Electronic Journal of Differential Equations, Vol. 2008(2008), No. 13, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

GLOBAL STRUCTURE OF POSITIVE SOLUTIONS FOR SUPERLINEAR SINGULAR m-POINT BOUNDARY-VALUE PROBLEMS

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Abstract. Using topological methods and a well known generalization of the Birkhoff-Kellogg theorem, we study the global structure of a class of superlinear singular m-point boundary value problem.

1. Introduction

We are concerned with the nonlinear second-order singular m-point boundary-value problem

$$\begin{gathered} -(L\varphi)(x) = \lambda f(x, \varphi(x)), \quad 0 < x < 1, \\ \varphi(0) = 0, \quad \varphi(1) = \sum_{i=1}^{m-2} a_i \varphi(\xi_i), \end{gathered} \tag{1.1}$$

where

$$(L\varphi)(x) = (p(x)\varphi'(x))' + q(x)\varphi(x),$$

$\xi_i \in (0,1)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_i \in [0, +\infty)$, $f \in [C(0,1) \times (0, +\infty), R^+]$, $\lambda \in R^+ = [0, +\infty)$, $f(x, u)$ may be singular not only at $x = 0, x = 1$ but also at $u = 0$.

The existence of solutions for nonlinear singular multi-point boundary value problems has been studied extensively in the literature (see [4, 6, 7] and references therein). However, up to now, there are few papers consider the global structure of solutions for singular m-point boundary-value problem. In this paper, we use the topological method and the generalization of the well known Birkhoff-Kellogg theorem to get the global structure of the closure of positive solution set of (1.1) (denoted by $\overline{L}$) when $f(x, \varphi)$ satisfying superlinear condition at ∞ where

$$L := \{(\lambda, \varphi) \in (0, +\infty) \times P \setminus \{\theta\} : (\lambda, \varphi) \text{ satisfying (1.1)}\}. \tag{1.2}$$

Under some suplinear conditions, we get that $\overline{L}$ possesses a maximal and unbounded subcontinuum C (i.e., a maximal closed connected subsets of solution) which comes from $(0, \theta)$ and tends to $(0, +\infty)$ eventually.

2000 *Mathematics Subject Classification.* 34B10,34B16.
Key words and phrases. Superlinear; singular; m-point boundary value problem; global structure.

Submitted May 31, 2007. Published January 31, 2008.
Supported by grant 10671167 from the National Natural Science Foundation of China and by grant 31805 from Science Foundation of Liaocheng University.

The basic space used in this paper is $E = R \times C[I, R]$. As is known, $C[I, R]$ is a Banach space with the norm $\|\varphi\| = \max_{x\in I} |\varphi(x)|$ for $\varphi \in C[I, R]$. Furthermore, E is also a Banach space if we endowed a norm $\|(\lambda, \varphi)\| = \max\{|\lambda|, \|\varphi\|\}$ for $(\lambda, \varphi) \in E$. (λ, φ) is called a solution of (1.1), if $\lambda > 0$, $\varphi \in C[I, R] \cap C^2[(0, 1), R]$ satisfying (1.1), where $I = [0, 1]$. In addition, if $\lambda > 0$, $\varphi(x) > 0$ holds for any $x \in (0, 1)$, then (λ, φ) is called a positive solution of (1.1).

The rest of this paper is organized as follows. Section 2 gives some necessary lemmas. Section 3 is devoted to the main result and its proof. An example is worked out in Section 4 to indicate the application of our main result.

2. Preliminary Lemmas

Throughout this paper, we always suppose

(H1) $p(x) \in C^1[0, 1]$, $p(x) > 0$, $q(x) \in C[0, 1]$, $q(x) \le 0$.

Lemma 2.1 ([7]). *Assume that* (H1) *holds. Let $\phi_1(x), \phi_2(x)$ be the solution of*

$$\begin{gathered}(L\varphi)(x) = 0, \quad 0 < x < 1,\\ \varphi(0) = 0, \quad \varphi(1) = 1,\end{gathered} \tag{2.1}$$

and

$$\begin{gathered}(L\varphi)(x) = 0, \quad 0 < x < 1,\\ \varphi(0) = 1, \quad \varphi(1) = 0,\end{gathered} \tag{2.2}$$

respectively. Then

(i) *$\phi_1(x)$ is increasing on [0,1] and $\phi_1(x) > 0, x \in (0, 1]$;*
(ii) *$\phi_2(x)$ is decreasing on [0,1] and $\phi_2(x) > 0, x \in [0, 1)$.*

Let

$$k(x, y) = \begin{cases} \frac{1}{\rho}\phi_1(x)\phi_2(y), & 0 \le x \le y \le 1,\\ \frac{1}{\rho}\phi_1(y)\phi_2(x), & 0 \le y \le x \le 1,\end{cases} \tag{2.3}$$

where $\rho = \phi_1'(0)$. By Lemma 2.1 we know that $\phi_1' > 0$. Let

$$K(x, y) = k(x, y) + D^{-1}\phi_1(x)\sum_{i=1}^{m-2} a_i k(\xi_i, y), \quad 0 \le x, y \le 1 \tag{2.4}$$

where $D = 1 - \sum_{i=1}^{m-2} a_i\phi_1(\xi_i)$.

Lemma 2.2 ([7]). *Assume* (H1) *holds. Then $k(x, y)$ defined by* (2.3) *possesses the following properties:*

(i) *$k(x, y)$ is continuous and symmetrical over $[0, 1] \times [0, 1]$;*
(ii) *$k(x, y) \ge 0$, and $k(x, y) \le k(y, y)$, for all $0 \le x, y \le 1$;*
(iii) *There exist constants $k_1, k_2 > 0$ such that*

$$k_1x(1 - x) \le k(x, x) \le k_2x(1 - x), x \in [0, 1].$$

We make the following assumptions:

(H2) $\sum_{i=1}^{m-2} a_i\phi_1(\xi_i) < 1$, where $\phi_1(x)$ is the unique solution of (2.1).
(H3) $f : (0, 1) \times (0, +\infty) \to R^+$ is continuous (it may be singular at $x = 0, 1$ and $\varphi = 0$) and for any $R > r > 0$, $\int_0^1 K_1(y, y)f_{r,R}(y)\mathrm{d}y < +\infty$ where

$$K_1(y, y) = y(1 - y) + D^{-1}\sum_{i=1}^{m-2} a_i k(\xi_i, y);$$

$f_{r,R}(y) := \sup\{f(y,\varphi) : \varphi \in [\rho k_1 y(1-y)r, R], y \in (0,1)\}$, k_1 has the same meaning as in Lemma 2.2.

(H4) For every $R > 0$, there exists $\psi_R \in C[I, R^+]$ ($\psi_R \not\equiv \theta$) such that

$$f(x,\varphi) \geq \psi_R(x), \quad \text{for } x \in (0,1), \varphi \in (0,R].$$

(H5) There exists $[a,b] \subset (0,1)$ such that

$$\lim_{\varphi \to +\infty} \frac{f(x,\varphi)}{\varphi} = +\infty \quad \text{uniformly for } x \in [a,b].$$

Set

$$(A\varphi)(x) = \int_0^1 K(x,y)\widetilde{p}(y)f(x,\varphi(y))\mathrm{d}y, \quad x \in [0,1], \tag{2.5}$$

where

$$\widetilde{p}(y) = \frac{1}{p(y)} \exp\Big(\int_0^y \frac{p'(s)}{p(s)}\mathrm{d}s\Big). \tag{2.6}$$

Let

$$P = \{\varphi \in C[0,1] : \varphi(x) \geq 0, \varphi(x) \geq \|\varphi\|\rho k_1 x(1-x), \rho k_1 < 4, x \in [0,1]\}.$$

where k_1 has the same meaning as in Lemma 2.2. It is easy to check that P is a cone in $C[0,1]$.

The following theorem is the generalization of the well known Birkhoff-Kellogg.

Lemma 2.3 ([1, 5]). *Let X be an infinite-dimensional Banach space, P a cone of X, and $A : P \to P$ a completely continuous operator. Suppose that there exists a bounded open set Ω in $X, \theta \in \Omega$ such that*

$$\inf_{x \in P \cap \partial\Omega} \|Ax\| > 0.$$

Then the closure of the set of nonzero solutions of the equation $\varphi = \lambda A\varphi$, i.e.,

$$\Sigma := \overline{\{(\lambda,\varphi) : \lambda \in R_+, \varphi \in P, \varphi \neq \theta, \varphi = \lambda A\varphi\}}$$

possesses a maximal subcontinuum C (i.e., a maximal closed connected subsets of $\sum$), which is unbounded and there exists $\overline{\lambda} > 0$ (for example we may choose $\overline{\lambda} > \sup_{x \in P\cap\partial\Omega} \|x\| / \inf_{x \in P \cap \partial\Omega} \|Ax\|$) such that

(i) $C \cap ((0,+\infty) \times P \setminus ((\overline{\lambda},+\infty) \times \overline{\Omega}))$ *is unbounded;*
(ii) $C \cap ([\overline{\lambda},+\infty) \times \partial\Omega) = \emptyset$, $C \cap (\{0\} \times (P \setminus \{\theta\})) = \emptyset$*; and either*
(iii) $C \cap ([\overline{\lambda},+\infty) \times \Omega)$ *is unbounded, or*
(iii)* $C \cap ([0,+\infty) \times \{\theta\}) \neq \emptyset$,

where θ denotes zero element of X.

3. Main Result

First, we consider the following approximating problem of BVP (1.1)

$$\begin{gathered} -(L\varphi)(x) = \lambda f_n(x,\varphi(x)), \quad 0 < x < 1, \\ \varphi(0) = 0, \quad \varphi(1) = \sum_{i=1}^{m-2} a_i \varphi(\xi_i), \end{gathered} \tag{3.1}$$

where $f_n(x,\varphi(x)) = f(x, \max\{\frac{1}{n}, \varphi(x)\})$. Obviously, $f_n(x,\varphi(x))$ only has the singularity at $x = 0, 1$ and has no singularity at $\varphi = 0$ any more. Define an operator A_n on the cone P by

$$(A_n\varphi)(x) = \int_0^1 K(x,y)\widetilde{p}(y)f_n(y,\varphi(y))\mathrm{d}y \quad \text{for any } \varphi \in P, \tag{3.2}$$

where $K(x,y)$ and $\widetilde{p}(y)$ are defined as in (2.4)(2.6) respectively. It follows from (H3) and the definition of $K(x,y)$ that A_n is well defined on P for each $n \in N$.

Lemma 3.1. *Assume* (H2), (H3) *hold. Then for each* $n \geq 1$, (3.1) *has a positive solution belonging to* $C^2[(0,1),R] \cap C[I,R]$ *if and only* λA_n *has a fixed point in* $P \setminus \{\theta\}$.

Proof. Sufficiency is obvious. Now we are in position to prove necessity.

Suppose $(\lambda,\varphi) = (\lambda,\varphi(x))$ is a positive solution of (3.1). Then, $\lambda > 0, \varphi \in C^2[(0,1),R^+] \cap C[I,R^+]$ and for any $x \in (0,1), \varphi(x) > 0$. It is obvious, $\varphi(x) = \lambda A_n\varphi(x)$. Take $x_0 \in [0,1]$ such that $\varphi(x_0) = \|\varphi\|$. From [7], for any $x, y \in [0,1]$ we have $k(x,y) \geq k(x_0,y)\phi_1(x)\phi_2(x)$. So, we have

$$\begin{aligned}
\varphi(x) = \lambda &\int_0^1 k(x,y)\widetilde{p}(y)f_n(y,\varphi(y))\mathrm{d}y \\
&+ \lambda D^{-1}\phi_1(x)\sum_{i=1}^{m-2} a_i \int_0^1 k(\xi_i,y)\widetilde{p}(y)f_n(y,\varphi(y))\mathrm{d}y \\
&\geq \lambda\phi_1(x)\phi_2(x)\int_0^1 k(x_0,y)\widetilde{p}(y)f_n(y,\varphi(y))\mathrm{d}y \\
&\quad + \lambda D^{-1}\phi_1(x)\sum_{i=1}^{m-2} a_i \int_0^1 k(\xi_i,y)\widetilde{p}(y)f_n(y,\varphi(y))\mathrm{d}y \\
&\geq \lambda\phi_1(x)\phi_2(x)\Big[\int_0^1 k(x_0,y)\widetilde{p}(y)f_n(y,\varphi(y))\mathrm{d}y \\
&\quad + D^{-1}\sum_{i=1}^{m-2} a_i \int_0^1 k(\xi_i,y)\widetilde{p}(y)f_n(y,\varphi(y))\mathrm{d}y\Big] \\
&\geq \lambda\phi_1(x)\phi_2(x)\Big[\int_0^1 k(x_0,y)\widetilde{p}(y)f_n(y,\varphi(y))\mathrm{d}y \\
&\quad + D^{-1}\phi_1(x_0)\sum_{i=1}^{m-2} a_i \int_0^1 k(\xi_i,y)\widetilde{p}(y)f_n(y,\varphi(y))\mathrm{d}y\Big] \\
&= \varphi(x_0)\phi_1(x)\phi_2(x) = \|\varphi\|\rho k_1 x(1-x)
\end{aligned}$$

As a consequence, $\varphi \in P \setminus \{\theta\}$. □

Lemma 3.2. *Assume* (H1)–(H3) *hold. Then* $A_n : P \to P$ *is continuous for each* $n \in N$.

The proof of the above lemma is obvious, so we omit it. Let

$$L_n := \{(\lambda,\varphi) \in R^+ \times P : \varphi = \lambda A_n\varphi\} \text{ for all } n \geq 1.$$

Lemma 3.3. *Suppose* (H1)–(H4) *hold. Then for each* n, L_n *is locally compact in* $[0,+\infty)\times P$ *and*

$$L_n = \overline{\{(\lambda,\varphi)\in R^+\times P : \varphi=\lambda A_n\varphi, \varphi\neq\theta\}}.$$

Proof. For every $R>0$, let $L_n^R := \{(\lambda,\varphi)\in L_n : |\lambda|\le R, |\varphi|\le R\}$. If $(\lambda,\varphi)\in L_n$ and $\varphi=\theta$, then by (H4) we get $\lambda=0$. So, we need only to prove that L_n^R is relatively compact and closed.

In fact, for each $(\lambda,\varphi)\in L_n^R$, from the construction of P we have

$$f_n(x,\varphi(x))\le f_{\frac{1}{n},R}(x) \quad \text{for all } x\in(0,1),$$

and

$$\varphi(x)=\lambda\int_0^1 K(x,y)\widetilde{p}(y)f_n(y,\varphi(y))\mathrm{d}y, \quad x\in[0,1].$$

Combining with (H3), it is easy to know that $\{\varphi=\varphi(x) : (\lambda,\varphi)\in L_n^R\}$ are equicontinuous on I. Thus, from Ascoli-Arzela theorem we get that L_n^R is relatively compact. On the other hand, (H3) and Lebesgue dominated convergence theorem guarantee that L_n^R is closed. □

The next theorem gives the global structure of L_n.

Theorem 3.4. *Suppose that* (H1)–(H5) *hold. Then for each* $n\ge 1, L_n$ *possesses a maximal and unbounded subcontinuum* C_n*, which comes from* $(0,\theta)$ *and tends to* $(0,+\infty)$ *eventually satisfying*

(1) $(0,\theta)\in C_n$;
(2) *There exists* $\lambda_n^0\in(0,+\infty)$ *such that*

$$C_n\subset[0,\lambda_n^0]\times P, \quad C_n\cap(\{\lambda\}\times P)\neq\emptyset, \quad \forall\,\lambda\in[0,\lambda_n^0];$$

(3) C_n *is unbounded in* $[0,\lambda_n^0]\times P$;
(4) $\lambda=0$ *is an unique asymptotic bifurcation point of* A_n;
(5) *There exists* $\lambda_n^*\in(0,\lambda_n^0]$ *such that for each* $\lambda\in(0,\lambda_n^*)$, (3.1) *has at least two positive solution* $\varphi_{n\lambda}^*$ *and* $\varphi_{n\lambda}^{**}$ *satisfying*

$$\|\varphi_{n\lambda}^*\|\le\|\varphi_{n\lambda}^{**}\|, \quad (\lambda,\varphi_{n\lambda}^*),(\lambda,\varphi_{n\lambda}^{**})\in C_n;$$

(6)

$$\lim_{\lambda\to0^+,\,(\lambda,\varphi_{n\lambda}^*)\in C_n}\|\varphi_{n\lambda}^*\|=0, \quad \lim_{\lambda\to0^+,\,(\lambda,\varphi_{n\lambda}^{**})\in C_n}\|\varphi_{n\lambda}^{**}\|=+\infty.$$

Proof. First we prove that for every $\overline{\lambda}>0$, there exists $\overline{R}>0$ such that

$$L_n\cap([\overline{\lambda},+\infty)\times(P\setminus\overline{P}_{\overline{R}}))=\emptyset, \quad n=1,2,\dots, \tag{3.3}$$

where $P_{\overline{R}}=\{\varphi\in P : \|\varphi\|<\overline{R}\}$.

In fact, take a positive number l satisfying

$$l>\Big(\rho k_1\overline{\lambda}\max_{x\in I}\int_a^b K(x,y)\widetilde{p}(y)y(1-y)\mathrm{d}y\Big)^{-1}>0, \tag{3.4}$$

where a,b are as the same as in (H5). Then there exists $R'>1$ such that

$$f(x,u)\ge lu \quad \text{for all } x\in[a,b], u>R'. \tag{3.5}$$

Choose a number $\overline{R}$ with $\overline{R}>\frac{R'}{\rho k_1 a(1-b)}$. It follows from the definition of cone P that

$$\varphi(y)\ge\|\varphi\|\rho k_1 y(1-y)\ge\rho k_1 a(1-b)\overline{R}>R' \quad \text{for all } y\in[a,b], \varphi\in P\setminus\overline{P}_{\overline{R}}. \tag{3.6}$$

Therefore, by (3.5) and (3.6) for $\lambda \geq \overline{\lambda}$ and $\varphi \in P \setminus \overline{P}_{\overline{R}}$

$$\begin{aligned}\lambda A_n\varphi(x) &= \lambda \int_0^1 K(x,y)\widetilde{p}(y)f_n(y,\varphi(y))\mathrm{d}y \\ &\geq \overline{\lambda}\int_a^b K(x,y)\widetilde{p}(y)f(y,\varphi(y))\mathrm{d}y \\ &\geq l\overline{\lambda}\int_a^b K(x,y)\widetilde{p}(y)\varphi(y)\mathrm{d}y \\ &\geq \rho k_1 l\overline{\lambda}\|\varphi\| \int_a^b K(x,y)\widetilde{p}(y)y(1-y)\mathrm{d}y\end{aligned}$$

Combining with (3.4), we have

$$\|\lambda A_n\varphi\| \geq \rho k_1 l\overline{\lambda}\|\varphi\| \max_{x\in I}\int_a^b K(x,y)\widetilde{p}(y)y(1-y)\mathrm{d}y > \|\varphi\|, \tag{3.7}$$

for all $\lambda \geq \overline{\lambda}$, $\varphi \in P \setminus \overline{P}_{\overline{R}}$, which implies that (3.3) holds.

On the other hand, from the definition of $f_n(x,\varphi(x))$, for fixed $n \geq 1$ we have $0 < \varphi(y) \leq \overline{R}$, for all $\varphi \in \overline{P}_{\overline{R}}$. Consequently, by (H4) we know

$$A_n\varphi(x) = \int_0^1 K(x,y)\widetilde{p}(y)f_n(x,\varphi(y))\mathrm{d}y \geq \int_0^1 K(x,y)\widetilde{p}(y)\psi_{\overline{R}}(y)\mathrm{d}y, \forall\ \varphi \in \overline{P}_{\overline{R}}. \tag{3.8}$$

Let $r < \min\left\{R, \overline{\lambda}\max_{x\in I}\int_0^1 K(x,y)\widetilde{p}(y)\psi_{\overline{R}}(y)\mathrm{d}y\right\}$. This together with (3.8) implies that for any $\varphi \in \overline{P}_r, \lambda > \overline{\lambda}$

$$\begin{aligned}\|\lambda A_n\varphi\| &= \lambda \max_{x\in I}\int_0^1 K(x,y)\widetilde{p}(y)f_n(y,\varphi(y))\mathrm{d}y \\ &> \overline{\lambda}\max_{x\in I}\int_0^1 K(x,y)\widetilde{p}(y)f_n(y,\varphi(y))\mathrm{d}y \geq r = \|\varphi\|,\end{aligned} \tag{3.9}$$

which yields

$$L_n \cap ((\overline{\lambda},+\infty)\times P_r) = \emptyset. \tag{3.10}$$

Note that (3.7) implies

$$\begin{gathered}\inf_{\varphi\in\partial P_{\overline{R}}}\|A_n\varphi\| \geq \rho k_1 l\overline{R}\max_{x\in I}\int_a^b K(x,y)\widetilde{p}(y)y(1-y)\mathrm{d}y > 0, \\ \overline{\lambda} > \sup_{\varphi\in\partial P_{\overline{R}}}\|\varphi\|/\inf_{\varphi\in\partial P_{\overline{R}}}\|A_n\varphi\|.\end{gathered}$$

As a consequence, by (3.3) (3.10) and Lemma 2.3 we get that L_n possesses a maximal and unbounded subcontinuum C_n satisfying that

$$\begin{gathered}C_n \cap ((0,+\infty)\times P)\setminus((\overline{\lambda},+\infty)\times\overline{P}_{\overline{R}}) \text{ is unbounded and} \\ C_n \cap ((\overline{\lambda},+\infty)\times\{P_r \cup (P\setminus\overline{P}_{\overline{R}})\}) = \emptyset.\end{gathered} \tag{3.11}$$

Next, for $(\lambda,\varphi) \in L_n \cap ([\overline{\lambda},+\infty)\times(\overline{P}_{\overline{R}}\setminus P_r))$, noticing that $\rho k_1 r x(1-x) \leq \varphi(x) \leq \overline{R}$ for $x \in I$, by (H4) we can get

$$\varphi(x) = \lambda(A_n\varphi)(x) = \lambda\int_0^1 K(x,y)\widetilde{p}(y)f_n(x,\varphi(y))\mathrm{d}y \geq \lambda\int_0^1 K(x,y)\widetilde{p}(y)\psi_{\overline{R}}(y)\mathrm{d}y$$

This means

$$\lambda \le \overline{R}\Big(\max_{x\in I}\int_a^b K(x,y)\widetilde{p}(y)\psi_{\overline{R}}(y)\mathrm{d}y\Big)^{-1}, \tag{3.12}$$

which implies $L_n \cap ([\overline{\lambda}, +\infty) \times (\overline{P}_{\overline{R}} \setminus P_r))$ is bounded. This together with (3.3) and (3.10) guarantees that

$$L_n \cap ([\overline{\lambda}, +\infty) \times P) \text{ is bounded }, \forall\ \overline{\lambda} > 0. \tag{3.13}$$

Thus, by (3.11)(3.13) we know that $C_n \cap ((0, \overline{\lambda}] \times P)$ is unbounded. Furthermore, by virtue of (iii) and (iii)* of Lemma 2.3 and (3.11) (3.12) one can get

$$C_n \cap ([0, +\infty) \times \{\theta\}) \neq \emptyset.$$

Now we show that

$$C_n \cap ([0, +\infty) \times \{\theta\}) = \{(0, \theta)\}.$$

Suppose $(\lambda_0, \theta) \in C_n \cap ([0, +\infty) \times \{\theta\})$, then there exist $\lambda_m \in R^+$ and $\varphi_m \in P \setminus \{\theta\}, m = 1, 2, \dots$ such that

$$\varphi_m(x) = \lambda_m(A_n\varphi_m)(x), \quad \lambda_m \to \lambda_0, \quad \varphi_m \to \theta \quad (m \to +\infty).$$

Without loss of generality, assume $\varphi_m \in P_{\overline{R}} \setminus \{\theta\}$. Then

$$(A_n\varphi_m)(x) \ge \int_0^1 K(x,y)\widetilde{p}(y)\psi_{\overline{R}}(y)\mathrm{d}y.$$

Therefore,

$$|\lambda_m| \le \frac{\|\varphi_m\|}{\max_{x\in I}\int_0^1 K(x,y)\widetilde{p}(y)\psi_{\overline{R}}(y)\mathrm{d}y} \to 0 \quad (m \to +\infty).$$

So, $\lambda_0 = 0$, i.e., $C_n \cap ([0, +\infty) \times \{\theta\}) = \{(0, \theta)\}$. As a consequence, (1) holds. By Lemma 2.3 we know C_n is a maximal and unbounded subcontinuum which comes from $(0, \theta)$.

On the other hand, suppose $\lambda_0 \in (0, \overline{\lambda}]$ is an asymptotic bifurcation point of the operator A_n. Then there exist $\lambda_m \in R^+$ and $\varphi_m \in P \setminus P_{\overline{R}}$ such that $\varphi_m = \lambda_m A_n \varphi_m$ and $\lambda_m \to \lambda_0, \|\varphi_m\| \to +\infty$ as $m \to +\infty$.

From (H5), as in the proof of (3.7), one obtain

$$\frac{1}{\lambda_m} = \frac{\|A_n\varphi_m\|}{\|\varphi_m\|} \to +\infty \quad (\|\varphi_m\| \to +\infty).$$

This means that $\lambda_0 = 0$ is the unique asymptotic bifurcation point. Therefore, C_n tends to $(0, +\infty)$; i.e., (4) holds.

Let $\mathcal{L} := \{\lambda : \text{there exists } \varphi \in P \setminus \{\theta\} \text{ such that } \varphi = \lambda A_n\varphi\}$. Obviously, $\mathcal{L} \neq \emptyset$. Let $\lambda_n^0 := \sup\{\lambda : \lambda \in \mathcal{L}\}$. By virtue of (3.11) (3.12) we know $\lambda_n^0 \in (0, +\infty)$. Suppose $(\lambda_m, \varphi_m) \in L_n$ satisfying $\lambda_m \to \lambda_n^0,\ m \to \infty$. It follows from (3.13) that $\{\varphi_m\}$ is bounded. By Lemma 3.3 there exists $\varphi \in P \setminus \{\theta\}$ such that $(\lambda_n^0, \varphi) \in L_n$. Consequently, noticing C_n is unbounded, by virtue of the connection of subcontinuum one can get (2) holds. Consequently, we have

$$L_n \cap ((\lambda_n^0, +\infty) \times P) = \emptyset. \tag{3.14}$$

Considering C_n is unbounded and 0 is the unique asymptotic bifurcation point, it is not difficult to know from (3.14) that (3) also holds.

To get (5) and (6), noticing that for $(\lambda,\varphi)\in L_n\cap((0,+\infty)\times(\overline{P}_R\setminus P_r))$ $(R>1>r>0)$, we have

$$\begin{aligned}\varphi(x)=\lambda(A_n\varphi)(x)&=\lambda\int_0^1 K(x,y)\widetilde{p}(y)f_n(y,\varphi(y))\mathrm{d}y\\&\le\lambda\int_0^1 K(x,y)\widetilde{p}(y)f_{r,R}(y)\mathrm{d}y,\end{aligned}$$

This together with (3.12), we get

$$\begin{aligned}\lambda' &:= r\Big(\max_{x\in I}\int_0^1 K(x,y)\widetilde{p}(y)f_{r,R}(y)\mathrm{d}y\Big)^{-1}\le\lambda\\&\le R\Big(\max_{x\in I}\int_0^1 K(x,y)\widetilde{p}(y)\psi_R(y)\mathrm{d}y\Big)^{-1}:=\lambda''.\end{aligned}\tag{3.15}$$

Thus

$$C_n\cap((0,+\infty)\times(\overline{P}_R\setminus P_r))\subset[\lambda',\lambda'']\times\overline{P}_R\setminus P_r.\tag{3.16}$$

Since C_n is a maximal and unbounded subcontinuum which comes from $(0,\theta)$ and tends to $(0,+\infty)$ eventually, for any $\lambda\in(0,\lambda')$ from (3.15) and (3.16) one can get that there exist at least two points $\varphi^*_{n\lambda}$ and $\varphi^{**}_{n\lambda}\in P\setminus\{\theta\}$ such that $(\lambda,\varphi^*_{n\lambda}),(\lambda,\varphi^{**}_{n\lambda})\in C_n$ with $\|\varphi^{**}_{n\lambda}\|>R>r>\|\varphi^*_{n\lambda}\|>0$. Notice that R and r satisfying $R>1>r>0$ are arbitrary. Thus, it is easy to know (5) and (6) hold. □

From (3.3) (3.10) and (3.12) in above Theorem 3.4, one can obtain the following corollary.

Corollary 3.5. *Assume* (H1)–(H5) *hold. Then for every $\varepsilon>0$, there exist positive number $R_\varepsilon>1>r_\varepsilon>0,\lambda_\varepsilon>0$ such that*

$$L_n\cap([\varepsilon,+\infty)\times P)\subset[\varepsilon,\lambda_\varepsilon]\times(\overline{P}_{R_\varepsilon}\setminus P_{r_\varepsilon}),\forall\ n\ge1,\tag{3.17}$$

where R_ε and λ_ε are nonincreasing and r_ε is nondecreasing with respect to $\varepsilon\in(0,+\infty)$.

The next theorem gives a result for L and (1.1).

Theorem 3.6. *Let* (H1)–(H5) *be satisfied. Then $\overline{L}$ possesses a maximal and unbounded subcontinuum C, which comes from $(0,\theta)$ and tends to $(0,+\infty)$ eventually such that*

- (i) *There exists $\lambda^0>0$ satisfying $L\cap([\lambda^0,+\infty)\times P)=\emptyset$;*
- (ii) *For each $\overline{\lambda}>0$, $C\cap([0,\overline{\lambda}]\times P)$ is unbounded;*
- (iii) *There exist $\lambda^*\in(0,\lambda^0)$ such that for all $\lambda\in(0,\lambda^*)$,* (1.1) *has at least two positive solution φ^1_λ and φ^2_λ satisfying*
$$(\lambda,\varphi^1_\lambda),(\lambda,\varphi^2_\lambda)\in C,\quad\|\varphi^2_\lambda\|>\|\varphi^1_\lambda\|;$$
- (iv)
$$\lim_{\lambda\to0^+,\,(\lambda,\varphi^1_\lambda)\in C}\|\varphi^1_\lambda\|=0,\qquad\lim_{\lambda\to0^+,\,(\lambda,\varphi^2_\lambda)\in C}\|\varphi^2_\lambda\|=+\infty.$$

Proof. Firstly, we prove that $L\neq\emptyset$. By Theorem 3.4 and (3.15), we know that there exists $\lambda_0>0$ such that for each n, L_n possesses a maximal and unbounded subcontinuum C_n containing $(0,\theta)$, which satisfies

$$C_n\cap(\{\lambda_0\}\times P)\neq\emptyset,\quad\forall\ n\ge1.\tag{3.18}$$

On the other hand, from Corollary 3.5, one can get that there exist $\overline{R} > 1 > \overline{r} > 0$ such that

$$L_n \cap (\{\lambda_0\} \times P) \subset \{\lambda_0\} \times (\overline{P}_{\overline{R}} \setminus P_{\overline{r}}) \quad \text{for all } n \geq 1. \tag{3.19}$$

For every n, by (3.18) one can take $\varphi_n \in C_n \cap (\{\lambda_0\} \times P)$. Then it follows from (3.19) that $\varphi_n \in \overline{P}_{\overline{R}} \setminus P_{\overline{r}}$. By (H3) we know

$$f_n(x, \varphi_n(x)) \leq f_{\overline{r},\overline{R}}(x) \quad \text{for all } x \in (0,1),\ n \geq 1. \tag{3.20}$$

Similar to the proof of Lemma 3.3, it is easy to know that $\{\varphi_n\}$ is uniformly bounded and equicontinuous on $I = [0,1]$. As a consequence, Ascoli-Arzela theorem generates the compactness of $\{\varphi_n\}$. So there exists a subsequence (without loss of generality, we may assume this sequence is $\{\varphi_n\}$ as well) and $\varphi^* \in \overline{P}_{\overline{R}} \setminus P_{\overline{r}}$ such that $\varphi_n \to \varphi^*$ as $n \to +\infty$. (3.2) and Lebesgue dominated convergence theorem guarantee $(\lambda_0, \varphi^*) \in L$, that is, $L \neq \emptyset$.

Secondly, define an operator A on $P \setminus \{\theta\}$ as follows:

$$(A\varphi)(x) = \int_0^1 K(x,y)\widetilde{p}(y) f(y, \varphi(y))\mathrm{d}y \quad \text{for all } x \in I,\ \varphi \in P \setminus \{\theta\}. \tag{3.21}$$

By (H3), A is well defined on $P \setminus \{\theta\}$. It is easy to see that to seek a positive solution of (1.1) is equivalent to find a fixed point of λA on $P \setminus \{\theta\}$. Similar to Theorem 3.4, one can get (i) holds.

To obtain (ii), noticing that for any $\varepsilon \in (0, \lambda_0)$, it follows from Corollary 3.5 that there exist $R_\varepsilon, \lambda_\varepsilon$, and r_ε such that

$$L_n \cap ([\varepsilon, +\infty) \times P) \subset \overline{Q}_\varepsilon \quad \text{for all } n \geq 1.$$

where $R_\varepsilon, \lambda_\varepsilon$ are nonincreasing and r_ε is nondecreasing functions with respect to ε, $Q_\varepsilon := (\varepsilon, \lambda_\varepsilon] \times P_{R_\varepsilon}$.

On the other hand,

$$\Big(\bigcup_{n=1}^{+\infty} L_n\Big) \bigcap \overline{Q}_\varepsilon \subset \Big(\bigcup_{n=1}^{+\infty} L_n\Big) \bigcap ([\varepsilon, \lambda_\varepsilon] \times (\overline{P}_{R_\varepsilon} \setminus P_{r_\varepsilon})).$$

This together with Lemma 3.3 and its proof implies that

$$\Big(\bigcup_{n=1}^{+\infty} L_n\Big) \bigcap \overline{Q}_\varepsilon \quad \text{are relatively compact.} \tag{3.22}$$

Recall that a maximal subcontinuum is a maximal, closed and connected set. In what follows, we denote by C_n^ε the subcontinuum of $C_n \cap \overline{Q}_\varepsilon$ containing (λ_0, φ_n). Let

$$\begin{aligned} F_\varepsilon := \big\{y : &\text{ there exist the subsequence } \{n_k\} \text{ of } \{n\} \\ &\text{and } y_{n_k} \in C_{n_k}^\varepsilon \text{ satisfying } \lim_{k \to +\infty} y_{n_k} = y\big\}. \end{aligned} \tag{3.23}$$

Combining with (3.22) and Lebesgue dominated convergence theorem one can get

$$F_\varepsilon \subset L \quad \text{and} \quad (\lambda_0, \varphi^*) \in F_\varepsilon. \tag{3.24}$$

Now we prove that F_ε is connected. Otherwise, there exist subsets V_1 and V_2 such that $\overline{V}_1 \cap V_2 = \emptyset, V_1 \cap \overline{V}_2 = \emptyset$ and $F_\varepsilon = V_1 \cup V_2$. Since F_ε is closed, $F_\varepsilon = V_1 \cup \overline{V}_2$, and consequently, $V_2 = \overline{V}_2$. Similarly, $V_1 = \overline{V}_1$. Therefore, V_1 and V_2 are compact. Noticing $V_1 \cap V_2 = \emptyset$, there exists $\delta > 0$, such that $\rho(V_1, V_2) = \delta$. Let

$$U(V_1, \frac{\delta}{3}) := \{(\lambda, \varphi) \in R^+ \times C[I, P] : d((\lambda, \varphi); V_1) < \frac{\delta}{3}\};$$
$$U(V_2, \frac{\delta}{3}) := \{(\lambda, \varphi) \in R^+ \times C[I, P] : d((\lambda, \varphi); V_2) < \frac{\delta}{3}\};$$

where $d(\cdot, \cdot)$ denotes the distance between two sets in $E = R \times C[I, P]$.

Without loss of generality, suppose $P_1 = (\lambda_0, \varphi^*) \in V_1$, and choose $P_2 \in V_2$. Obviously, $P_{1n} := (\lambda_0, \varphi_n) \to P_1$ as $n \to +\infty$ and there exists a subsequence $\{n_k\}$ of $\{n\}$ and $P_{2,n_k} \in C^{\varepsilon}_{n_k}$ such that $\lim_{k\to+\infty} P_{2,n_k} = P_2$. As a consequence, there exists $N > 0$ such that $P_{1,n_k} \in U(V_1, \frac{\delta}{3}), P_{2,n_k} \in U(V_2, \frac{\delta}{3})$ for $n_k \geq N$. Notice that $C^{\varepsilon}_{n_k}$ is connected. Then there exists $P_{n_k} \in C^{\varepsilon}_{n_k} \cap \partial U(V_1, \frac{\delta}{3})$ for each $n_k \geq n$. Since $\{P_{n_k}\}$ are relatively compact, without loss of generality, we may assume $\lim_{k\to+\infty} P_{n_k} = P^*$ as well. Then $P^* \in \partial U(V_1, \frac{\delta}{3})$ and $P^* \in F_\varepsilon$, which contradicts $F_\varepsilon \cap \partial U(V_1, \frac{\delta}{3}) = \emptyset$. Consequently, F_ε is connected .

Let

$$C := \bigcup_{0<\varepsilon<\lambda_0} F_\varepsilon.$$

Now we are in position to show that C meets our requirements. Noticing that F_ε is connected, it follows from (3.24) that $(\lambda_0, \varphi^*) \in F_\varepsilon$ for any $\varepsilon \in (0, \lambda_0)$. Thus, C is connected.

For every pair of positive numbers $R > r > 0$, $\lambda \in (0, \lambda')(\lambda'$ is the same as in (3.15), $n \geq 1$, by virtue of (3.15) and the connectivity of C_n there exist $\varphi_{1n}, \varphi_{2n} \in P \setminus \{\theta\}$ such that

$$(\lambda, \varphi_{1n}), (\lambda, \varphi_{2n}) \in C_n, \quad \|\varphi_{1n}\| \leq r \quad \text{with } \|\varphi_{2n}\| \geq R \text{ for each } n \geq 1.$$

Using Corollary 3.5, we know that $\{\varphi_{2n}\}$ is bounded. Moreover, notice that $\bigcup_{n=1}^{+\infty} L_n \bigcap(\{\lambda\} \times P)$ are relatively compact. This together with (3.23) guarantees that there exist φ_1^* and φ_2^* such that

$$(\lambda, \varphi_1^*), (\lambda, \varphi_2^*) \in C, \quad \|\varphi_1^*\| \leq r, \|\varphi_2^*\| \geq R.$$

Since R and r are arbitrary, we can easily know that C is an unbounded subcontinuum. Consequently, (ii) holds.

On the other hand, similar to the proof of Theorem 3.4, it is not difficult to see that C comes from $(0, \theta)$ and tends to $(0, +\infty)$ eventually. Thus, (iii) and (iv) hold. $\square$

Example. Consider the singular m-point boundary-value problem

$$\begin{gathered}\varphi''(x) + \lambda f(x, \varphi) = \frac{1}{\sqrt{x(1-x)}}(1 + \varphi^{\frac{3}{2}} + \frac{1}{\sqrt[3]{\varphi}}), \quad 0 < x < 1, \\ \varphi(0) = 0, \quad \varphi(1) = \frac{1}{2}\varphi\Big(\frac{1}{2}\Big).\end{gathered} \tag{3.25}$$

Note that Theorem 3.6 applies to this problem, with $f(x, \varphi) = \frac{1}{\sqrt{x(1-x)}}(1 + \varphi^{\frac{3}{2}} + \frac{1}{\sqrt[3]{\varphi}})$, $a_1 = \xi_1 = \frac{1}{2}$, $\phi_1(x) = x$, $\varphi_2(x) = 1 - x$, $\rho = \varphi'(0) = 1$, $D = \frac{3}{4}$. Certainly, (H1) holds with $p(x) \equiv 1, q(x) \equiv 0$. Also (H2) is obviously satisfied.

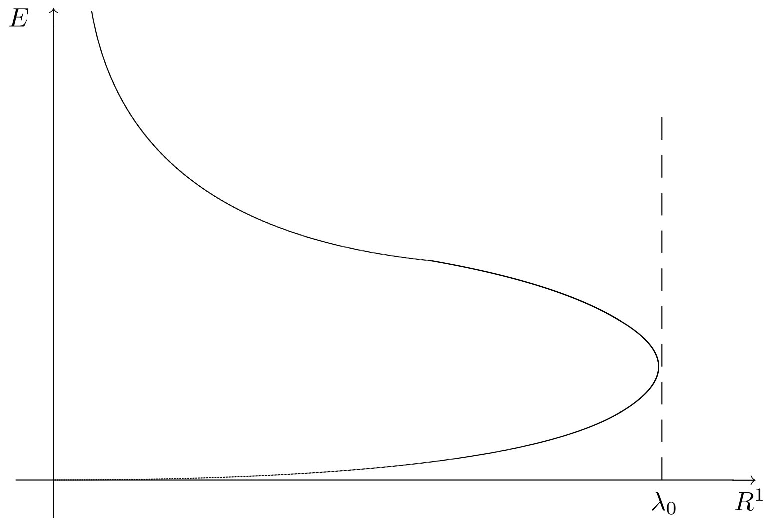

FIGURE 1. Graph of continuum C

To show that (H3) holds, we take $k_1 = 1$, then

$$f_{r,R}(y) = \frac{1}{\sqrt{y(1-y)}}\Big(1 + R^{\frac{3}{2}} + \frac{1}{\sqrt[3]{y(1-y)r}}\Big).$$

Thus, we can easily get

$$\begin{aligned}
&\int_0^1 K_1(y,y) f_{r,R}(y)\mathrm{d}y \\
&= \int_0^1 y(1-y)\frac{1}{\sqrt{y(1-y)}}\Big(1 + R^{\frac{3}{2}} + \frac{1}{\sqrt[3]{y(1-y)r}}\Big)\mathrm{d}y \\
&\quad + \frac{4}{3}\int_0^{\frac{1}{2}} \frac{1}{2} y \frac{1}{\sqrt{y(1-y)}}\Big(1 + R^{\frac{3}{2}} + \frac{1}{\sqrt[3]{y(1-y)r}}\Big)\mathrm{d}y \\
&\quad + \frac{4}{3}\int_{\frac{1}{2}}^1 \frac{1}{2}(1-y) \frac{1}{\sqrt{y(1-y)}}\Big(1 + R^{\frac{3}{2}} + \frac{1}{\sqrt[3]{y(1-y)r}}\Big)\mathrm{d}y < +\infty.
\end{aligned}$$

It is clear, (H4) holds with $\psi_R(x) = \frac{1}{\sqrt{x(1-x)}}$. Also (H5) is satisfied. So, that Theorem 3.6 guarantees that the closure of positive solution set for (3.25) possesses a maximal and unbounded subcontinuum C, which comes from $(0, \theta)$ and tends to $(0, +\infty)$ eventually and meets (i)-(iv) in Theorem 3.6.

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