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EXISTENCE OF SOLUTIONS TO THIRD-ORDER m-POINT BOUNDARY-VALUE PROBLEMS

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Abstract. This paper concerns the third-order m-point boundary-value problem

$$u'''(t) + f(t, u(t), u'(t), u''(t)) = 0, \quad \text{a.e. } t \in (0,1),$$
$$u(0) = u'(0) = 0, \quad u''(1) = \sum_{i=1}^{m-2} k_i u''(\xi_i),$$

where $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is L_p-Carathéodory, $1 \le p < +\infty$, $0 = \xi_0 < \xi_1 < \cdots < \xi_{m-2} < \xi_{m-1} = 1$, $k_i \in \mathbb{R}$ $(i = 1, 2, \dots, m-2)$ and $\sum_{i=1}^{m-2} k_i \neq 1$. Some criteria for the existence of at least one solution are established by using the well-known Leray-Schauder Continuation Principle.

1. Introduction

Third-order differential equations arise in a variety of areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three layer beam, electromagnetic waves or gravity driven flows and so on [8].

Recently, third-order two-point or three-point boundary-value problems (BVPs for short) have received much attention [1, 2, 3, 6, 7, 9, 10, 11, 12, 13, 14]. In particular, for two-point BVPs, Yao and Feng [14] employed the upper and lower solution method to prove the existence of solutions for the problem

$$\begin{gathered} u'''(t) + f(t, u(t)) = 0, \quad 0 \le t \le 1, \\ u(0) = u'(0) = u'(1) = 0. \end{gathered} \tag{1.1}$$

El-Shahed [6] considered the existence of at least one positive solution for the problem

$$\begin{gathered} u'''(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \quad \alpha u'(1) + \beta u''(1) = 0 \end{gathered} \tag{1.2}$$

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by using the Guo-Krasnoselskii fixed point theorem. Hopkins and Kosmatov [10] obtained the existence of at least one solution for the problem

$$\begin{gathered} u'''(t) = f(t, u(t), u'(t), u''(t)), \quad \text{a.e. } t \in (0,1), \\ u(0) = u'(0) = u''(1) = 0. \end{gathered} \tag{1.3}$$

Their main tool was the Leray-Schauder Continuation Principle. For three-point BVPs, Anderson [1] studied the existence and multiplicity of positive solutions for the problem

$$\begin{gathered} x'''(t) = f(t, x(t)), \quad t_1 \le t \le t_3, \\ x(t_1) = x'(t_2) = 0, \quad \gamma x(t_3) + \delta x''(t_3) = 0 \end{gathered} \tag{1.4}$$

by using the Guo-Krasnoselskii and Leggett-Williams fixed point theorems. Guo, Sun and Zhao [9] considered the existence of at least one positive solution to the problem

$$\begin{gathered} u'''(t) + a(t)f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\eta). \end{gathered} \tag{1.5}$$

The main tool was the Guo-Krasnoselskii fixed point theorem.

Although there are many excellent works on third-order two-point or three-point BVPs, a little work has been done for more general third-order m-point BVP [4, 5]. Moreover, almost all of the existing literatures assumed that the nonlinear term was continuous.

Motivated by the above-mentioned works, in this paper we investigate the third-order m-point BVP

$$\begin{gathered} u'''(t) + f(t, u(t), u'(t), u''(t)) = 0, \quad \text{a.e. } t \in (0,1), \\ u(0) = u'(0) = 0, \quad u''(1) = \sum_{i=1}^{m-2} k_i u''(\xi_i). \end{gathered} \tag{1.6}$$

Throughout this paper, we assume that $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is L_p-Carathéodory, $1 \le p < +\infty$, $0 = \xi_0 < \xi_1 < \dots < \xi_{m-2} < \xi_{m-1} = 1$, $k_i \in \mathbb{R}$ $(i = 1, 2, \dots, m-2)$ and $\sum_{i=1}^{m-2} k_i \ne 1$. Firstly, Green's function for associated linear BVP is constructed. Secondly, some useful properties of the Green's function are obtained. Finally, existence results of at least one solution for the BVP (1.6) are established by applying the well-known Leray-Schauder Continuation Principle [15], which we state here for convenience of the reader.

Theorem 1.1. *Let X be a Banach space and $T : X \to X$ be a compact map. Suppose that there exists an $R > 0$ such that if $u = \lambda T u$ for $\lambda \in (0,1)$, then $\|u\| \le R$. Then T has a fixed point.*

In the remainder of this section, we introduce some fundamental definitions.

Definition 1.2. We say that a map $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$, $(t, x) \mapsto f(t, x)$ is L_p-Carathéodory, if the following conditions are satisfied:

(1) for each $x \in \mathbb{R}^n$, the mapping $t \mapsto f(t, x)$ is Lebesgue measurable;
(2) for a.e. $t \in [0,1]$, the mapping $x \mapsto f(t, x)$ is continuous on $\mathbb{R}^n$;
(3) for each $r > 0$, there exists an $\alpha_r \in L_p[0,1]$ such that for a.e. $t \in [0,1]$ and every x with $|x| \le r$, $|f(t, x)| \le \alpha_r(t)$.

Let $X = C^2[0,1]$. For $x \in X$, we use the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty\}$, where $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. We denote the usual Lebesgue norm in $L_p[0,1]$ by $\|\cdot\|_p$ and the space of absolutely continuous functions on the interval $[0,1]$ by $AC[0,1]$. We also use the Sobolev space

$$\begin{gathered}W^{3,p}[0,1] = \big\{u : [0,1] \to \mathbb{R} : u, u', u'' \in AC[0,1],\ u(0) = u'(0) = 0,\\ u''(1) = \sum_{i=1}^{m-2} k_i u''(\xi_i), u''' \in L_p[0,1]\big\}.\end{gathered}$$

2. Main results

Lemma 2.1. *Let $y \in L_p[0,1]$. Then the BVP*

$$\begin{gathered}u'''(t) + y(t) = 0, \quad a.e.\ t \in (0,1),\\ u(0) = u'(0) = 0, \quad u''(1) = \sum_{i=1}^{m-2} k_i u''(\xi_i)\end{gathered} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G_0(t,s) y(s) ds,$$

which satisfies

$$u'(t) = \int_0^1 G_1(t,s) y(s) ds, \quad u''(t) = \int_0^1 G_2(t,s) y(s) ds,$$

where, for $j = 1, 2, \dots, m-1$,

$$G_0(t,s) = \begin{cases} \frac{\sum_{i=1}^{j-1} k_i}{2(1-\sum_{i=1}^{m-2} k_i)} t^2 + ts - \frac{1}{2}s^2, & s \le t,\ \xi_{j-1} < s \le \xi_j,\\ \frac{1-\sum_{i=j}^{m-2} k_i}{2(1-\sum_{i=1}^{m-2} k_i)} t^2, & s > t, \quad \xi_{j-1} < s \le \xi_j, \end{cases} \tag{2.2}$$

$$G_1(t,s) = \begin{cases} \frac{\sum_{i=1}^{j-1} k_i}{1-\sum_{i=1}^{m-2} k_i} t + s, & s \le t,\ \xi_{j-1} < s \le \xi_j,\\ \frac{1-\sum_{i=j}^{m-2} k_i}{1-\sum_{i=1}^{m-2} k_i} t, & s > t,\ \xi_{j-1} < s \le \xi_j, \end{cases} \tag{2.3}$$

$$G_2(t,s) = \begin{cases} \frac{\sum_{i=1}^{j-1} k_i}{1-\sum_{i=1}^{m-2} k_i}, & s \le t,\ \xi_{j-1} < s \le \xi_j,\\ \frac{1-\sum_{i=j}^{m-2} k_i}{1-\sum_{i=1}^{m-2} k_i}, & s > t,\ \xi_{j-1} < s \le \xi_j, \end{cases} \tag{2.4}$$

are called Green function. Here, if $l' < l$, then we let $\sum_{i=l}^{l'} k_i = 0$.

Proof. In view of (2.1) and the boundary condition $u''(1) = \sum_{i=1}^{m-2} k_i u''(\xi_i)$, we have

$$u''(t) = -\int_0^t y(s) ds + \frac{1}{1-\sum_{i=1}^{m-2} k_i} \int_0^1 y(s) ds - \frac{1}{1-\sum_{i=1}^{m-2} k_i} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} y(s) ds.$$

If $0 \le t \le \xi_1$, then

$$u''(t) = \int_t^{\xi_1} y(s) ds + \sum_{j=2}^{m-2} \int_{\xi_{j-1}}^{\xi_j} \frac{1-\sum_{i=j}^{m-2} k_i}{1-\sum_{i=1}^{m-2} k_i} y(s) ds + \int_{\xi_{m-2}}^1 \frac{1}{1-\sum_{i=1}^{m-2} k_i} y(s) ds,$$

which together with the boundary conditions $u(0)=u'(0)=0$ imply

$$\begin{aligned} u'(t) &= \int_0^t sy(s)ds + \int_t^{\xi_1} ty(s)ds + \sum_{j=2}^{m-2}\int_{\xi_{j-1}}^{\xi_j} \frac{1-\sum_{i=j}^{m-2}k_i}{1-\sum_{i=1}^{m-2}k_i} ty(s)ds \\ &\quad + \int_{\xi_{m-2}}^1 \frac{1}{1-\sum_{i=1}^{m-2}k_i} ty(s)ds \end{aligned}$$

and

$$\begin{aligned} u(t) &= \int_0^t (ts-\frac{1}{2}s^2)y(s)ds + \int_t^{\xi_1} \frac{1}{2}t^2 y(s)ds \\ &\quad + \sum_{j=2}^{m-2} k_i \int_{\xi_{j-1}}^{\xi_j} \frac{1-\sum_{i=j}^{m-2}k_i}{2(1-\sum_{i=1}^{m-2}k_i)} t^2 y(s)ds + \int_{\xi_{m-2}}^1 \frac{1}{2(1-\sum_{i=1}^{m-2}k_i)} t^2 y(s)ds. \end{aligned}$$

If $\xi_{l-1} < t \le \xi_l \quad (l=2,3,\dots,m-2)$, then

$$\begin{aligned} u''(t) &= \sum_{j=2}^{l-1}\int_{\xi_{j-1}}^{\xi_j} \frac{\sum_{i=1}^{j-1}k_i}{1-\sum_{i=1}^{m-2}k_i} y(s)ds + \int_{\xi_{l-1}}^t \frac{\sum_{i=1}^{l-1}k_i}{1-\sum_{i=1}^{m-2}k_i} y(s)ds \\ &\quad + \int_t^{\xi_l} \frac{1-\sum_{i=l}^{m-2}k_i}{1-\sum_{i=1}^{m-2}k_i} y(s)ds + \sum_{j=l+1}^{m-2}\int_{\xi_{j-1}}^{\xi_j} \frac{1-\sum_{i=j}^{m-2}k_i}{1-\sum_{i=1}^{m-2}k_i} y(s)ds \\ &\quad + \int_{\xi_{m-2}}^1 \frac{1}{1-\sum_{i=1}^{m-2}k_i} y(s)ds, \end{aligned}$$

which together with the boundary conditions $u(0)=u'(0)=0$ imply

$$\begin{aligned} u'(t) &= \int_0^{\xi_1} sy(s)ds + \sum_{j=2}^{l-1}\int_{\xi_{j-1}}^{\xi_j} \Big(\frac{\sum_{i=1}^{j-1}k_i}{1-\sum_{i=1}^{m-2}k_i} t + s\Big) y(s)ds \\ &\quad + \int_{\xi_{l-1}}^t \Big(\frac{\sum_{i=1}^{l-1}k_i}{1-\sum_{i=1}^{m-2}k_i} t + s\Big) y(s)ds + \int_t^{\xi_l} \frac{1-\sum_{i=l}^{m-2}k_i}{1-\sum_{i=1}^{m-2}k_i} ty(s)ds \\ &\quad + \sum_{j=l+1}^{m-2}\int_{\xi_{j-1}}^{\xi_j} \frac{1-\sum_{i=j}^{m-2}k_i}{1-\sum_{i=1}^{m-2}k_i} ty(s)ds + \int_{\xi_{m-2}}^1 \frac{1}{1-\sum_{i=1}^{m-2}k_i} ty(s)ds \end{aligned}$$

and

$$\begin{aligned} & u(t) \\ &= \int_0^{\xi_1} (ts-\frac{1}{2}s^2)y(s)ds + \sum_{j=2}^{l-1} k_i \int_{\xi_{j-1}}^{\xi_j} \Big(\frac{\sum_{i=1}^{j-1}k_i}{2(1-\sum_{i=1}^{m-2}k_i)} t^2 + ts - \frac{1}{2}s^2\Big) y(s)ds \\ &\quad + \int_{\xi_{l-1}}^t \Big(\frac{\sum_{i=1}^{l-1}k_i}{2(1-\sum_{i=1}^{m-2}k_i)} t^2 + ts - \frac{1}{2}s^2\Big) y(s)ds + \int_t^{\xi_l} \frac{1-\sum_{i=l}^{m-2}k_i}{2(1-\sum_{i=1}^{m-2}k_i)} t^2 y(s)ds \\ &\quad + \sum_{j=l+1}^{m-2}\int_{\xi_{j-1}}^{\xi_j} \frac{1-\sum_{i=j}^{m-2}k_i}{2(1-\sum_{i=1}^{m-2}k_i)} t^2 y(s)ds + \int_{\xi_{m-2}}^1 \frac{1}{2(1-\sum_{i=1}^{m-2}k_i)} t^2 y(s)ds. \end{aligned}$$

Similarly, if $\xi_{m-2} < t \le 1$, then we get

$$\begin{aligned} u''(t) &= \sum_{j=2}^{m-2} \int_{\xi_{j-1}}^{\xi_j} \frac{\sum_{i=1}^{j-1} k_i}{1-\sum_{i=1}^{m-2} k_i} y(s)ds + \int_{\xi_{m-2}}^{t} \frac{\sum_{i=1}^{m-2} k_i}{1-\sum_{i=1}^{m-2} k_i} y(s)ds \\ &\quad + \int_t^1 \frac{1}{1-\sum_{i=1}^{m-2} k_i} y(s)ds, \end{aligned}$$

$$\begin{aligned} u'(t) &= \int_0^{\xi_1} sy(s)ds + \sum_{j=2}^{m-2} \int_{\xi_{j-1}}^{\xi_j} \Big(\frac{\sum_{i=1}^{j-1} k_i}{1-\sum_{i=1}^{m-2} k_i} t + s \Big) y(s)ds \\ &\quad + \int_{\xi_{m-2}}^{t} \Big(\frac{\sum_{i=1}^{m-2} k_i}{1-\sum_{i=1}^{m-2} k_i} t + s \Big) y(s)ds + \int_t^1 \frac{1}{1-\sum_{i=1}^{m-2} k_i} ty(s)ds \end{aligned}$$

and

$$\begin{aligned} &u(t) \\ &= \int_0^{\xi_1} (ts - \frac{1}{2}s^2) y(s)ds + \sum_{j=2}^{m-2} k_i \int_{\xi_{j-1}}^{\xi_j} \Big(\frac{\sum_{i=1}^{j-1} k_i}{2(1-\sum_{i=1}^{m-2} k_i)} t^2 + st - \frac{1}{2}s^2 \Big) y(s)ds \\ &\quad + \int_{\xi_{m-2}}^{t} \Big(\frac{\sum_{i=1}^{m-2} k_i}{2(1-\sum_{i=1}^{m-2} k_i)} t^2 + st - \frac{1}{2}s^2 \Big) y(s)ds + \int_t^1 \frac{1}{2(1-\sum_{i=1}^{m-2} k_i)} t^2 y(s)ds. \end{aligned}$$

Summing up, we obtain the relationships:

$$u^{(i)}(t) = \int_0^1 G_i(t,s) y(s)ds, \quad t \in [0,1],\ i = 0,1,2.$$

□

Lemma 2.2. *Let*

$$A_0 = \frac{\sum_{i=1}^{m-2} |k_i| + \max\big\{ |1-\sum_{i=1}^{m-2} k_i|, \quad 1 \big\}}{2|1-\sum_{i=1}^{m-2} k_i|}, \quad A_1 = A_2 = 2A_0.$$

Then the Green functions $G_i(t,s) \quad (i = 0,1,2)$ satisfy

$$|G_i(t,s)| \le A_i, \quad (t,s) \in [0,1] \times [0,1]. \tag{2.5}$$

Proof. Since the proof of (2.5) is very similar for $i = 0,1,2$, we only prove the case when $i = 0$. In fact, for $j = 1,2,\dots,m-1$,

$$|G_0(t,s)| \le \begin{cases} \frac{\sum_{i=1}^{j-1} |k_i|}{2|1-\sum_{i=1}^{m-2} k_i|} t^2 + |ts - \frac{1}{2}s^2| \\ \le \frac{\sum_{i=1}^{m-2} |k_i|}{2|1-\sum_{i=1}^{m-2} k_i|} t^2 + \frac{1}{2}t^2 \le A_0, & s \le t,\ \xi_{j-1} < s \le \xi_j, \\ \frac{1+\sum_{i=j}^{m-2} |k_i|}{2|1-\sum_{i=1}^{m-2} k_i|} t^2 \le \frac{1+\sum_{i=1}^{m-2} |k_i|}{2|1-\sum_{i=1}^{m-2} k_i|} t^2 \le A_0, & s > t,\ \xi_{j-1} < s \le \xi_j\,. \end{cases}$$

□

Lemma 2.3. *Let $y \in L_p[0,1]$. Then the unique solution of* (2.1) *satisfies*

$$\|u^{(i)}\|_\infty \le A_i \|y\|_p, \quad i = 0,1,2, \tag{2.6}$$

where A_i ($i = 0,1,2$) is defined as in Lemma 2.2.

Proof. We divide the proof into two cases: $p > 1$ and $p = 1$.
Case 1: $p > 1$**.** Let $\frac{1}{p} + \frac{1}{q} = 1$. Then by Hölder's inequality,

$$|u^{(i)}(t)| \le \int_0^1 |G_i(t,s)||y(s)|ds \le \|G_i(t,\cdot)\|_q \|y\|_p \le \max_{0\le t\le 1} \|G_i(t,\cdot)\|_q \|y\|_p,$$

for $t \in [0,1]$, $i = 0,1,2$. In view of Lemma 2.2, we have

$$\|G_i(t,\cdot)\|_q^q = \int_0^1 |G_i(t,s)|^q ds \le \int_0^1 A_i^q ds = A_i^q, \quad t \in [0,1],$$

which implies that $\max_{0\le t\le 1} \|G_i(t,\cdot)\|_q \le A_i$. So,

$$\|u^{(i)}\|_\infty \le A_i \|y\|_p, \quad i = 0,1,2.$$

Case 2: $p = 1$**.** By Lemma 2.2, we have

$$|u^{(i)}(t)| \le \int_0^1 |G_i(t,s)||y(s)|ds \le A_i \int_0^1 |y(s)|ds = A_i \|y\|_1,$$

for $t \in [0,1]$, $i = 0,1,2$, which shows that

$$\|u^{(i)}\|_\infty \le A_i \|y\|_1, \quad i = 0,1,2.$$

The proof is complete. □

Now, if we define the integral operator $T : X \to X$ by

$$Tu(t) = \int_0^1 G_0(t,s) f(s, u(s), u'(s), u''(s))ds, \quad t \in [0,1],$$

then it is obvious that if u is a fixed point of T in X, then u is a solution of (1.6).

Lemma 2.4. *The mapping $T : X \to X$ is compact.*

Proof. At first, since T is so-called the Hammerstein operator and f is a L_p-Carathéodory function, we know that T is continuous.

Now, let $D \subset X$ be a bounded set, we will prove that $T(D)$ is relatively compact in X. Suppose that $\{w_k\}_{k=1}^\infty \subset T(D)$ is an arbitrary sequence. Then there is $\{u_k\}_{k=1}^\infty \subset D$ such that $T(u_k) = w_k$. Set

$$r = \sup_{u\in D} \|u\|.$$

Since $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is L_p-Carathéodory, there exists $\alpha_r \in L_p[0,1]$ such that

$$|f(t, u_k(t), u_k'(t), u_k''(t))| \le \alpha_r(t), \quad \text{a.e. } t \in [0,1], \quad k \in \mathbb{N}.$$

Since the proof is similar for $p = 1$, we only prove the case when $p > 1$. First, it follows from Hölder's inequality and Lemma 2.2 that

$$\begin{aligned}
|w_k(t)| &= |Tu_k(t)| \\
&= \Big|\int_0^1 G_0(t,s) f(s, u_k(s), u_k'(s), u_k''(s)) ds\Big| \\
&\le \int_0^1 |G_0(t,s)||f(s, u_k(s), u_k'(s), u_k''(s))| ds \\
&\le \int_0^1 |G_0(t,s)|\alpha_r(s) ds \\
&\le \max_{t\in[0,1]} \|G_0(t,\cdot)\|_q \|\alpha_r\|_p \\
&\le A_0 \|\alpha_r\|_p, \quad t \in [0,1],
\end{aligned}$$

which implies that $\{w_k\}_{k=1}^\infty$ is uniformly bounded. Similarly, we get

$$\begin{aligned}
|w_k'(t)| &= |Tu_k'(t)| \\
&= \Big|\int_0^1 G_1(t,s) f(s, u_k(s), u_k'(s), u_k''(s)) ds\Big| \\
&\le \max_{t\in[0,1]} \|G_1(t,\cdot)\|_q \|\alpha_r\|_p \\
&\le A_1 \|\alpha_r\|_p, \quad t \in [0,1],
\end{aligned}$$

which shows that $\{w_k'\}_{k=1}^\infty$ is also uniformly bounded. Therefore, $\{w_k\}_{k=1}^\infty$ is equicontinuous. By the Arzela-Ascoli theorem, $\{w_k\}_{k=1}^\infty$ has a convergent subsequence. Without loss of generality, we may assume that $\{w_k\}_{k=1}^\infty$ converges on $[0,1]$.

Next, for all $t \in [0,1]$, we have

$$\begin{aligned}
|w_k''(t)| &= |Tu_k''(t)| \\
&= \Big|\int_0^1 G_2(t,s) f(s, u_k(s), u_k'(s), u_k''(s)) ds\Big| \\
&\le \max_{t\in[0,1]} \|G_2(t,\cdot)\|_q \|\alpha_r\|_p \\
&\le A_2 \|\alpha_r\|_p,
\end{aligned}$$

that is to say, $\{w_k''\}_{k=1}^\infty$ is uniformly bounded, and so $\{w_k'\}_{k=1}^\infty$ is equicontinuous. As a result, without loss of generality, we may put that $\{w_k'\}_{k=1}^\infty$ is also convergent.

Finally, for any $\varepsilon > 0$, we can choose $\delta = \varepsilon^q/\|\alpha_r\|_p^q$ such that for any $k \in N$, t_1, $t_2 \in [0,1]$ and $|t_2 - t_1| < \delta$,

$$\begin{aligned}
|w_k''(t_2) - w_k''(t_1)| &= |Tu_k''(t_2) - Tu_k''(t_1)| \\
&= \Big|\int_{t_1}^{t_2} f(s, u_k(s), u_k'(s), u_k''(s)) ds\Big| \\
&\le |t_2 - t_1|^{1/q} \|\alpha_r\|_p < \varepsilon,
\end{aligned}$$

which shows that $\{w_k''\}_{k=1}^\infty$ is equicontinuous. Again, by the Arzela-Ascoli theorem, we know that $\{w_k''\}_{k=1}^\infty$ has a convergent subsequence. We establish that $\{w_k\}_{k=1}^\infty$ has a convergent subsequence in X. $\square$

Now, we apply the Leray-Schauder Continuation Principle to obtain the existence of at least one solution for (1.6).

Theorem 2.5. *Assume that there exist* $\alpha_0, \alpha_1, \alpha_2$ *and* $\delta \in L_p[0,1]$ *such that*

$$|f(t, x_0, x_1, x_2)| \le \sum_{i=0}^{2} \alpha_i(t) x_i + \delta(t), \quad a.e.\ t \in (0,1)\,, \tag{2.7}$$

$$\sum_{i=0}^{2} A_i \|\alpha_i\|_p < 1, \tag{2.8}$$

where A_i *(*$i = 0, 1, 2$*) is defined as in Lemma 2.2. Then* (1.6) *has at least one solution.*

Proof. To complete the proof, it suffices to verify that the set of all possible solutions of the BVP

$$\begin{gathered} u'''(t) + \lambda f(t, u(t), u'(t), u''(t)) = 0, \quad \text{a.e. } t \in (0,1), \\ u(0) = u'(0) = 0, \quad u''(1) = \sum_{i=1}^{m-2} k_i u''(\xi_i) \end{gathered} \tag{2.9}$$

is a priori bounded in X by a constant independent of $\lambda \in [0,1]$.

Suppose that $u \in W^{3,p}[0,1]$ is a solution of (2.9). Then it follows from (2.7), Lemma 2.2 and Lemma 2.3 that

$$\begin{aligned} \|u'''\|_p &= \lambda \|f(t, u, u', u'')\|_p \\ &\le \|f(t, u, u', u'')\|_p \\ &\le \sum_{i=0}^{2} \|\alpha_i u^{(i)}\|_p + \|\delta\|_p \\ &\le \sum_{i=0}^{2} \|\alpha_i\|_p \|u^{(i)}\|_\infty + \|\delta\|_p \\ &\le \sum_{i=0}^{2} A_i \|\alpha_i\|_p \|u'''\|_p + \|\delta\|_p, \end{aligned}$$

which implies

$$\|u'''\|_p \le \frac{\|\delta\|_p}{1 - \sum_{i=0}^{2} A_i \|\alpha_i\|_p},$$

and so,

$$\begin{aligned} \|u\| &= \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\} \\ &\le \max\{A_0, A_1, A_2\} \|u'''\|_p \\ &\le \frac{A_1 \|\delta\|_p}{1 - \sum_{i=0}^{2} A_i \|\alpha_i\|_p}. \end{aligned}$$

It is now immediate from Theorem 1.1 that T has at least one fixed point, which is a desired solution of (1.6). $\square$

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