k}<\frac{m}{300\times 800^k}<\frac{1}{800^k}.$$

After some simple calculation we have

$$\begin{gathered}f(u)\ge\varphi(mr_k)=m^2r_k^2\quad\text{for } u\in[\mu_k r_k,r_k];\\ f(u)\le\varphi(MR_k)=M^2R_k^2\quad\text{for}\quad u\in[0,R_k].\end{gathered}$$

Then by Theorem 4.1, the boundary-value problem (5.1) and (5.2) has infinitely many solutions $\{u_k\}_{k=1}^{\infty}$ such that

$$\frac{1}{300\times 800^k}\le\|u_k\|\le\frac{1}{800^k},\quad k=1,2,\dots.$$

Remmark 5.3. From the Example 5.2, we can see that φ is not odd, then the boundary value problem with p-Laplacian operator [7, 17] do not apply to Example 5.2. So, we generalize a p-Laplace operator for some $p>1$ and the function φ which we defined above is more comprehensive and general than p-Laplace operator.

Acknowledgment. The authors are indebted to the anonymous referee whose suggestions have improved greatly this article.
This project was supported by supported by NSFC, Foundation of Major Project of Science and Technology of Chinese Education Ministry, SRFDP of Higher Education, NSF of Education Committee of Jiangsu Province and Project of Graduate Education Innovation of Jiangsu Province.

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