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CONSTRUCTION OF ALMOST PERIODIC SEQUENCES WITH GIVEN PROPERTIES

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Abstract. We define almost periodic sequences with values in a pseudometric space $\mathcal{X}$ and we modify the Bochner definition of almost periodicity so that it remains equivalent with the Bohr definition. Then, we present one (easily modifiable) method for constructing almost periodic sequences in $\mathcal{X}$. Using this method, we find almost periodic homogeneous linear difference systems that do not have any non-trivial almost periodic solution. We treat this problem in a general setting where we suppose that entries of matrices in linear systems belong to a ring with a unit.

1. Introduction

First of all we mention the article [9] by Fan which considers almost periodic sequences of elements of a metric space and the article [22] by Tornehave about almost periodic functions of the real variable with values in a metric space. In these papers, it is shown that many theorems that are valid for complex valued sequences and functions are no longer true. For continuous functions, it was observed that the important property is the local connection by arcs of the space of values and also its completeness. However, we will not use their results or other theorems and we will define the notion of the almost periodicity of sequences in pseudometric spaces without any conditions, i.e., the definition is similar to the classical definition of Bohr, the modulus being replaced by the distance. We also refer to [31] (or [28]). We add that the concept of almost periodic functions of several variables with respect to Hausdorff metrics can be found in [19] which is an extension of [8].

In Banach spaces, a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ is almost periodic if and only if any sequence of translates of $\{\varphi_k\}$ has a subsequence which converges and its convergence is uniform with respect to k in the sense of the norm. In 1933, the continuous case of the previous result was proved by Bochner in [4], where the fundamental theorems of the theory of almost periodic functions with values in a Banach space are proved too – see, e.g., [2], [3, pp. 3–25] or [15], where the theorems have been redemonstrated by the methods of the functional analysis. We add that the discrete

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version of this result can be proved similarly as in [4]. We also mention directly the papers [26] and [17].

In pseudometric spaces, the above result is not generally true. Nevertheless, by a modification of the Bochner proof of this result, we will prove that a necessary and sufficient condition for a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ to be almost periodic is that any sequence of translates of $\{\varphi_k\}$ has a subsequence satisfying the Cauchy condition, uniformly with respect to k.

The paper is organized as follows. The next section presents the definition of almost periodic sequences in a pseudometric space, the above necessary and sufficient condition for the almost periodicity of a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$, and some basic properties of almost periodic sequences in pseudometric spaces (see also, e.g., [16]).

In Section 3, we show the way one can construct almost periodic sequences in pseudometric spaces. We present it in the below given theorems. In Theorem 3.1, we consider almost periodic sequences for $k \in \mathbb{N}_0$; in Theorem 3.3 and Corollary 3.4, sequences for $k \in \mathbb{Z}$ obtained from almost periodic sequences for $k \in \mathbb{N}_0$; and, in Theorems 3.5 and 3.6, sequences for $k \in \mathbb{Z}$. We remark that our process is comprehensible and easily modifiable. We add that methods of generating almost periodic sequences are mentioned also in [16, Section 4].

Then, in Section 4, we use results from the second and the third section of this paper to obtain a theorem which will play important role in the article [23], where it is proved that the almost periodic homogeneous linear difference systems which do not have any nonzero almost periodic solutions form a dense subset of the set of all considered systems. Using our method, one can get generalizations of statements from [21] and [24], where unitary (and orthogonal) systems are studied (see also [25]).

We will analyse systems of the form

$$x_{k+1} = A_k \cdot x_k, \quad k \in \mathbb{Z} \quad (\text{or } k \in \mathbb{N}_0),$$

where $\{A_k\}$ is almost periodic. We want to prove that there exists a system of the above form which does not have an almost periodic solution other than the trivial one. (See Theorem 4.7.) A closer examination of the methods used in constructions reveals that the problem can be treated in possibly the most general setting:

(1) almost periodic sequences attain values in a pseudometric space;
(2) the entries of almost periodic matrices are elements of an infinite ring with a unit.

We note that many theorems about the existence of almost periodic solutions of almost periodic difference systems of general forms are published in [11, 12, 18, 28, 29, 31] and several these existence theorems are proved there in terms of discrete Lyapunov functions. Here, we can also refer to the monograph [27] and [32, Theorems 3.6, 3.7, 3.8]. We add that the existence of an almost periodic homogeneous linear *differential* system, which has nontrivial bounded solutions and, at the same time, all the systems from some neighbourhood of it have no nontrivial almost periodic solutions, is proved in [20].

As usual, $\mathbb{R}$ denotes the real line, $\mathbb{R}_0^+$ the set of all nonnegative reals, $\mathbb{C}$ the complex plane, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N}$ the set of natural numbers, and $\mathbb{N}_0$ the set of positive integers including the zero 0.

Let $\mathcal{X} \neq \emptyset$ be an arbitrary set and let $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_0^+$ have these properties:

(i) $d(x, x) = 0$ for all $x \in \mathcal{X}$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$,
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in \mathcal{X}$.

We say that d is *a pseudometric* on $\mathcal{X}$ and $(\mathcal{X}, d)$ *a pseudometric space.*

For given $\varepsilon > 0$, $x \in \mathcal{X}$, in the same way as in metric spaces, we define *the ε-neighbourhood* of x in $\mathcal{X}$ as the set $\{y \in \mathcal{X}; d(x, y) < \varepsilon\}$. It will be denoted by $\mathcal{O}_\varepsilon(x)$.

All sequences, which we will consider, will be subsets of $\mathcal{X}$. The scalar (and vector) valued sequences will be denoted by the lower-case letters, the matrix valued sequences by the capital letters ($\mathcal{X}$ is a set of matrices in this case), and each one of the scalar and matrix valued sequences by the symbols $\{\varphi_k\}$, $\{\psi_k\}$, $\{\chi_k\}$.

2. Almost periodic sequences in pseudometric spaces

Now we introduce a “natural” generalization of the almost periodicity. We remark that our approach is very general and that the theory of almost periodic sequences presented here does not distinguish between $x \in \mathcal{X}$ and $y \in \mathcal{X}$ if $d(x, y) = 0$.

Definition 2.1. A sequence $\{\varphi_k\}$ is called *almost periodic* if, for any $\varepsilon > 0$, there exists a positive integer $p(\varepsilon)$ such that any set consisting of $p(\varepsilon)$ consecutive integers (nonnegative integers if $k \in \mathbb{N}_0$) contains at least one integer l with the property that

$$d(\varphi_{k+l}, \varphi_k) < \varepsilon, \quad k \in \mathbb{Z} \quad (\text{or } k \in \mathbb{N}_0).$$

In the above definition, l is called an *ε-translation number* of $\{\varphi_k\}$.

Consider again $\varepsilon > 0$. Henceforward, the set of all ε-translation numbers of a sequence $\{\varphi_k\}$ will be denoted by $\mathfrak{T}(\{\varphi_k\}, \varepsilon)$.

Remark 2.2. If $\mathcal{X}$ is a Banach space ($d(x, y)$ is given by $\|x - y\|$), then a necessary and sufficient condition for a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ to be almost periodic is it to be *normal*; i.e., $\{\varphi_k\}$ is almost periodic if and only if any sequence of translates of $\{\varphi_k\}$ has a subsequence, uniformly convergent for $k \in \mathbb{Z}$ in the sense of the norm. This statement and the below given Theorem 2.3 are not valid if $\{\varphi_k\}$ is defined for $k \in \mathbb{N}_0$ and if we consider only translates to the right – see the example $\mathcal{X} = \mathbb{R}$, $\varphi_0 = 1$, and $\varphi_k = 0$, $k \in \mathbb{N}$. But, if we consider translates to the left, then both of results are valid for $k \in \mathbb{N}_0$ too.

It is seen that the above result is no longer valid if the space of values fails to be complete. Especially, in a pseudometric space $(\mathcal{X}, d)$, it is not generally true that a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ is almost periodic if and only if it is normal. Nevertheless, applying the methods from any one of the proofs of the results [6, Theorem 1.10, p. 16], [10, Theorem 1.14, pp. 9–10], and [3, Statement (ζ), pp. 8–9], one can easily prove that every normal sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ is almost periodic. Further, we can prove the next theorem which we will need later. We add that its proof is a modification of the proof of [6, Theorem 1.26, pp. 45–46].

Theorem 2.3. *Let $\{\varphi_k\}_{k\in\mathbb{Z}}$ be given. For an arbitrary sequence $\{h_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}$, there exists a subsequence $\{\tilde{h}_n\}_{n\in\mathbb{N}} \subseteq \{h_n\}_{n\in\mathbb{N}}$ with the Cauchy property with respect to $\{\varphi_k\}$, i.e., for any $\varepsilon > 0$, there exists $M \in \mathbb{N}$ for which the inequality*

$$d\big(\varphi_{k+\tilde{h}_i}, \varphi_{k+\tilde{h}_j}\big) < \varepsilon$$

holds for all $i, j, k \in \mathbb{Z}$, $i, j > M$, if and only if $\{\varphi_k\}_{k\in\mathbb{Z}}$ is almost periodic.

Proof. If any sequence of translates of $\{\varphi_k\}$ has a subsequence which has the Cauchy property, then $\{\varphi_k\}$ is almost periodic. It can be proved similarly as [6, Theorem 1.10, p. 16], where it is not used that $\mathcal{X}$ is complete. To prove the opposite implication, we will assume that $\{\varphi_k\}$ is almost periodic, and we will use the known method of the diagonal extraction.

Let $\{h_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}$ and $\vartheta > 0$ be arbitrary. By Definition 2.1, there exists a positive integer p such that, in any set $\{h_n - p, h_n - p + 1, \dots, h_n\}$, there exists a ϑ-translation number l_n. We know that $0 \le h_n - l_n \le p$ for all $n \in \mathbb{N}$. We put $k_n := h_n - l_n$, $n \in \mathbb{N}$. Clearly, $k_n = c = \text{const.}$ (a constant value from $\{0, 1, \dots, p\}$) for infinitely many values of n. Since

$$d(\varphi_{k+h_n}, \varphi_{k+k_n}) = d(\varphi_{(k+h_n-l_n)+l_n}, \varphi_{k+h_n-l_n}) < \vartheta, \quad k \in \mathbb{Z},$$

there exists a subsequence $\{h_n^1\}$ of $\{h_n\}$ and an integer c_1 such that

$$d(\varphi_{k+h_n^1}, \varphi_{k+c_1}) < \vartheta, \quad k \in \mathbb{Z},\ n \in \mathbb{N}. \tag{2.1}$$

Consider now a sequence of positive numbers $\vartheta_1 > \vartheta_2 > \dots > \vartheta_n > \dots$ converging to 0. We extract from the sequence $\{\varphi_{k+h_n}\}$ a subsequence $\{\varphi_{k+h_n^1}\}$ which satisfies (2.1) for $\vartheta = \vartheta_1$. From this sequence we extract a subsequence $\{\varphi_{k+h_n^2}\}$ for which an inequality analogous to (2.1) is valid. Of course, c will not be the same, but will depend on the subsequence. We proceed further in the same way. Next, we form the sequence $\{\varphi_{k+h_n^n}\}_{n\in\mathbb{N}}$. Assume that $\varepsilon > 0$ is given and that we have $2\vartheta_m < \varepsilon$ for $m \in \mathbb{N}$. As a result, for $i, j > m$, $i, j \in \mathbb{N}$, we obtain

$$d\big(\varphi_{k+h_i^i}, \varphi_{k+h_j^j}\big) \le d\big(\varphi_{k+h_i^i}, \varphi_{k+c_m}\big) + d\big(\varphi_{k+c_m}, \varphi_{k+h_j^j}\big) < \varepsilon, \quad k \in \mathbb{Z},$$

where c_m is the number corresponding to the sequence $\{\varphi_{k+h_n^m}\}_{n\in\mathbb{N}}$ and ϑ_m. □

Theorem 2.4. *Let $\mathcal{X}_1, \mathcal{X}_2$ be arbitrary pseudometric spaces and $\Phi : \mathcal{X}_1 \to \mathcal{X}_2$ be a uniformly continuous map. If $\{\varphi_k\}$ is almost periodic, then the sequence $\{\Phi \circ \varphi_k\}$ is almost periodic too.*

Proof. Taking $\varepsilon > 0$ arbitrarily, let $\delta(\varepsilon) > 0$ be the number corresponding to ε from the definition of the uniform continuity of Φ. Now, Theorem 2.4 follows from the fact that the set of all ε-translation numbers of $\{\Phi \circ \varphi_k\}$ contains the set of all $\delta(\varepsilon)$-translation numbers of $\{\varphi_k\}$, i.e., from the inclusion

$$\mathfrak{T}(\{\varphi_k\}, \delta(\varepsilon)) \subseteq \mathfrak{T}(\{\Phi \circ \varphi_k\}, \varepsilon).$$

□

We note that we can prove many theorems which are valid in the classical case also for pseudometric spaces. For example, we mention the following result:

(a) For every sequence of almost periodic sequences $\{\varphi_k^1\}, \dots, \{\varphi_k^n\}, \dots$, the sequence of $\lim_{n\to\infty} \varphi_k^n$ is almost periodic if the convergence is uniform with respect to k.

Taking $n \in \mathbb{N}$ and using Theorem 2.3 (and Remark 2.2) n-times, one can easily prove also:

(b) If $(\mathcal{X}_1, d_1), \dots, (\mathcal{X}_n, d_n)$ are pseudometric spaces and $\{\varphi_k^1\}, \dots, \{\varphi_k^n\}$ are arbitrary almost periodic sequences with values in $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively, then the sequence $\{\psi_k\}$, with values in $\mathcal{X}_1 \times \dots \times \mathcal{X}_n$ given by

$$\psi_k := (\varphi_k^1, \dots, \varphi_k^n) \quad \text{for all considered } k,$$

is also almost periodic.

(c) Let the sequences $\{\varphi_k^1\}, \dots, \{\varphi_k^n\}$ ($k \in \mathbb{Z}$ or $k \in \mathbb{N}_0$) be given. Then, the sequence $\{\psi_k\}$ which is defined by
$$\psi_k := \varphi_j^{i+1} \quad \text{for all considered } k,$$
where $k = jn + i$, $j \in \mathbb{Z}$ ($j \in \mathbb{N}_0$), $i \in \{0, \dots, n-1\}$, is almost periodic if and only if all sequences $\{\varphi_k^1\}, \dots, \{\varphi_k^n\}$ are almost periodic.

For the first result (in $\mathbb{C}$), see [6, Theorem 6.4, p. 139]. We remark that one can use the above theorems to obtain more general versions of Theorems 3.1, 3.5, 3.6.

3. Construction of almost periodic sequences

Now we prove several theorems which facilitate to find almost periodic sequences having certain specific properties:

Theorem 3.1. *Let $m \in \mathbb{N}_0$, $\varphi_0, \dots, \varphi_m \in \mathcal{X}$, and $j \in \mathbb{N}$ be arbitrarily given. Let $\{r_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of nonnegative real numbers such that*
$$\sum_{n=1}^{\infty} r_n < +\infty. \tag{3.1}$$
Then, any sequence $\{\varphi_k\}_{k\in\mathbb{N}_0} \subseteq \mathcal{X}$, where ($n > 2, n \in \mathbb{N}$)
$$\begin{gathered}
\varphi_k \in \mathcal{O}_{r_1}(\varphi_{k-(m+1)}), \quad k \in \{m+1, \dots, 2m+1\}, \\
\varphi_k \in \mathcal{O}_{r_1}(\varphi_{k-2(m+1)}), \quad k \in \{2(m+1), \dots, 3(m+1)-1\}, \\
\dots \\
\varphi_k \in \mathcal{O}_{r_1}(\varphi_{k-j(m+1)}), \quad k \in \{j(m+1), \dots, (j+1)(m+1)-1\}, \\
\varphi_k \in \mathcal{O}_{r_2}(\varphi_{k-(j+1)(m+1)}), \quad k \in \{(j+1)(m+1), \dots, 2(j+1)(m+1)-1\}, \\
\varphi_k \in \mathcal{O}_{r_2}(\varphi_{k-2(j+1)(m+1)}), \quad k \in \{2(j+1)(m+1), \dots, 3(j+1)(m+1)-1\}, \\
\dots \\
\varphi_k \in \mathcal{O}_{r_2}(\varphi_{k-j(j+1)(m+1)}), \quad k \in \{j(j+1)(m+1), \dots, (j+1)^2(m+1)-1\}, \\
\dots \\
\varphi_k \in \mathcal{O}_{r_n}(\varphi_{k-(j+1)^{n-1}(m+1)}), \quad k \in \big\{(j+1)^{n-1}(m+1), \\
\dots, 2(j+1)^{n-1}(m+1)-1\big\}, \\
\varphi_k \in \mathcal{O}_{r_n}(\varphi_{k-2(j+1)^{n-1}(m+1)}), \quad k \in \big\{2(j+1)^{n-1}(m+1), \\
\dots, 3(j+1)^{n-1}(m+1)-1\big\}, \\
\dots \\
\varphi_k \in \mathcal{O}_{r_n}(\varphi_{k-j(j+1)^{n-1}(m+1)}), \quad k \in \big\{j(j+1)^{n-1}(m+1), \\
\dots, (j+1)^n(m+1)-1\big\}, \dots
\end{gathered}$$
are arbitrary too, is almost periodic.

Proof. Consider an arbitrary $\varepsilon > 0$. We need to prove that the set of all ε-translation numbers of $\{\varphi_k\}$ is relative dense in $\mathbb{N}_0$. Using (3.1), one can find $n(\varepsilon)$ for which
$$\sum_{n=n(\varepsilon)}^{\infty} r_n < \frac{\varepsilon}{2}. \tag{3.2}$$

We see that

$$\begin{gathered}\varphi_{k+(j+1)^{n(\varepsilon)-1}(m+1)} \in \mathcal{O}_{r_{n(\varepsilon)}}(\varphi_k),\\ \varphi_{k+2(j+1)^{n(\varepsilon)-1}(m+1)} \in \mathcal{O}_{r_{n(\varepsilon)}}(\varphi_k),\\ \dots\\ \varphi_{k+j(j+1)^{n(\varepsilon)-1}(m+1)} \in \mathcal{O}_{r_{n(\varepsilon)}}(\varphi_k)\end{gathered} \tag{3.3}$$

for

$$0 \le k < (j+1)^{n(\varepsilon)-1}(m+1).$$

Next, from (iii) and (3.3) it follows ($i \in \{(j+1)^n, \dots, (j+1)^{n+1}-1\}$, $n \in \mathbb{N}$)

$$\begin{gathered}\varphi_{k+(j+1)(j+1)^{n(\varepsilon)-1}(m+1)} \in \mathcal{O}_{r_{n(\varepsilon)}+r_{n(\varepsilon)+1}}(\varphi_k),\\ \dots\\ \varphi_{k+((j+1)^2-1)(j+1)^{n(\varepsilon)-1}(m+1)} \in \mathcal{O}_{r_{n(\varepsilon)}+r_{n(\varepsilon)+1}}(\varphi_k),\\ \dots\\ \varphi_{k+i(j+1)^{n(\varepsilon)-1}(m+1)} \in \mathcal{O}_{r_{n(\varepsilon)}+r_{n(\varepsilon)+1}+\dots+r_{n(\varepsilon)+n}}(\varphi_k),\\ \dots\end{gathered}$$

for $k \in \{0, \dots, (j+1)^{n(\varepsilon)-1}(m+1)-1\}$. Therefore (consider (3.2)), we have

$$\varphi_{k+l(j+1)^{n(\varepsilon)-1}(m+1)} \in \mathcal{O}_{\frac{\varepsilon}{2}}(\varphi_k), \quad 0 \le k < (j+1)^{n(\varepsilon)-1}(m+1),\, l \in \mathbb{N}_0. \tag{3.4}$$

We put

$$q(\varepsilon) := (j+1)^{n(\varepsilon)-1}(m+1). \tag{3.5}$$

Any $p \in \mathbb{N}_0$ can be expressed uniquely in the form

$$p = k(p) + l(p)q(\varepsilon) \quad \text{for some } k(p) \in \{0, \dots, q(\varepsilon)-1\} \text{ and } l(p) \in \mathbb{N}_0.$$

Applying (3.4), we obtain

$$\begin{aligned} d(\varphi_p, \varphi_{p+lq(\varepsilon)}) &= d(\varphi_{k(p)+l(p)q(\varepsilon)}, \varphi_{k(p)+l(p)q(\varepsilon)+lq(\varepsilon)})\\ &\le d(\varphi_{k(p)+l(p)q(\varepsilon)}, \varphi_{k(p)}) + d(\varphi_{k(p)}, \varphi_{k(p)+(l+l(p))q(\varepsilon)})\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned} \tag{3.6}$$

where $p, l \in \mathbb{N}_0$ are arbitrary; i.e., $lq(\varepsilon)$ is an ε-translation number of $\{\varphi_k\}$ for all $l \in \mathbb{N}_0$. The fact that the set $\{lq(\varepsilon); l \in \mathbb{N}_0\}$ is relative dense in $\mathbb{N}_0$ proves the theorem. □

Remark 3.2. From the proof of Theorem 3.1 (see (3.5) and (3.6)), for any $\varepsilon > 0$ and any sequence $\{\varphi_k\}$ considered there, we get the existence of $n(\varepsilon) \in \mathbb{N}$ such that the set of all ε-translation numbers of $\{\varphi_k\}$ contains $\{l(j+1)^{n(\varepsilon)-1}(m+1); l \in \mathbb{N}\}$; i.e., we have

$$T(\{\varphi_k\}, n(\varepsilon)) := \{l(j+1)^{n(\varepsilon)-1}(m+1);\, l \in \mathbb{N}\} \subseteq \mathfrak{T}(\{\varphi_k\}, \varepsilon) \tag{3.7}$$

for every $\varepsilon > 0$.

Theorem 3.3. *Let $\{\varphi_k\}_{k\in\mathbb{N}_0}$ be an almost periodic sequence and let the sequences $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_0^+$ and $\{l_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ be such that*

$$r_n l_n \to 0 \quad \text{as } n \to \infty. \tag{3.8}$$

If, for all n, there exists a set $T(r_n)$ of some r_n-translation numbers of $\{\varphi_k\}$ which is relative dense in $\mathbb{N}_0$ and, for every nonzero $l = l(r_n) \in T(r_n)$, there exists $i = i(l) \in \{1, \dots, l_n + 1\}$ with the property that

$$\varphi_{(i-1)l+k} \in \mathcal{O}_{r_n l_n}(\varphi_{il-k}), \quad k \in \{0, \dots, l\}, \tag{3.9}$$

then the sequence $\{\psi_k\}_{k\in\mathbb{Z}}$, given by the formula

$$\psi_k := \varphi_k \text{ for } k \in \mathbb{N}_0 \quad \text{and} \quad \psi_k := \varphi_{-k} \text{ for } k \in \mathbb{Z} \setminus \mathbb{N}_0, \tag{3.10}$$

is almost periodic.

If, for all n, there exists a set $\widetilde{T}(r_n)$ of some r_n-translation numbers of $\{\varphi_k\}$ which is relative dense in $\mathbb{N}_0$ and, for every nonzero $m = m(r_n) \in \widetilde{T}(r_n)$, there exists $i = i(m) \in \{1, \dots, l_n + 1\}$ with the property that

$$\varphi_{(i-1)m+k} \in \mathcal{O}_{r_n l_n}(\varphi_{im-k-1}), \quad k \in \{0, \dots, m-1\}, \tag{3.11}$$

then the sequence $\{\chi_k\}_{k\in\mathbb{Z}}$, given by the formula

$$\chi_k := \varphi_k \text{ for } k \in \mathbb{N}_0 \quad \text{and} \quad \chi_k := \varphi_{-(k+1)} \text{ for } k \in \mathbb{Z} \setminus \mathbb{N}_0, \tag{3.12}$$

is almost periodic.

Proof. We will prove only the first part of Theorem 3.3. The proof of the second case (the almost periodicity of $\{\chi_k\}$) is analogical. Let $\varepsilon > 0$ be arbitrarily small. Consider $n \in \mathbb{N}$ satisfying (see (3.8))

$$r_n l_n < \frac{\varepsilon}{3}. \tag{3.13}$$

We will prove that the set $\mathfrak{T}(\{\psi_k\}, \varepsilon)$ of all ε-translation numbers of $\{\psi_k\}$ contains the numbers $\{\pm l; l \in T(r_n)\}$; i.e., we will get the inequality

$$d(\psi_k, \psi_{k\pm l}) < \varepsilon, \quad l \in T(r_n),\ k \in \mathbb{Z} \tag{3.14}$$

which proves the theorem because $\{\pm l;\ l \in T(r_n)\}$ is relative dense in $\mathbb{Z}$.

First of all we choose arbitrary $l \in T(r_n)$. From the theorem, we have $i = i(l)$. Without loss of the generality, we can consider only $+l$. (For $-l$, we can proceed similarly.) Because of $l_n \in \mathbb{N}$ and $l \in T(r_n)$, from (3.10) and (3.13) it follows

$$d(\psi_k, \psi_{k+l}) < \frac{\varepsilon}{3}, \quad k \notin \{-l, \dots, -1\},\ k \in \mathbb{Z}. \tag{3.15}$$

Let $k \in \{-l, \dots, -1\}$ be also arbitrarily chosen. Evidently, we have

$$k + (1-i)l \in \{-il, \dots, -(i-1)l - 1\}$$

and

$$\begin{aligned} d(\psi_k, \psi_{k+l}) &\le d(\psi_k, \psi_{k+(1-i)l}) + d(\psi_{k+(1-i)l}, \psi_{k+l}) \\ &= d(\varphi_{-k}, \varphi_{(i-1)l-k}) + d(\varphi_{(i-1)l-k}, \varphi_{l+k}). \end{aligned} \tag{3.16}$$

The number $(i-1)l$ is an $\frac{\varepsilon}{3}$-translation number of $\{\varphi_k\}$. Indeed, it follows from (iii), (3.13), and from $i \le l_n + 1$. Therefore, we have

$$d(\varphi_{-k}, \varphi_{(i-1)l-k}) < \frac{\varepsilon}{3}. \tag{3.17}$$

Using (3.9) and (3.13), we get

$$d(\varphi_{(i-1)l-k}, \varphi_{il+k}) < r_n l_n < \frac{\varepsilon}{3}.$$

Thus, it holds

$$d(\varphi_{(i-1)l-k}, \varphi_{l+k}) < \frac{2\varepsilon}{3}. \tag{3.18}$$

Indeed, $(i-1)l$ is an $\frac{\varepsilon}{3}$-translation number of $\{\varphi_k\}$ (consider again (iii), (3.13), and the inequality $i-1 \le l_n$).

Altogether, from (3.16), (3.17), and (3.18), we obtain

$$d(\psi_k, \psi_{k+l}) < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon. \tag{3.19}$$

Since the choice of k, l was arbitrary (see (3.15)), (3.19) gives (3.14). □

Corollary 3.4. *Let $m \in \mathbb{N}_0$, $j \in \mathbb{N}$, and the sequence $\{\varphi_k\}_{k\in\mathbb{N}_0}$ be from Theorem 3.1 and $M > 0$ be arbitrary. If, for all $n > M$, $n \in \mathbb{N}$, there exists at least one $i \in \{1,\dots,j\}$ satisfying*

$$\varphi_{i(j+1)^n(m+1)+k} = \varphi_{(i+1)(j+1)^n(m+1)-k}, \quad k \in \{0,\dots,(j+1)^n(m+1)\}, \tag{3.20}$$

then the sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ given by (3.10) *is almost periodic. If, for all $n > M$, $n \in \mathbb{N}$, there exists at least one $i \in \{1,\dots,j\}$ satisfying*

$$\varphi_{i(j+1)^n(m+1)+k} = \varphi_{(i+1)(j+1)^n(m+1)-k-1}, \quad k \in \{0,\dots,(j+1)^n(m+1)-1\}, \tag{3.21}$$

then the sequence $\{\chi_k\}_{k\in\mathbb{Z}}$ given by (3.12) *is almost periodic.*

Proof. We put

$$r_n := \frac{1}{n}, \quad l_n := 1, \quad T(r_n) := T(\{\varphi_k\}, n(\frac{r_n}{2})) \quad \text{for all } n \in \mathbb{N},$$

where $T(\{\varphi_k\}, n(\varepsilon))$ is defined by (3.7). Because we can assume that $n(\frac{1}{2}) > M-1$, it suffices to consider Theorem 3.3 and Remark 3.2 (from (iii), using (3.20) and (3.21), we get (3.9) and (3.11), respectively). □

For $k \in \mathbb{Z}$, we can prove (analogously as Theorem 3.1) the following two theorems:

Theorem 3.5. *Let $m \in \mathbb{N}_0$, $\psi_0,\dots,\psi_m \in \mathcal{X}$, $j \in \mathbb{N}$, and the sequences of non-negative real numbers $\{r_n^1\}_{n\in\mathbb{N}},\dots,\{r_n^j\}_{n\in\mathbb{N}}$ be arbitrarily given so that*

$$\sum_{n=1}^{\infty} r_n^i < +\infty, \quad i \in \{1,\dots,j\} \tag{3.22}$$

holds. Then, every sequence $\{\psi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ for which it is true

$$\begin{gathered}
\psi_k \in \mathcal{O}_{r_1^1}(\psi_{k-(m+1)}), \quad k \in \{m+1,\dots,2(m+1)-1\};\\
\dots\\
\psi_k \in \mathcal{O}_{r_1^j}(\psi_{k-j(m+1)}), \quad k \in \{j(m+1),\dots,(j+1)(m+1)-1\};\\
\psi_k \in \mathcal{O}_{r_2^1}(\psi_{k+(j+1)(m+1)}), \quad k \in \{-(j+1)(m+1),\dots,-1\};\\
\dots\\
\psi_k \in \mathcal{O}_{r_2^j}(\psi_{k+j(j+1)(m+1)}),\\
k \in \{-j(j+1)(m+1),\dots,-(j-1)(j+1)(m+1)-1\};\\
\psi_k \in \mathcal{O}_{r_3^1}(\psi_{k-(j+1)^2(m+1)}),\\
k \in \{(j+1)(m+1),\dots,(j+1)(m+1)+(j+1)^2(m+1)-1\};\\
\dots
\end{gathered}$$

$$
\begin{aligned}
&\psi_k \in \mathcal{O}_{r_3^j}(\psi_{k-j(j+1)^2(m+1)}),\\
&k \in \{(j+1)(m+1)+(j-1)(j+1)^2(m+1),\dots,\\
&\quad (j+1)(m+1)+j(j+1)^2(m+1)-1\};\\
&\psi_k \in \mathcal{O}_{r_4^1}(\psi_{k+(j+1)^3(m+1)}),\\
&k \in \{-(j+1)^3(m+1)-j(j+1)(m+1),\dots,-j(j+1)(m+1)-1\};\\
&\dots\\
&\psi_k \in \mathcal{O}_{r_4^j}(\psi_{k+j(j+1)^3(m+1)}),\\
&k \in \{-j(j+1)^3(m+1)-j(j+1)(m+1),\dots,\\
&\quad -(j-1)(j+1)^3(m+1)-j(j+1)(m+1)-1\};\\
&\dots\\
&\psi_k \in \mathcal{O}_{r_{2n}^1}(\psi_{k+(j+1)^{2n-1}(m+1)}),\\
&k \in \{-((j+1)^{2n-1}+\dots+j(j+1)^3+j(j+1))(m+1),\\
&\quad \dots,-((j+1)^{2n-3}+\dots+j(j+1)^3+j(j+1))(m+1)-1\};\\
&\dots\\
&\psi_k \in \mathcal{O}_{r_{2n}^j}(\psi_{k+j(j+1)^{2n-1}(m+1)}),\\
&k \in \{-(j(j+1)^{2n-1}+\dots+j(j+1)^3+j(j+1))(m+1),\\
&\quad \dots,-((j-1)(j+1)^{2n-1}+\dots+j(j+1)^3+j(j+1))(m+1)-1\};\\
&\psi_k \in \mathcal{O}_{r_{2n+1}^1}(\psi_{k-(j+1)^{2n}(m+1)}),\\
&k \in \{(j+1)(m+1)+j(j+1)^2(m+1)+\dots+j(j+1)^{2n-2}(m+1),\\
&\quad \dots,(j+1)(m+1)+j(j+1)^2(m+1)+\\
&\quad \dots+j(j+1)^{2n-2}(m+1)+(j+1)^{2n}(m+1)-1\};\\
&\dots\\
&\psi_k \in \mathcal{O}_{r_{2n+1}^j}(\psi_{k-j(j+1)^{2n}(m+1)}),\\
&k \in \{(j+1)(m+1)+j(j+1)^2(m+1)+\\
&\quad \dots+j(j+1)^{2n-2}(m+1)+(j-1)(j+1)^{2n}(m+1),\\
&\quad \dots(j+1)(m+1)+j(j+1)^2(m+1)+\dots+j(j+1)^{2n}(m+1)-1\};\\
&\dots
\end{aligned}
$$

is almost periodic.

Proof. We put $r_n := \max_{1\le i\le j} r_n^i$, $n \in \mathbb{N}$. Then (3.22) implies $\sum_{n=1}^{\infty} r_n < +\infty$. Let the number $n(\varepsilon) \in \mathbb{N}$ satisfy the condition (3.2). We can show that, for arbitrarily given $\varepsilon > 0$, in any set $Z \subseteq \mathbb{Z}$ consisting of

$$(j+1)^{n(\varepsilon)-1}(m+1) \text{ consecutive integers,}$$

there exists an ε-translation number of $\{\psi_k\}$. From it follows the theorem. $\square$

Theorem 3.6. *Let $\varphi_0, \dots, \varphi_m \in \mathcal{X}$ be given, $\{r_i\}_{i\in\mathbb{N}} \subset \mathbb{R}_0^+$, $\{j_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$, and $n \in \mathbb{N}_0$ be arbitrary such that $m+n$ is even and*

$$\sum_{i=1}^{\infty} r_i j_i < +\infty. \tag{3.23}$$

For any $\varphi_{m+1}, \dots, \varphi_{m+n}$, if we set

$$\psi_k := \varphi_{k+\frac{m+n}{2}}, \quad k \in \{-\frac{m+n}{2}, \dots, \frac{m+n}{2}\},$$
$$M := \frac{m+n}{2}, \quad N := m+n$$

and we choose arbitrarily

$$\psi_k \in \mathcal{O}_{r_1}(\psi_{k+N+1}), \quad k \in \{-N-M-1, \dots, -M-1\},$$
$$\dots$$
$$\psi_k \in \mathcal{O}_{r_1}(\psi_{k+N+1}), \quad k \in \{-j_1N-M-1, \dots, -(j_1-1)N-M-1\},$$
$$\psi_k \in \mathcal{O}_{r_1}(\psi_{k-N-1}), \quad k \in \{M+1, \dots, N+M+1\},$$
$$\dots$$
$$\psi_k \in \mathcal{O}_{r_1}(\psi_{k-N-1}), \quad k \in \{(j_1-1)N+M+1, \dots, j_1N+M+1\},$$
$$\dots$$
$$\psi_k \in \mathcal{O}_{r_i}(\psi_{k+p_i}), \quad k \in \{-p_i-p_{i-1}-\dots-p_1, \dots, -p_{i-1}-\dots-p_1\},$$
$$\dots$$
$$\psi_k \in \mathcal{O}_{r_i}(\psi_{k+p_i}),$$
$$k \in \{-j_ip_i-p_{i-1}-\dots-p_1, \dots, -(j_i-1)p_i-p_{i-1}-\dots-p_1\},$$
$$\psi_k \in \mathcal{O}_{r_i}(\psi_{k-p_i}), \quad k \in \{p_{i-1}+\dots+p_1, \dots, p_i+p_{i-1}+\dots+p_1\},$$
$$\dots$$
$$\psi_k \in \mathcal{O}_{r_i}(\psi_{k-p_i}), \quad k \in \{(j_i-1)p_i+p_{i-1}+\dots+p_1, \dots, j_ip_i+p_{i-1}+\dots+p_1\},$$
$$\dots$$

where

$$p_1 := (j_1N+M+1)+1, \quad p_2 := 2(j_1N+M+1)+1,$$
$$p_3 := (2j_2+1)p_2, \dots, p_i := (2j_{i-1}+1)p_{i-1}, \dots,$$

then the resulting sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ is almost periodic.

Proof. Consider arbitrary $\varepsilon > 0$ and a positive integer $n(\varepsilon) \geq 2$ for which (see (3.23)) $\sum_{i=n(\varepsilon)}^{\infty} r_i\, j_i < \frac{\varepsilon}{4}$. One can show that $\{lp_{n(\varepsilon)}; l \in \mathbb{Z}\} \subseteq \mathfrak{T}(\{\psi_k\}, \varepsilon)$ which completes the proof. $\square$

4. An application

Let $m \in \mathbb{N}$ be arbitrarily given. We will analyse almost periodic systems of m homogeneous linear difference equations of the form

$$x_{k+1} = A_k \cdot x_k, \quad k \in \mathbb{Z} \text{ (or } k \in \mathbb{N}_0), \tag{4.1}$$

where $\{A_k\}$ is almost periodic. Let $\mathfrak{X}$ denote *the set of all systems* (4.1). Our aim is to study the existence of a system $\widehat{\mathfrak{S}} \in \mathfrak{X}$ which does not have any nontrivial almost periodic solutions. We are going to treat this problem in a very general setting and this motivates our requirements on the set of values of matrices A_k.

We need the set of entries of A_k to be a subset of a set R with two operations and unit elements such that R with them is a ring because the multiplication of matrices A_k has to be associative (consider the natural expression of solutions of (4.1)). We also need the set of all considered A_k to form a set X which has the below given properties (4.5), and we need that there exists at least one of the below mentioned functions F_1, $F_2 : [-1,1] \to X$ – see (4.6), (4.7), respectively. The conditions (4.6) are common, natural, and simple. However, the main theorem of this article (the existence of the above system $\widehat{\mathfrak{S}} \in \mathfrak{X}$) is true for many subsets of the set of all unitary or orthogonal matrices which contain of matrices that have the eigenvalue $\lambda = 1$. Thus, we will consider the existence of F_2.

We remark that it is possible to obtain results about the nonexistence of nontrivial almost periodic solutions using different methods than those presented in our paper. For example, if the zero solution of a system $\mathfrak{S}$ of the form (4.1) is asymptotically (or even exponentially) stable, then it is obviously that we can choose $\widehat{\mathfrak{S}} := \mathfrak{S}$. See [14] and more general [7], [13], and [32], where the method of Lyapunov function(al)s is used.

Let $R = (R, \oplus, \odot)$ be an infinite ring with a unit and a zero denoted as e_1 and e_0, respectively. The symbol $\mathcal{M}(R, m)$ will denote the set of all $m \times m$ matrices with elements from R. If we consider the i-th column of $U \in \mathcal{M}(R, m)$, then we write U_i; and R^m if we consider the set of all $m \times 1$ vectors with entries attaining values from R. As usual, we define the multiplication $\cdot$ of matrices from $\mathcal{M}(R, m)$ (and $U \cdot v$, $U \in \mathcal{M}(R, m)$, $v \in R^m$) by $\oplus$ and $\odot$. Let d be a pseudometric on R and suppose that

$$\text{the operations } \oplus \text{ and } \odot \text{ are continuous with respect to } d. \tag{4.2}$$

It gives the pseudometrics in R^m and $\mathcal{M}(R, m)$ because $\mathcal{M}(R, m)$ can be expressed as $R^{m\times m}$; i.e., d in R^m and $\mathcal{M}(R, m)$ is the sum of m and m^2 nonnegative numbers given by d in R, respectively. For simplicity, we will also denote these pseudometrics as d.

The vector $v \in R^m$ is called *nonzero* (or *nontrivial*) if $d\big(v, (e_0, \dots, e_0)^T\big) > 0$. We say that a nonzero vector $(r_1, \dots, r_m)^T$, where $r_1, \dots, r_m \in R$, is an e_1-*eigenvector* of $U \in \mathcal{M}(R, m)$ if

$$d\left(U \cdot \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}, \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}\right) = 0,$$

and that $V \in \mathcal{M}(R, m)$ is *regular for a nonzero vector* $(r_1, \dots, r_m)^T \in R^m$ if

$$d\left(V \cdot \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}, \begin{pmatrix} e_0 \\ \vdots \\ e_0 \end{pmatrix}\right) > 0. \tag{4.3}$$

Next, we set

$$\mathcal{I} := \begin{pmatrix} e_1 & e_0 & \dots & e_0 \\ e_0 & e_1 & \dots & e_0 \\ \vdots & \vdots & \ddots & \vdots \\ e_0 & e_0 & \dots & e_1 \end{pmatrix} \in \mathcal{M}(R, m).$$

If, for given $U \in \mathcal{M}(R,m)$ and $X \subseteq \mathcal{M}(R,m)$, there exists the unique matrix $V \in X$ (we put $V = W$ if $d(V,W) = 0$) for which

$$U \cdot V = V \cdot U = \mathcal{I},$$

then we define $U^{-1} := V$ and we say that V is *the inverse matrix* of U in X.

For any function $H : [a,b] \to X$ ($a \le 0 < b$, $a,b \in \mathbb{R}$) and $s \in \mathbb{R}$, we extend its domain of definition as follows

$$H(s) := \begin{cases} H(\sigma) \cdot (H(b))^l & \text{for } s \ge 0, \\ (H(a))^l \cdot H(\sigma) & \text{for } s < 0 \text{ if } a < 0, \end{cases} \tag{4.4}$$

where $s = lb + \sigma$ for $l \in \mathbb{N}_0$, $\sigma \in [0,b)$ or $s = la + \sigma$ for $l \in \mathbb{N}_0$, $\sigma \in (a,0]$. Hereafter, we will restrict coefficients A_k in (4.1) to be elements of an infinite set $X \subseteq \mathcal{M}(R,m)$ with the following properties:

$$\mathcal{I} \in X; \quad U, V \in X \implies U \cdot V \in X,\ U^{-1} \text{ exists in } X; \tag{4.5}$$

and either

there exists a continuous function $F_1 : [-1,1] \to X$ satisfying

$$F_1(0) = \mathcal{I}; \quad F_1(t) = F_1^{-1}(-t), \quad t \in [0,1]; \tag{4.6}$$

and the matrix $F_1(1)$ has no e_1-eigenvector

or

there exist continuous $F_2 : [-1,1] \to X$, $t_1, \dots, t_q \in (0,1]$, $\delta > 0$ such that

$$F_2(0) = \mathcal{I}; \quad F_2\Big(\sum_{i=1}^{p} s_i\Big) = \prod_{i=1}^{p} F_2(s_i), \quad s_1, \dots, s_p \in [-1,1]; \tag{4.7}$$

and, for any $v \in R^m$, one can find $j \in \{1, \dots, q\}$ for which v is not an e_1-eigenvector of $F_2(t)$, $t \in (\max\{0, t_j - \delta\}, \min\{t_j + \delta, 1\})$.

We mention that, for $U_1, \dots, U_p \in X$ ($p \in \mathbb{N}$), we define

$$\prod_{i=1}^{p} U_i := U_1 \cdot U_2 \cdots U_p, \quad \prod_{i=p}^{1} U_i := U_p \cdot U_{p-1} \cdots U_1.$$

For the above function H, we also use the conventional notation

$$(H(s))^0 := \mathcal{I}, \quad H^{-1}(s) := \big(H(s)\big)^{-1} \quad \text{for all considered } s \in \mathbb{R}.$$

We remark that, because of (4.2), the requirement for the existence of $\delta > 0$ (in (4.7)) can be dropped. Now we comment our assumptions on R and X: We note that R does not need to be commutative, and thus the set of all solutions of (4.1) is not generally a modulus over R with the scalar multiplication given by

$$r \begin{pmatrix} x_k^1 \\ \vdots \\ x_k^m \end{pmatrix} := \begin{pmatrix} r \odot x_k^1 \\ \vdots \\ r \odot x_k^m \end{pmatrix},$$

where $\{(x_k^1, \dots, x_k^m)^T\}$ is a solution of (4.1), $r \in R$, $k \in \mathbb{Z}$ ($k \in \mathbb{N}_0$).

We need a sequence $\{a_k\}_{k\in\mathbb{N}_0}$ of real numbers, which has special properties (mentioned in the below given Lemmas 4.1–4.4), to prove the main theorem of this paper. We define the sequence $\{a_k\}_{k\in\mathbb{N}_0}$ by the recurrent formula

$$a_0 := 1, \quad a_1 := 0, \quad a_{2^n+k} := a_k - \frac{1}{2^n}, \quad k = 0, \dots, 2^n - 1,\ n \in \mathbb{N}. \tag{4.8}$$

For this sequence, we have the following results:

Lemma 4.1. *The sequence $\{a_k\}$ is almost periodic.*

The above lemma follows from Theorem 3.1 where we set $\varphi_k = a_k$ $(k \in \mathbb{N})$ and

$$\mathcal{X} = \mathbb{R}, \quad m = 0, \quad j = 1, \quad \varphi_0 = 1, \quad r_n = \frac{4}{2^n},\ n \in \mathbb{N}.$$

Lemma 4.2. *The following holds*

$$a_{2^{n+2}-1-i} = -a_{2^{n+1}+i} \tag{4.9}$$

for any $n \in \mathbb{N}_0$ and $i \in \{0, \dots, 2^n - 1\}$; i.e., $i \in \{0, \dots, 2^{n+1} - 1\}$.

Before proving this statement, observe that (4.9) is equivalent to

$$\sum_{k=2^{n+1}+i}^{2^{n+2}-1-i} a_k = 0, \quad n \in \mathbb{N}_0,\ i \in \{0, \dots, 2^n - 1\};$$

i.e., to

$$\sum_{k=0}^{2^{n+1}-1+i} a_k = \sum_{k=0}^{2^{n+2}-1-i} a_k, \quad n \in \mathbb{N}_0,\ i \in \{0, \dots, 2^n - 1\}.$$

Proof of Lemma 4.2. Obviously, (4.9) is true for $n \in \{0, 1\}$ because

$$a_2 = -a_3 = \frac{1}{2}, \quad a_4 = -a_7 = \frac{3}{4}, \quad a_5 = -a_6 = -\frac{1}{4};$$

i.e.,

$$\sum_{k=0}^{1} a_k = \sum_{k=0}^{3} a_k = \sum_{k=0}^{7} a_k = 1, \quad \sum_{k=0}^{4} a_k = \sum_{k=0}^{6} a_k = \frac{7}{4}.$$

Suppose that (4.9) is true also for $2, \dots, n-1$. We choose $i \in \{0, \dots, 2^n - 1\}$ arbitrarily. (We have $2^{n+2} - 1 - i \geq 2^{n+1} + 2^n$.) From (4.8) and the induction hypothesis it follows

$$a_{2^{n+2}-1-i} + a_{2^{n+1}+2^n+i} = -\frac{1}{2^n}, \quad a_{2^{n+1}+i} - a_{2^{n+1}+2^n+i} = \frac{1}{2^n}.$$

Summing the above equalities, we get (4.9). $\square$

Lemma 4.3. *We have*

$$\sum_{k=0}^{n} a_k \geq 1, \quad n \in \mathbb{N}_0. \tag{4.10}$$

Proof. Evidently, $a_0 = a_0 + a_1 = 1$. It means that (4.10) is true for $n = 0$ and $n = 1 = 2^1 - 1$. Let it be valid for arbitrarily given $2^p - 1$ and all $n < 2^p - 1$, i.e., let

$$\sum_{k=0}^{n} a_k \geq 1, \quad n \leq 2^p - 1,\ n \in \mathbb{N}_0.$$

Considering the definition of $\{a_k\}$, we obtain

$$\sum_{k=0}^{2^p+j-1} a_k = \sum_{k=0}^{2^p-1} a_k + \sum_{k=2^p}^{2^p+j-1} a_k \geq 1 + \sum_{k=0}^{j-1} a_k - j\frac{1}{2^p} \geq 1+1-1 = 1$$

for any $j \in \{1, \dots, 2^p\}$. Lemma 4.3 now follows by the induction. $\square$

Lemma 4.4. *We have*

$$\sum_{k=0}^{2^n-1} a_k = 1, \tag{4.11}$$

$$\sum_{k=0}^{2^{n+i}+2^n-1} a_k = 2 - \frac{1}{2^i}, \tag{4.12}$$

where $n \in \mathbb{N}_0$*,* $i \in \mathbb{N}$*.*

Proof. It is possible to prove this result by means of Lemma 4.2, but we prove it directly using (4.8) and the induction principle. We have

$$a_0 = 1, \quad a_0 + a_1 = 1, \quad a_0 + a_1 + a_2 + a_3 = 1.$$

If we assume that

$$\sum_{k=0}^{2^{n-1}-1} a_k = 1,$$

then we get (see (4.8))

$$\begin{aligned}
\sum_{k=0}^{2^n-1} a_k &= \sum_{k=0}^{2^{n-1}-1} a_k + \sum_{k=2^{n-1}}^{2^n-1} a_k \\
&= \sum_{k=0}^{2^{n-1}-1} a_k + \sum_{k=0}^{2^{n-1}-1} \Big(a_k - \frac{1}{2^{n-1}}\Big) \\
&= 2\sum_{k=0}^{2^{n-1}-1} a_k - 1 = 1.
\end{aligned}$$

Therefore, (4.11) is proved. Analogously, applying (4.8) and (4.11), one can compute

$$\begin{aligned}
\sum_{k=0}^{2^{n+i}+2^n-1} a_k &= \sum_{k=0}^{2^{n+i}-1} a_k + \sum_{k=2^{n+i}}^{2^{n+i}+2^n-1} a_k \\
&= 1 + \sum_{k=0}^{2^n-1} \Big(a_k - \frac{1}{2^{n+i}}\Big) \\
&= 1 + \big(1 - \frac{1}{2^i}\big)
\end{aligned}$$

what gives (4.12). $\square$

Applying the matrix valued functions F_1, F_2, we obtain the next lemma.

Lemma 4.5. *For each* $j \in \{1,2\}$*, any* $n \in \mathbb{N}_0$*, and* $i \in \{0,\dots,2^n-1\}$*, it holds*

$$F_j(a_{2^{n+2}-1-i}) = F_j^{-1}(a_{2^{n+1}+i})$$

and, consequently,

$$\prod_{k=2^{n+1}+i}^{2^{n+2}-1-i} F_j(a_k) = \prod_{k=2^{n+2}-1-i}^{2^{n+1}+i} F_j(a_k) = \mathcal{I}.$$

Proof. Clearly, this is a corollary of Lemma 4.2. Consider (4.6), (4.7), and the fact that the multiplication of matrices (in $\mathcal{M}(R,m)$) is associative. □

Immediately, from Lemma 4.4 (see (4.7)), we have the following formulas for the function F_2:

Lemma 4.6. *The equalities*

$$\prod_{k=0}^{2^n-1} F_2(a_k) = F_2(1), \qquad \prod_{k=0}^{2^{n+i}+2^n-1} F_2(a_k) = F_2\big(2-\frac{1}{2^i}\big)$$

hold for all $n \in \mathbb{N}_0$ *and* $i \in \mathbb{N}$*.*

Now we can prove the main statement of our paper.

Theorem 4.7. *There exists a system of the form* (4.1) *that does not possess a nonzero almost periodic solution.*

Proof. First we suppose that the coefficients A_k belong to X such that there exists a function F_1 from (4.6). Using Theorem 2.4, we get the almost periodicity of the sequence $\{F_1 \circ a_k\}_{k\in\mathbb{N}_0}$, where $\{a_k\}$ is given by (4.8). We want to show that all nonzero solutions of the system $\mathfrak{S}_1 \in \mathfrak{X}$ determined by $\{F_1 \circ a_k\}$ are not almost periodic.

By contradiction, suppose that there exist $c_1,\dots,c_m \in R$ such that the vector valued sequence

$$\{f_k\} := \Big\{P_k \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}\Big\}, \quad k \in \mathbb{N}_0, \tag{4.13}$$

where $\{P_k\}_{k\in\mathbb{N}_0}$ is the principal fundamental matrix of $\mathfrak{S}_1$, is nontrivial and almost periodic; i.e., suppose that $\mathfrak{S}_1$ has a nontrivial almost periodic solution $\{f_k\}$. Since $\{f_k\}$ is almost periodic, $(c_1,\dots,c_m)^T$ is nonzero, and, because of $a_0 = 1$, it is valid

$$f_i = U_i \cdot F_1(1) \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \quad \text{for any } i \in \mathbb{N} \text{ and some } U_i \in X, \tag{4.14}$$

we know that (see (4.3))

$$F_1(1) \text{ is regular for } c := (c_1,\dots,c_m)^T. \tag{4.15}$$

Considering (4.8), the uniform continuity of F_1 and the continuity of the multiplication of matrices (see (4.2)), (iii), Lemma 4.5, and (4.13), from the first part of Theorem 3.3 (see the proof of Corollary 3.4 and again Lemma 4.5), one can obtain that the sequence $\{g_k\}_{k\in\mathbb{Z}}$, where

$$g_k := f_k, \quad k \in \mathbb{N}_0, \quad g_k := f_{-k}, \quad k \in \mathbb{Z}\setminus\mathbb{N}_0, \tag{4.16}$$

is almost periodic too. Now we use Theorem 2.3 for $\{\varphi_k\} \equiv \{g_k\}$ and $\{h_n\}_{n\in\mathbb{N}} \equiv \{2^n\}_{n\in\mathbb{N}}$ (we can also consider directly $\{\varphi_k\} \equiv \{f_k\}$ and use Remark 2.2). This theorem implies that, for any $\varepsilon > 0$, there exists an infinite set $N(\varepsilon) \subseteq \mathbb{N}$ such that the inequality

$$d(g_{k+2^{n_1}}, g_{k+2^{n_2}}) < \varepsilon, \quad k \in \mathbb{Z} \tag{4.17}$$

holds for all $n_1, n_2 \in N(\varepsilon)$.

Using (4.6) (twice), we get $d(c, F_1(1)\cdot c) > 0$, and consequently (consider (4.15))

$$\vartheta := d(F_1(1)\cdot c, F_1(1)\cdot F_1(1)\cdot c) > 0. \tag{4.18}$$

From Lemma 4.5 (for $i = 0$), (4.13), and (4.16) (see also (4.14)), we have

$$g_0 = c, \quad g_1 = F_1(1)\cdot c, \quad \dots, \quad g_{2^n} = F_1(1)\cdot c, \tag{4.19}$$

where $n \in \mathbb{N}$ is arbitrary, and hence, considering (4.8), it holds

$$d(g_{2^i+2^n}, F_1(1)\cdot F_1(1)\cdot c) \to 0 \quad \text{as } n \to \infty \tag{4.20}$$

and for every $i \in \mathbb{N}$ because F_1 is uniformly continuous and the multiplication of matrices is continuous. We also have

$$d(g_{2^{n_2}+2^{n_1}}, F_1(1)\cdot c) < \frac{\vartheta}{2} \tag{4.21}$$

for all $n_1, n_2 \in N(\frac{\vartheta}{2})$. Indeed, put $k = 2^{n_2}$ in (4.17) and consider (4.19) for $n = n_2 + 1$. If we choose $n_1 \in N(\frac{\vartheta}{2})$ and put $i = n_1$ in (4.20), then there exists $n_0 \in \mathbb{N}$ such that, for any $n \ge n_0$, it holds

$$d(g_{2^{n_1}+2^n}, F_1(1)\cdot F_1(1)\cdot c) < \frac{\vartheta}{2}.$$

Thus, for arbitrarily given $n_2 \ge n_0$, $n_2 \in N(\frac{\vartheta}{2})$, we get

$$d(g_{2^{n_2}+2^{n_1}}, F_1(1)\cdot F_1(1)\cdot c) < \frac{\vartheta}{2}. \tag{4.22}$$

Finally, applying (4.18), (iii), (4.21), and (4.22), we have

$$\vartheta \le d(F_1(1)\cdot c, g_{2^{n_2}+2^{n_1}}) + d(g_{2^{n_2}+2^{n_1}}, F_1(1)\cdot F_1(1)\cdot c) < \vartheta.$$

This contradiction gives the proof when we consider (4.6) for $k \in \mathbb{N}_0$.

Let $k \in \mathbb{Z}$. Then, we can consider the system $\widetilde{\mathfrak{S}}_1$ determined by the sequence

$$B_k := F_1(a_k), \quad k \in \mathbb{N}_0, \quad B_k := F_1(-a_{-k-1}), \quad k \in \mathbb{Z} \setminus \mathbb{N}_0. \tag{4.23}$$

Since the sequence $\{|\, a_k\,|\}_{k\in\mathbb{N}_0}$ is almost periodic (see Theorem 2.4) and has the form of $\{\varphi_k\}_{k\in\mathbb{N}_0}$ from Theorem 3.1 and since it is valid (see (4.9))

$$|\, a_{2^{n+2}-1-i}\,| = |\, a_{2^{n+1}+i}\,|, \quad n \in \mathbb{N}_0,\ i \in \{0, \dots, 2^n - 1\},$$

the fact that $\{B_k\}$ is almost periodic follows from the second part of Corollary 3.4, from (c) (mentioned in Section 2), and Theorem 2.4. Next, the process is the same as for $k \in \mathbb{N}_0$. Let $\{P_k\}_{k\in\mathbb{Z}}$ be the principal fundamental matrix of $\widetilde{\mathfrak{S}}_1$ and $g_k := f_k$, $k \in \mathbb{Z}$. Also now we have (4.17), and consequently we get the same contradiction.

Let the coefficients A_k belong to X such that there exists a function F_2 from (4.7). Consider the numbers $t_1, \dots, t_q \in (0, 1]$ and $\delta > 0$ from (4.7). Without loss of the generality, we can assume

$$\delta < t_1 < \dots < t_q \quad \text{and} \quad t_q < 1 - \delta. \tag{4.24}$$

Indeed, if $t_j = 1$, then we can put $t_j := 1 - \frac{\delta}{2}$ and redefine δ. We repeat that any vector $v \in R^m$ determines some $j \in \{1, \dots, q\}$ (see again (4.7)) such that v is not an e_1-eigenvector of $F_2(t)$ for $t \in (t_j - \delta, t_j + \delta)$.

From Theorem 2.4 it follows that the sequence $\{F_2 \circ a_k\}_{k \in \mathbb{N}_0}$ is almost periodic. Thus, it determines the system of the form (4.1). We will denote it as $\mathfrak{S}_2$. Suppose that $\mathfrak{S}_2$ has a nontrivial almost periodic solution $\{x_k\}_{k \in \mathbb{N}_0}$. For the principal fundamental matrix $\{P_k\}$ of the system $\mathfrak{S}_2$, we have

$$x_k = P_k \cdot x_0, \quad k \in \mathbb{N}_0,$$

where the vector x_0 is nonzero. Using this fact and taking into account Lemma 4.3 and (4.7), we obtain

$$x_n = F_2(t) \cdot F_2^i(1) \cdot x_0 \quad \text{for some } i \in \mathbb{N},\ t \in [0, 1), \tag{4.25}$$

and for arbitrary $n \in \mathbb{N}$. From Lemma 4.6, we also get

$$x_{2^n} = F_2(1) \cdot x_0 \quad \text{for all } n \in \mathbb{N}_0 \tag{4.26}$$

and

$$x_{2^{n+i}+2^n} = F_2\Big(1 - \frac{1}{2^i}\Big) \cdot F_2(1) \cdot x_0 \quad \text{for all } n \in \mathbb{N}_0,\ i \in \mathbb{N}. \tag{4.27}$$

Analogously as for $\{f_k\}$, one can extend $\{x_k\}_{k \in \mathbb{N}_0}$ by the formula

$$x_k := x_{-k}, \quad k \in \mathbb{Z} \setminus \mathbb{N}_0$$

for all $k \in \mathbb{Z}$ so that the sequence $\{x_k\}_{k \in \mathbb{Z}}$ is almost periodic too. Now we apply Theorem 2.3 for the sequences $\{x_k\}_{k \in \mathbb{Z}}$ and $\{2^n\}_{n \in \mathbb{N}}$. For any $\varepsilon > 0$, there exists an infinite set $M(\varepsilon) \subseteq \mathbb{N}$ such that, for any $n_1, n_2 \in M(\varepsilon)$, we have

$$d(x_{k+2^{n_1}}, x_{k+2^{n_2}}) < \varepsilon, \quad k \in \mathbb{Z}. \tag{4.28}$$

Since F_2 is uniformly continuous and the multiplication of matrices is continuous, for arbitrary $i \in \mathbb{N}_0$ and $\varepsilon > 0$, we have from (4.8) and (4.26) that

$$d(x_{2^i+2^n}, F_2(1) \cdot F_2(1) \cdot x_0) < \varepsilon \quad \text{for sufficiently large } n \in \mathbb{N}. \tag{4.29}$$

Because of the almost periodicity of $\{x_k\}$ and (4.25), the matrix $F_2(1)$ has to be regular for x_0. Let $\varepsilon > 0$ be arbitrarily small and $n_1 \in M(\varepsilon)$ arbitrarily large. From (4.28) and (4.29), where we choose $k = 2^{n_1 - j}$ and $i = n_1 - j$ for $j \in \{0, \dots, n_1\}$, it follows that, for given n_1, there exists sufficiently large $n_2 \in M(\varepsilon)$ for which

$$\begin{aligned} &d(x_{2^{n_1-j}+2^{n_1}}, F_2(1) \cdot F_2(1) \cdot x_0) \\ &\le d(x_{2^{n_1-j}+2^{n_1}}, x_{2^{n_1-j}+2^{n_2}}) + d(x_{2^{n_1-j}+2^{n_2}}, F_2(1) \cdot F_2(1) \cdot x_0) < 2\varepsilon. \end{aligned} \tag{4.30}$$

Since ε (in (4.30)) is arbitrarily small and, choosing $j = 0$, we have

$$d(x_{2^{n_1+1}}, F_2(1) \cdot F_2(1) \cdot x_0) < 2\varepsilon,$$

we know (see (4.26)) that $F_2(1) \cdot x_0$ is an e_1-eigenvector of $F_2(1)$, i.e., we get

$$d(F_2(1) \cdot x_0, F_2(1) \cdot F_2(1) \cdot x_0) = 0. \tag{4.31}$$

If we choose $j = 1$, then we obtain (consider (4.27))

$$d(F_2(\frac{1}{2}) \cdot F_2(1) \cdot x_0, F_2(1) \cdot F_2(1) \cdot x_0) = 0.$$

Analogously, for any j (the number n_1 is arbitrarily large), we get

$$d(F_2\big(1 - \frac{1}{2^j}\big) \cdot F_2(1) \cdot x_0, F_2(1) \cdot F_2(1) \cdot x_0) = 0.$$

Thus,

$$d(F_2\big(2-\frac{1}{2^j}\big)\cdot x_0, F_2(1)\cdot F_2(1)\cdot x_0) = 0, \quad j \in \mathbb{N}. \tag{4.32}$$

Hence, we have

$$d(F_2\big(2-\frac{1}{2^j}\big)\cdot x_0, F_2\Big(2-\frac{1}{2^{j-1}}\Big)\cdot x_0) = 0, \quad j \in \mathbb{N}. \tag{4.33}$$

Because of

$$F_2\big(2-\frac{1}{2^j}\big) = F_2\big(\frac{1}{2^j}+2-\frac{1}{2^{j-1}}\big) = F_2\big(\frac{1}{2^j}\big)\cdot F_2\big(2-\frac{1}{2^{j-1}}\big)$$

and (see (4.31) and (4.32))

$$d\Big(F_2\big(2-\frac{1}{2^{j-1}}\big)\cdot x_0, F_2(1)\cdot x_0\Big) = 0,$$

from (4.33) it follows

$$d\Big(F_2\big(\frac{1}{2^j}\big)\cdot F_2(1)\cdot x_0, F_2(1)\cdot x_0\Big) = 0 \quad \text{for all } j \in \mathbb{N},$$

i.e., $F_2(1)\cdot x_0$ is an e_1-eigenvector of $F_2(2^{-j})$ for all $j \in \mathbb{N}$.

Since any number $t \in [0,1]$ can be expressed in the form

$$\sum_{i=1}^{\infty} \frac{a_i}{2^i}, \quad \text{where } a_i \in \{0,1\},$$

for considered $\delta > 0$, there exists $n \in \mathbb{N}$ such that, for every $t \in [0,1]$, there exist $a_1,\dots,a_n \in \{0,1\}$ satisfying

$$\big|t-\sum_{i=1}^{n}\frac{a_i}{2^i}\big| < \delta.$$

Thus, $F_2(1)\cdot x_0$ is an e_1-eigenvector of $F_2(t_j+s_j)$ for some $s_j \in (-\delta,\delta)$ and any $j \in \{1,\dots,q\}$ which cannot be true. This contradiction shows that $\{x_k\}_{k\in\mathbb{N}_0}$ is not almost periodic.

If one considers the system $\widetilde{\mathfrak{S}}_2$ obtained from $\mathfrak{S}_2$ as in (4.23) (after replacing $\mathfrak{S}_1$ by $\mathfrak{S}_2$), then, analogously as for F_1 and $k \in \mathbb{N}_0$, one can prove that $\widetilde{\mathfrak{S}}_2 \in \mathfrak{X}$ and that any its nontrivial solution $\{x_k\}_{k\in\mathbb{Z}}$ is not almost periodic. □

Remark 4.8. Let a nonzero $F_1(1)\cdot v \in R^m$ not be an e_1-eigenvector of the matrix $F_1(1)$ from (4.6); i.e., the condition (4.6) be weakened in this way. Then, from the first part of the proof of Theorem 4.7, we obtain that the sequence $\{f_k\}$, given by (4.13), is not almost periodic for $(c_1,\dots,c_m)^T = v$. It means that there exists a system $\mathfrak{S}^1 \in \mathfrak{X}$ with the principal fundamental matrix $\{P_k^1\}$ such that the sequence $\{P_k^1\cdot v\}_{k\in\mathbb{N}_0}$ or $\{P_k^1\cdot v\}_{k\in\mathbb{Z}}$ is not almost periodic. Analogously, it is seen: If one requires in (4.7) only that, for a nonzero vector $v \in R^m$, there exists $t \in (0,1]$ for which $F_2(1)\cdot v$ is not an e_1-eigenvector of $F_2(t)$, then there exists a system $\mathfrak{S}^2 \in \mathfrak{X}$ satisfying that the sequence $\{P_k^2\cdot v\}_{k\in\mathbb{Z}}$ (or $\{P_k^2\cdot v\}_{k\in\mathbb{N}_0}$), where $\{P_k^2\}_{k\in\mathbb{Z}}$ (or $\{P_k^2\}_{k\in\mathbb{N}_0}$) is the principal fundamental matrix of $\mathfrak{S}^2$, is not almost periodic.

The condition

$$F_2\Big(\sum_{i=1}^{p} s_i\Big) = \prod_{i=1}^{p} F_2(s_i), \quad s_1,\dots,s_p \in [-1,1],\ p \in \mathbb{N} \tag{4.34}$$

in (4.7) is "strong". For example, from it follows that the multiplication of matrices from the set $\{F_2(t);\, t \in \mathbb{R}\}$ is commutative. At the same time, we say that, for many subsets of unitary or orthogonal matrices, it is not a limitation and that the method in the proof of Theorem 4.7 can be simplified in many cases. We will show it in two important special cases.

Example 4.9. If, for any nontrivial vector $v \in R^m$, there exists $\varepsilon(v) > 0$ with the property that

$$F_2(t) \cdot v \notin \mathcal{O}_{\varepsilon(v)}(v) \quad \text{for all } t \geq 1 \text{ (see (4.4))},$$

then the fact, that the systems $\mathfrak{S}_2$ and $\widetilde{\mathfrak{S}}_2$ from the proof of Theorem 4.7 do not have nontrivial almost periodic solutions, follows directly from Lemma 4.3 and (4.34). Indeed, the set $\mathfrak{T}(\{x_k\}, \varepsilon(x_0)) \setminus \{0\}$ is empty for any nonzero solution $\{x_k\}$.

Example 4.10. Let the function F_2, in addition to (4.7), satisfy

$$F_2(s) = F_2(0) = \mathcal{I} \tag{4.35}$$

for some positive irrational number s, (4.24) hold, and $p \in \mathbb{N}$ be arbitrary. Then, the system $\mathfrak{S}$ determined by the sequence

$$\{A_k\} := \{F_2(k/p)\},$$

where $k \in \mathbb{N}_0$ or $k \in \mathbb{Z}$, has no nontrivial almost periodic solutions.

The function $F_2(t/p)$, $t \in \mathbb{R}$ is continuous and periodic with a period ps (see (4.34), (4.35)). Using the compactness of the interval $[0, ps]$, (4.34), and Theorem 2.3, we get that $\{F_2(k/p)\}_{k\in\mathbb{Z}}$ is almost periodic. The almost periodicity of $\{F_2(k/p)\}_{k\in\mathbb{N}_0}$ is now obvious.

Suppose, by contradiction, that $\{x_k\} \equiv \{P_k \cdot x_0\}$ is a nontrivial almost periodic solution of $\mathfrak{S}$. We mention that there exists $\delta > 0$ satisfying that, for any nonzero $v \in R^m$, one can find $j \in \mathbb{N}$ such that there exists a positive number $\vartheta(v)$ for which

$$\vartheta(v) \leq d(F_2(\frac{j}{p} + t) \cdot v, v), \quad t \in (-\delta, \delta), \tag{4.36}$$

because

$$\{F_2(k/p); k \in \mathbb{N}\} \text{ is dense in } \{F_2(t); t \in \mathbb{R}\} \tag{4.37}$$

which is proved (for a continuous periodic function F_2 satisfying (4.34) with the smallest period $s > 0$ that is an irrational number) in detail, e.g., in [25, pp. 44–46]. Evidently, (4.37) gives that

$$\{F_2(k/p); k \in N\} \text{ is dense in } \{F_2(t); t \in \mathbb{R}\} \tag{4.38}$$

for any set N what is relative dense in $\mathbb{N}$.

Because the multiplication of matrices is continuous, there exists $\varepsilon > 0$ which satisfies that every vector u with the property $d(u, x_0) < \varepsilon$ determines the same j in (4.36) as x_0 and one can find

$$\vartheta(u) \geq \frac{\vartheta(x_0)}{2}. \tag{4.39}$$

From (4.34), we see that

$$x_k = F_2\Big(\sum_{i=0}^{k-1} \frac{i}{p}\Big) \cdot x_0, \quad k \in \mathbb{N}. \tag{4.40}$$

Let l be an arbitrary positive $\frac{\vartheta(x_0)}{2}$-translation number of $\{x_k\}$, thus, let

$$d(x_{k+l}, x_k) < \frac{\vartheta(x_0)}{2} \quad \text{for all } k \in \mathbb{N}, \tag{4.41}$$

and let N be the set of all positive ε-translation numbers of $\{x_k\}$. Since

$$\sum_{i=0}^{k+l-1} \frac{i}{p} = \sum_{i=0}^{k-1} \frac{i}{p} + \frac{kl}{p} + \frac{l(l-1)}{2p}, \quad k \in \mathbb{N},$$

for all $k \in N$, we have (see again (4.34))

$$d(x_{k+l}, x_k) = d\Big(F_2\Big(\frac{kl}{p} + \frac{l(l-1)}{2p}\Big) \cdot F_2\Big(\sum_{i=0}^{k-1} \frac{i}{p}\Big) \cdot x_0, F_2\Big(\sum_{i=0}^{k-1} \frac{i}{p}\Big) \cdot x_0\Big). \tag{4.42}$$

From (4.38), if we replace $\frac{1}{p}$ by $\frac{l}{p}$, we get the choice of $k \in N$ such that

$$\Big|\frac{j}{p} - \frac{kl}{p} - \frac{l(l-1)}{2p}\Big| < \delta \pmod s \tag{4.43}$$

for j in (4.36) determined by x_0. From (4.36), (4.39) (consider the definition of ε), (4.40), (4.42), and (4.43), we have

$$d(x_{k+l}, x_k) \geq \frac{\vartheta(x_0)}{2}$$

for at least one $k \in \mathbb{N}$. But, at the same time, we have (4.41). This contradiction gives that $\{x_k\}$ cannot be almost periodic. See also the proof of the first part of [21, Proposition 2, p. 593], where almost periodic unitary systems are studied.

At the end, we remark that the last considered system $\mathfrak{S}$ (in Example 4.10) has no physical interpretations in any technical applications if we consider directly the sequence $\{k/p\}$; in contrast to $\mathfrak{S}_2$ and $\widetilde{\mathfrak{S}}_2$ (the sequence $\{a_k\}$). In applications, the following can be utilized: Let $\{u_k\}_{k\in\mathbb{Z}}$ (or $\{u_k\}_{k\in\mathbb{N}_0}$) be a sequence of arbitrary values and let the below considered function φ be defined on the set $\{u_k; k \in \mathbb{Z}\}$ or $\{u_k; k \in \mathbb{N}_0\}$. If we extend the definition of the discrete almost periodicity so that φ is *almost periodic* if, for every $\varepsilon > 0$, one can find $p(\varepsilon) \in \mathbb{N}_0$ with the property that any set, in the form $\{k_0, \dots, k_0 + p(\varepsilon)\}$, $k_0 \in \mathbb{Z}$ (or $k_0 \in \mathbb{N}_0$), contains a number l satisfying the inequality

$$d(\varphi(u_{k+l}), \varphi(u_k)) < \varepsilon$$

for all $k \in \mathbb{Z}$ (or $k \in \mathbb{N}_0$), then all results (mentioned in this article) about almost periodic sequences are still valid.

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