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REGULARIZATION OF THE BACKWARD HEAT EQUATION VIA HEATLETS

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Abstract. Shen and Strang [16] introduced heatlets in order to solve the heat equation using wavelet expansions of the initial data. The advantage of this approach is that heatlets, or the heat evolution of the wavelet basis functions, can be easily computed and stored. In this paper, we use heatlets to regularize the *backward* heat equation and, more generally, ill-posed Cauchy problems. Continuous dependence results obtained by Ames and Hughes [4] are applied to approximate stabilized solutions to ill-posed problems that arise from the method of quasi-reversibility.

1. Introduction

Shen and Strang [16] introduced heatlets in order to solve the heat equation using wavelet expansions of the initial data. The advantage of this approach is that heatlets, or the heat evolution of the wavelet basis functions, can be computed easily and stored. When the initial data is expanded in terms of the wavelet basis, the solution to the heat equation is then obtained from an expansion using the heatlets and the corresponding wavelet coefficients of the data. In this paper, we turn our attention to ill-posed problems, using heatlets, and the method of quasi-reversibility [8], to regularize the *backward* heat equation [11, 13, 17] as well as more general ill-posed problems.

Given an ill-posed problem, it is often convenient to define an approximate problem that is well-posed. Generally, we seek to ensure that a solution to the original problem, if it exists, will be appropriately close to the solution to the approximate problem. In our main results, we show that for a wide range of ill-posed problems, heatlets may be used to obtain such approximate solutions. In addition, applying the results of [4, 5], we obtain Hölder-continuous dependence results for the difference between solutions of the ill-posed and approximate well-posed problems. Previously, wavelets have been used by Liu et al. to decompose the regularized solution of inverse heat conduction problems using a sensitivity decomposition [9], but heatlets do not play a role in that work.

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We consider the backward heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} \quad \text{where } 0 < x < c,\ 0 < t < T, \\ u(0,t) = u(c,t) = 0, \quad 0 < t < T, \\ u(x,0) = k(x), \quad 0 < x < c, \end{aligned} \tag{1.1}$$

for suitable initial data $k(x)$. The continuous dependence results in [4, 5] use semigroup theory and the notion of C-semigroups [10, 12, 17]. If we let $A = -\Delta$ denote the self-adjoint Laplacian in $L^2(\mathbb{R})$, then the backward heat equation can be written as an *abstract Cauchy problem* [7]:

$$\begin{aligned} \frac{du}{dt} = Au, \\ u(0) = f. \end{aligned} \tag{1.2}$$

Following [2, 11], we define an approximate well-posed problem as follows:

$$\begin{aligned} \frac{dv}{dt} = (A - \epsilon A^2)v = -\frac{\partial^2 v}{\partial x^2} - \epsilon \frac{\partial^4 v}{\partial x^4}, \\ v(0) = f. \end{aligned} \tag{1.3}$$

This equation is well-posed, since the spectrum of $A - \epsilon A^2$ is bounded above. From the Spectral Theorem, it follows that solutions to the approximate well-posed problem are of the form

$$v(t) = e^{t(A-\epsilon A^2)} f. \tag{1.4}$$

Quasi-reversibility is a regularization technique for ill-posed problems that is designed to generate approximate solutions to the problem in question. The central idea of quasi-reversibility is to solve the original problem backward, after first replacing A by an approximate operator whose spectrum is bounded above. Miller [11, 12] refines the quasi-reversibility approach of Lattes and Lions, finding sufficient conditions on the approximate operator to guarantee Hölder continuous dependence on the data when the method is stabilized; he refers to his approach as an *SQR*-method. To implement the method of quasi-reversibility, we consider the well-posed final value problem

$$\begin{aligned} \frac{dw}{dt} = Aw, \\ w(T) = v(T) = e^{T(A-\epsilon A^2)} f, \end{aligned} \tag{1.5}$$

with solution

$$w(t) = e^{(t-T)A} e^{T(A-\epsilon A^2)} f = e^{tA} e^{-T\epsilon A^2} f. \tag{1.6}$$

We then have the following regularization result from [4].

Theorem 1.1 ([4, Theorem 2]). *If $u(t)$ and $w(t)$ are solutions to* (1.2) *and* (1.5) *respectively, and $\|u(T)\| \le k$, for some constant k, then there exist constants C and M, independent of $\epsilon > 0$, such that for $0 \le t < T$,*

$$\|u(t) - w(t)\| \le C\epsilon^{1-\frac{t}{T}} M^{t/T}.$$

In light of this result, we ask whether a heatlet decomposition of the initial data can be used to determine the regularization $w(t)$. First, we turn to the main result in [16], which deals with the well-posed *forward* heat equation

$$\begin{aligned} \frac{du}{dt} &= -Au, \\ u(0) &= f. \end{aligned} \tag{1.7}$$

Theorem 1.2 ([16, Theorem 3.1])**.** *Let $f \in L^2(\mathbb{R})$, and $\{\psi_{j,n}\}$ be an orthonormal wavelet basis. Then the corresponding heat evolution in $L^2(\mathbb{R})$ is given by*

$$u(x,t) = \sum_{j,n\in\mathbb{Z}} c_{j,n}\Psi^h_{j,n}(x,t),$$

where $c_{j,n}$ is the wavelet coefficient of $f(x)$ attached to $\psi_{j,n} = 2^{j/2}\psi(2^jx-n)$, and $\Psi^h_{j,n}(x,t)$ is the solution of (1.5) *with initial data $\psi_{j,n}$. Moreover, the infinite series converges in $L^2(\mathbb{R})$ uniformly with respect to t .*

Using quasi-reversibility, we determine that $w(t)$ can be obtained by evaluating a heatlet at time $T-t$. This will yield our main result, the *heatlet decomposition for the backward heat equation* (Theorem 3.3):

Theorem 1.3. *Let $f \in L^2(\mathbb{R})$. If $u(t)$ is a stabilized solution of* (1.2)*, so that $\|u(T)\| \le \tilde{M}$, we have*

$$\|u(t) - \sum_{j,n\in\mathbb{Z}} c_{j,n}e^{T(A-\epsilon A^2)}\Psi^h_{j,n}(x,T-t)\| \le C\epsilon^{1-\frac{t}{T}}M^{t/T},$$

for constants C and M that are independent of $\epsilon > 0$, and $c_{j,n}$ is the wavelet coefficient of $f(x)$ attached to $\psi_{j,n} = 2^{j/2}\psi(2^jx-n)$. Thus, for small values of $\epsilon > 0$,

$$\sum_{j,n\in\mathbb{Z}} c_{j,n}e^{T(A-\epsilon A^2)}\Psi^h_{j,n}(x,T-t)$$

is close to $u(t)$ in $L^2(\mathbb{R})$, for $0 \le t < T$.

The value of the above theorem lies in the fact that, as in the case of the well-posed heat equation, the heatlets may be computed and stored, and the approximation $w(t)$ will require evaluation of $e^{T(A-\epsilon A^2)}\Psi^h_{j,n}(x,T-t)$, rather than $e^{T(A-\epsilon A^2)}e^{(t-T)A}f$. Finally, in Section 4, we show that Theorem 1.3 may be framed in a more general setting, with other choices of the approximating operators. To pursue this generalization, we introduce the terminology of [4], and define *generalized heatlets*, that is, solutions of the *abstract* Cauchy problem with initial data consisting of elements of a wavelet basis. We then approximate the solution to the ill-posed problem using the wavelet coefficients in a manner analagous to that in Theorem 1.3 (Theorem 4.2).

2. Wavelets and Heatlets

In $L^2(\mathbb{R})$ we define the *mother wavelet* of the Haar basis as

$$\psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

For positive integers n, j define $\psi^j_n(x) = 2^{j/2}\psi(2^jx-n)$. Then according to a theorem of Haar, $\{\psi^j_n\}$ is an orthonormal basis for $L^2(\mathbb{R})$ (cf. [6]).

Definition. A *multiresolution analysis* of $L^2(\mathbb{R})$ is a chain of approximate spaces V_j such that $-\infty \le j \le \infty$. These closed subspaces satisfy the following properties:

(i) The V_j spaces are nested: $\dots V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \dots$

(ii) These spaces are complete; that is,
$$\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}) \quad (\text{i.e. } \lim_{j\to\infty} V_j = L^2(\mathbb{R})),$$
$$\cap_{j\in\mathbb{Z}} V_j = 0 \quad (\text{i.e. } \lim_{j\to-\infty} V_j = 0).$$

(iii) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$.

(iv) $f(x) \in V_0$ if and only if $f(x-k) \in V_0$.

(v) There exists a scaling function $\phi(x) \in V_0$ such that $\{\phi(x-k) : k \in \mathbb{Z}\}$ is an orthonormal basis of V_0 (cf. [6]).

To create a multiresolution, one needs to construct a scaling function $\phi(x)$. Then, using the properties of a multiresolution analysis, the entire chain can be constructed from $\phi(x)$. For example, we can let $V_0 = \{\phi(x-n) | n \in \mathbb{Z}\}$. Then
$$\begin{gathered}
V_1 = \{\phi(2x-n) : n \in \mathbb{Z}\}, \\
V_2 = \{\phi(2^2x-n) : n \in \mathbb{Z}\}, \\
V_{-1} = \{\phi(\frac{x}{2}-n) : n \in \mathbb{Z}\}.
\end{gathered}$$
This chain of approximate spaces V_j forms a multiresultion analysis of $L^2(\mathbb{R})$ [6].

The multiresolution analysis associated with the Haar basis is provided by
$$V_j = \{f \in L^2(\mathbb{R}) : f|_{[\frac{k}{2^j}, \frac{(k+1)}{2^j}]} = \text{constant},\ k \in \mathbb{Z}\}.$$

Next, we summarize the definitions and results from [16, Section 3].

Definition. Let $\phi(x)$ be the scaling function and $\psi(x)$ be the wavelet associated to a multiresolution analysis. Define the *heat evolutions* of $\phi(x)$ and $\psi(x)$ to be $\Phi^h(x,t)$ and $\Psi^h(x,t)$, where
$$\Phi^h_t = \Phi^h_{xx}, \quad \Phi^h(x,0) = \phi(x), \quad \text{for } t > 0,\ x \in \mathbb{R}.$$
Similarly,
$$\Psi^h_t = \Psi^h_{xx}, quad \Psi^h(x,0) = \psi(x), \quad \text{for } t > 0,\ x \in \mathbb{R}.$$
The function Ψ^h is called a *heatlet* and Φ^h is a *refinable heat*.

Proposition 2.1. *Assume that $\phi(x)$ and $\psi(x)$ satisfy the equations*
$$\begin{gathered}
\phi(x) = 2\sum_{n\in Z} h_n \phi(2x-n), \\
\psi(x) = 2\sum_{n\in Z} g_n \phi(2x-n),
\end{gathered}$$
where $(h_n), (g_n) \in l^2$. Then, the refinable heat and heatlet will satisfy
$$\begin{gathered}
\Phi^h(x,t) = 2\sum_{n\in Z} h_n \Phi^h(2x-n, 4t), \\
\Psi^h(x,t) = 2\sum_{n\in Z} g_n \Phi^h(2x-n, 4t).
\end{gathered}$$

Proposition 2.2. *Define $\Psi^h_{j,n}(x,t)$ to be the solution of* (1.5) *with initial data $\psi_{j,n}$. Then*
$$\Psi^h_{j,n}(x,t) = 2^{j/2}\Psi^h(2^j x - n, 4^j t).$$

The main theorem of Shen and Strang [16] is as follows.

Theorem 2.3 ([16])**.** *Let $f \in L^2(\mathbb{R})$. Then the corresponding heat evolution in $L^2(\mathbb{R})$ is given by*
$$u(x,t) = \sum_{j,n\in\mathbb{Z}} c_{j,n}\Psi^h_{j,n}(x,t),$$
where $c_{j,n}$ is the wavelet coefficient of $f(x)$ attached to $\psi_{j,n} = 2^{j/2}\psi(2^j x - n)$. Moreover, the infinite series converges in $L^2(\mathbb{R})$ uniformly with respect to t .

3. Regularization of the Backward Heat Equation

Consider the *final value* problem
$$\begin{gathered}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < t < T,\ x \in (0,l),\\ u(x,T) = \phi(x),\\ u(0,t) = u(l,t) = 0.\end{gathered}$$
This problem is ill-posed, and equivalent to (1.1). Following [13], we will stabilize the problem as follows. Define $\mathcal{M}$ to be the collection of all continuous functions $\phi(x,t)$ in $D \times [0,T)$ such that $\phi(x,t)$ is twice differentiable in x and continuously differentiable in t for $t \in (0,T)$. Furthermore, assume
$$\|\phi(T)\|^2 \le k^2$$
for some prescribed constant k which is a natural bound. The following stability result is well-known:

Theorem 3.1 ([13])**.** *If $u(x,t) \in \mathcal{M}$ is a solution to the backward heat equation and $\|u(T)\|^2 \le k^2$, then*
$$\|u(t)\|^2 \le \|f\|^{2(1-\frac{t}{T})} k^{\frac{2t}{T}}.$$

In addition, we have the previously mentioned Hölder-continuity result from [4] (Theorem 1.1).

Now, recall that for $f \in L^2(\mathbb{R})$, the corresponding heat evolution in $L^2(\mathbb{R})$ from f (for the well-posesd problem) is given by
$$u(x,t) = \sum_{j,n\in\mathbb{Z}} c_{j,n}\Psi^h_{j,n}(x,t),$$
where $c_{j,n}$ are the wavelet coefficients of $f(x)$ attached to $\psi_{j,n} = 2^{j/2}\psi(2^j x - n)$. Using quasireversibility, we find that $w(t)$ may be obtained by evaluating a heatlet at time $T - t$. This will yield the *heatlet decomposition for the backward heat equation.*

Theorem 3.2. *Let $f \in L^2(\mathbb{R})$, and let $c_{j,n}$ denote the wavelet coefficient of $f(x)$ attached to $\psi_{j,n} = 2^{j/2}\psi(2^j x - n)$. Assume that $u(t)$ is a stabilized solution of* (1.2)*. Then there exist constants C and M, independent of $\epsilon > 0$, such that*
$$\|u(t) - \sum_{j,n\in\mathbb{Z}} c_{j,n} e^{T(A-\epsilon A^2)}\Psi^h_{j,n}(x, T-t)\| \le C\epsilon^{1-\frac{t}{T}} M^{t/T}.$$

Thus, for small values of $\epsilon > 0$, $\sum_{j,n\in\mathbb{Z}} c_{j,n} e^{T(A-\epsilon A^2)} \Psi^h_{j,n}(x, T-t)$ *is close to* $u(t)$ *in* $L^2(\mathbb{R})$, *for* $0 \le t < T$.

Proof. Recall that the solution to (1.5) is

$$\begin{aligned} w(t) &= e^{(t-T)A} e^{T(A-\epsilon A^2)} f \\ &= \sum_{j,n\in\mathbb{Z}} c_{j,n} e^{(t-T)A} e^{T(A-\epsilon A^2)} \psi_{j,n} \\ &= \sum_{j,n\in\mathbb{Z}} c_{j,n} e^{T(A-\epsilon A^2)} \Psi^h_{j,n}(x, T-t), \end{aligned}$$

where for each j,n, $e^{(t-T)A}\psi_{j,n}$ is the heatlet $\Psi^h_{j,n}(x, T-t)$. We consider

$$\|u(t) - w(t)\| = \|e^{tA}\chi - e^{tA} e^{-\epsilon TA^2} f\| = \|(I - e^{-\epsilon TA^2}) e^{tA} f\|.$$

In order to obtain a convexity result, we set

$$\phi_n(\alpha) = (e^{\alpha^2}[e^{\alpha A} - e^{\alpha A} e^{-\epsilon TA^2}] f_n,\, h),$$

where $f_n = E(e_n)$, $E(\cdot)$ is the resolution of the identity for A, e_n is a bounded Borel function, and h is an arbitrary element of $\mathcal{H}$. Then

$$\begin{aligned} |\phi_n(\alpha)| &\le e^{t^2-\eta^2} \|e^{(t+i\eta)A} f_n - e^{(t+i\eta)A} e^{-\epsilon TA^2} f_n\| \, \|h\| \\ &\le e^{t^2-\eta^2} \|(I - e^{-\epsilon TA^2}) e^{tA} f_n\| \, \|h\| \\ &\le C_1\, e^{t^2-\eta^2} \epsilon \|A^2 e^{tA} f_n\| \, \|h\|. \end{aligned}$$

Thus $\phi_n(\alpha)$ is bounded in the strip $0 \le \Re\alpha \le T$, and so by the Three Lines Theorem, we obtain

$$|\phi_n(t)| \le M(0)^{1-t/T} M(T)^{t/T},$$

where $M(t) = \max_{\eta\in\mathbb{R}} |\phi(t + i\eta)|$. Since $M(0) \le C_1\epsilon\|A^2 f_n\| \, \|h\|$, and

$$M(T) \le e^{T^2} \|(I - e^{-\epsilon TA^2}) e^{TA} f_n\| \, \|h\| \le C_2\, e^{T^2} \|e^{TA} f_n\| \, \|h\|,$$

we obtain, taking the supremum over all $h \in \mathcal{H}$, with $\|h\| \le 1$,

$$\|u(t) - w(t)\| \le C\{\epsilon\|A^2 f_n\|\}^{1-t/T} \{\|e^{TA} f_n\|\}^{t/T}.$$

for a suitable constant C. If we take the limit as $n \to \infty$, and assume in addition that $\|e^{TA} f\| \le \tilde{M}$, from which it follows that $\|A^2 f\| \le \tilde{M}$, for a possibly different constant, we have

$$\|u(t) - \sum_{j,n\in\mathbb{Z}} c_{j,n} e^{T(A-\epsilon A^2)} \Psi^h_{j,n}(x, T-t)\| = \|u(t) - w(t)\| \le C\epsilon^{1-t/T} M^{t/T}.$$

Thus, for small values of $\epsilon > 0$, $\sum_{j,n\in\mathbb{Z}} c_{j,n} e^{T(A-\epsilon A^2)} \Psi^h_{j,n}(x, T-t)$ is close to $u(t)$ in $L^2(\mathbb{R})$, for $0 \le t < T$. □

4. Applications to Ill-Posed Problems

In this section, following [1, 2, 3, 4, 5], we consider ill-posed Cauchy problems in $L^2(\mathbb{R})$, where A is now any positive self-adjoint operator. Let ψ be associated with a multiresolution analysis, and let be the corresponding wavelet basis be $\{\psi_{j,n}\}$. We show that the choice of approximate problem can be generalized.

Definition. Let $f : [0,\infty) \to \mathbb{R}$ be a Borel function, and assume that there exists $\omega \in \mathbb{R}$ such that $f(\lambda) \le \omega$ for all $\lambda \in [0,\infty)$. Then f is said to satisfy

Condition (A) if there exist positive constants β, δ with $0 < \beta < 1$, for which $\mathrm{Dom}(A^{1+\delta}) \subseteq \mathrm{Dom}(f(A))$ and

$$\|(-A + f(A))\psi\| \le \beta\|A^{1+\delta}\psi\|.$$

Set $g(A) = -A + f(A)$.

As in the previous section, we also obtain an approximation $w(t)$ through quasireversibility: $w(t) = e^{(t-T)A}e^{Tf(A)}\chi$, where we replace the initial data f by χ, to avoid confusion.

Theorem 4.1 ([5, Theorem 2]). *Let A be a positive self-adjoint operator acting on $\mathcal{H}$, let f satisfy Condition (A), and assume that there exists a constant γ, independent of β, such that $(g(A)\psi, \psi) \le \gamma(\psi, \psi)$, for all $\psi \in \mathcal{H}$. If $u(t)$ and $w(t)$ are solutions of* (1.2) *and* (1.4)*, respectively, and $\|u(T)\| \le \tilde{M}$, then there exist constants C and M, independent of β, such that for $0 \le t < T$,*

$$\|u(t) - w(t)\| \le C\beta^{1-t/T}M^{t/T}.$$

Definition. For a self-adjoint operator A, we define a generalized heatlet to be the solution Ψ_n^j of the abstract Cauchy problem $\frac{du}{dt} = -Au$, with initial data $\psi_{j,n}$.

The next theorem follows in the same manner as Theorem 3.2 in the previous section, using the realization of $w(t)$ in terms of heatlets.

Theorem 4.2. *Let $\chi \in L^2(\mathbb{R})$, and let $c_{j,n}$ denote the wavelet coefficient of $\chi(x)$ attached to $\psi_{j,n} = 2^{j/2}\psi(2^j x - n)$. Assume that $u(t)$ is a stabilized solution of* (1.2)*, where A is a positive self-adjoint operator on $L^2(\mathbb{R})$, and that f satisfies Condition (A). Then there exist constants C and M, independent of $\epsilon > 0$, such that*

$$\|u(t) - \sum_{j,n\in\mathbb{Z}} c_{j,n}e^{Tf(A)}\Psi_{j,n}^h(x, T-t)\| \le C\epsilon^{1-\frac{t}{T}}M^{t/T}.$$

Thus, for small values of $\epsilon > 0$, $\sum_{j,n\in\mathbb{Z}} c_{j,n}e^{Tf(A)}\Psi_{j,n}^h(x, T-t)$ is close to $u(t)$ in $L^2(\mathbb{R})$, for $0 \le t < T$.

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