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A QUASISTATIC FRICTIONAL CONTACT PROBLEM WITH ADHESION FOR NONLINEAR ELASTIC MATERIALS

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Abstract. The aim of this paper is to study a quasistatic contact problem between a nonlinear elastic body and a foundation. The contact is adhesive and frictional and is modelled with a version of normal compliance condition and the associated Coulomb's law of dry friction. The evolution of the bonding field is described by a first order differential equation. We establish the variational formulation of the mechanical problem and prove an existence result of the weak solution if the coefficient of friction is sufficiently small by passing to the limit with respect to time. The proofs are based on arguments of time-discretization, compactness, lower semicontinuity and Banach fixed point.

1. Introduction

Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. Because of the importance of this process a considerable effort has been made in its modelling and numerical simulations. A first study of frictional contact problems within the framework of variational inequalities was made in [7]. The mathematical, mechanical and numerical state of the art can be found in [15]. Models for dynamic or quasistatic process of frictionless adhesive contact between a deformable body and a foundation have been studied in [3, 4, 10, 19]. In this paper we study a mathematical model which describes a frictional quasistatic contact problem with adhesion between a nonlinear elastic body and a foundation. The adhesive frictional contact is modelled with a version of normal compliance condition and the associated Coulomb's law of dry friction. As in [9, 10], we use the bonding field as an additional state variable β, defined on the contact surface of the boundary. The variable is restricted to values $0 \leq \beta \leq 1$, when $\beta = 0$ all the bonds are severed and there are no active bonds; when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$ it measures the fraction of active bonds and partial adhesion takes place. We refer the reader to the extensive bibliography on the subject in [11, 14, 16, 18, 19, 20]. In [2] a model of a contact problem with adhesion and friction was studied in which β represents a continuous transition between total adhesion and pure frictional states. In [5] the authors considered the interface model proposed in [2] in order to study

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a quasistatic unilateral contact problem with local friction and adhesion. They obtained an existence result under a smallness assumption of the coefficient of friction. In this work, as in [5] by applying an implicit time-discretization scheme, if the coefficient of friction is sufficiently small, we prove that the time-discretized problem has a unique solution for which appropriate estimations are established. We finally obtain the existence of a weak solution by passing to the limit with respect to time. The paper is structured as follows. In Section 2 we present some notations and give the variational formulation. In Section 3 we study a time-discretized problem which admits a unique solution if the coefficient of friction is sufficiently small, Proposition 3.1. In Section 4 we prove Theorem 2.1.

2. Problem Statement and Variational Formulation

Let $\Omega \subset \mathbb{R}^d; (d = 2, 3)$, be the domain initially occupied by an elastic body. Ω is supposed to be open, bounded, with a sufficiently regular boundary Γ. Γ is partitioned into three parts $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ where $\Gamma_1, \Gamma_2, \Gamma_3$ are disjoint open sets and *meas* $\Gamma_1 > 0$. The body is acted upon by a volume force of density φ_1 on Ω and a surface traction of density φ_2 on Γ_2. On Γ_3 the body is in adhesive frictional contact with a foundation.

Thus, the classical formulation of the mechanical problem is written as follows.

Problem P_1. Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0, T] \to [0, 1]$ such that

$$\operatorname{div} \sigma + \varphi_1 = 0 \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$\sigma = F\varepsilon(u) \quad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{2.3}$$

$$\sigma\nu = \varphi_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.4}$$

$$-\sigma_\nu = p(u_\nu) - c_\nu \beta^2 R_\nu(u_\nu) \quad \text{on } \Gamma_3 \times (0, T), \tag{2.5}$$

$$\begin{cases} |\sigma_\tau + c_\tau \beta^2 R_\tau(u_\tau)| \le \mu p(u_\nu), \\ |\sigma_\tau + c_\tau \beta^2 R_\tau(u_\tau)| < \mu p(u_\nu) \Longrightarrow \dot{u}_\tau = 0, \\ |\sigma_\tau + c_\tau \beta^2 R_\tau(u_\tau)| = \mu p(u_\nu) \Longrightarrow \\ \exists \lambda \ge 0 \text{ such that } \dot{u}_\tau = -\lambda(\sigma_\tau + c_\tau \beta^2 R_\tau(u_\tau)), \end{cases} \quad \text{on } \Gamma_3 \times (0, T), \tag{2.6}$$

$$\dot{\beta} = -\big[\beta\big(c_\nu \big(R_\nu(u_\nu)\big)^2 + c_\tau \big(|R_\tau(u_\tau)|\big)^2\big) - \varepsilon_a\big]_+ \quad \text{on } \Gamma_3 \times (0, T), \tag{2.7}$$

$$u(0) = u_0 \quad \text{in } \Omega, \tag{2.8}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{2.9}$$

Equation (2.1) represents the equilibrium equation. Equation (2.2) represents the elastic constitutive law of the material in which σ denotes the stress tensor F is a nonlinear elasticity operator and $\varepsilon(u)$ denotes the strain tensor; (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the stress vector. Condition (2.5) represents the normal compliance and adhesion. Condition (2.6) is the associated Coulomb's law of dry friction. $\dot{u}_\tau$ is the tangential velocity on the boundary Γ_3. Here p is a given function, μ is the coefficient of friction and the parameters c_ν, c_τ and ε_a are given adhesion coefficients which may depend on

$x \in \Gamma_3$. As in [20], R_ν, R_τ are truncation operators defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L \\ -s & \text{if } -L \le s \le 0 \\ 0 & \text{if } s > 0, \end{cases} \quad R_\tau(v) = \begin{cases} v & \text{if } |v| \le L \\ L\frac{v}{|v|} & \text{if } |v| > L, \end{cases}$$

where $L > 0$ is a characteristic length of the bonds. Equation (2.7) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [20] where $[s]_+ = \max(s, 0)$ for all $s \in \mathbb{R}$. Since $\dot\beta \le 0$ on $\Gamma_3 \times (0, T)$, once debonding occurs, bonding cannot be reestablished. Also we wish to make it clear that from [13] it follows that the model does not allow for complete debonding field in finite time. Finally, (2.8) and (2.9) are the initial conditions, in which u_0 and β_0 denotes respectively the initial displacement field and the initial bonding field. In (2.7) a dot above a variable represents its derivative with respect to time. We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and S_d are given by

$$u.v = u_i v_i, \quad |v| = (v.v)^{1/2} \quad \forall u, v \in \mathbb{R}^d,$$
$$\sigma.\tau = \sigma_{ij}\tau_{ij}, \quad |\tau| = (\tau.\tau)^{1/2} \quad \forall \sigma, \tau \in S_d,$$

where S_d is the space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$). Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H = (L^2(\Omega))^d, \quad Q = \{\tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega)\},$$
$$H_1 = (H^1(\Omega))^d, \quad Q_1 = \{\sigma \in Q : \operatorname{div}\sigma \in H\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$\langle u, v\rangle_H = \int_\Omega u_i v_i dx, \quad (\sigma, \tau)_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

The small strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = \{1, \dots, d\},$$

where $\operatorname{div}\sigma = (\sigma_{ij,j})$ is the divergence of σ. For every element $v \in H_1$ we denote by v_ν and v_τ the normal and the tangential components of v on the boundary Γ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu\nu.$$

Similary, for a regular tensor field $\sigma \in Q_1$, we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu\nu$$

and we recall that the following Green's formula holds:

$$(\sigma, \varepsilon(v))_Q + \langle \operatorname{div}\sigma, v\rangle_H = \int_\Gamma \sigma\nu.v da \quad \forall v \in H_1,$$

where da is the surface measure element. Let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1\}.$$

Since meas$\Gamma_1 > 0$, the following Korn's inequality holds,

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.10}$$

where the constant $c_\Omega > 0$ depends only on Ω and Γ_1; see [7]. We equip V with the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_Q$$

and $\|.\|_V$ is the associated norm. It follows from Korn's inequality (2.10) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent on V. Then $(V, \|.\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which depends only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.11}$$

For $p \in [1, \infty]$, we use the standard norm of $L^p(0, T; V)$. We also use the Sobolev space $W^{1,\infty}(0, T; V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

For every real Banach space $(X, \|.\|_X)$ and $T > 0$ we use the notation $C([0, T]; X)$ for the space of continuous functions from $[0, T]$ to X; recall that $C([0, T]; X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X.$$

We suppose that the body forces and surface tractions have the regularity

$$\varphi_1 \in W^{1,\infty}(0, T; H), \quad \varphi_2 \in W^{1,\infty}(0, T; (L^2(\Gamma_2))^d) \tag{2.12}$$

and we denote by $f(t)$ the element of V defined by

$$(f(t), v)_V = \int_\Omega \varphi_1(t).v dx + \int_{\Gamma_2} \varphi_2(t).v da \quad \forall v \in V, \quad \text{for } t \in [0, T]. \tag{2.13}$$

Using (2.12) and (2.13), we obtain $f \in W^{1,\infty}(0, T; V)$.

In the study of the mechanical problem P_1 we assume that $F : \Omega \times S_d \to S_d$, satisfies the following four conditions:

(a) there exists $M > 0$ such that
$$|F(x, \varepsilon_1) - F(x, \varepsilon_2)| \leq M|\varepsilon_1 - \varepsilon_2|$$
for all $\varepsilon_1, \varepsilon_2$ in S_d, a.e. x in Ω;

(b) there exists $m > 0$ such that
$$(F(x, \varepsilon_1) - F(x, \varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \geq m|\varepsilon_1 - \varepsilon_2|^2, \tag{2.14}$$
for all $\varepsilon_1, \varepsilon_2$ in S_d, a.e. x in Ω;

(c) the mapping $x \to F(x, \varepsilon)$ is Lebesgue measurable on Ω for any ε in S_d;

(d) $F(x, 0) = 0$ for a.e. x in Ω.

Also we assume that the normal compliance function p satisfies the following five conditions:

(a) $p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$;

(b) there exists $L_p > 0$ such that $|p(x, r_1) - p(x, r_2)| \leq L_p|r_1 - r_2|$ for all $r_1, r_2 \in \mathbb{R}$, a.e. x in Γ_3;

(c)
$$(p(x, r_1) - p(x, r_2))(r_1 - r_2) \geq 0 \quad \text{for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \text{ in } \Gamma_3; \tag{2.15}$$

(d) the mapping $x \to p(x,r)$ is measurable on Γ_3 for any $r \in \mathbb{R}$;
(e) $p(x,r) = 0$ for all $r \le 0$, a.e. $x \in \Gamma_3$.

We define the functional $j : V \times V \to \mathbb{R}$ by

$$j(u,v) = \int_{\Gamma_3} (p(u_\nu)v_\nu + \mu p(u_\nu)|v_\tau|)da \quad \forall (u,v) \in V \times V.$$

As in [19] we suppose that the adhesion coefficients c_ν, c_τ and ε_a satisfy the conditions

$$c_\nu, c_\tau \in L^\infty(\Gamma_3), \quad \varepsilon_a \in L^\infty(\Gamma_3), \quad c_\nu, c_\tau, \varepsilon_a \ge 0 \quad \text{a.e. on } \Gamma_3. \tag{2.16}$$

We suppose that μ satisfies

$$\mu \in L^\infty(\Gamma_3), \quad \text{and} \quad \mu \ge 0 \quad \text{a.e. on } \Gamma_3. \tag{2.17}$$

We need the following set for the bonding fields,

$$B = \big\{\beta \in L^\infty(0,T;L^2(\Gamma_3));\ 0 \le \beta(t) \le 1\ \forall t \in [0,T], \quad \text{a.e. on } \Gamma_3\big\},$$

and finally we assume that the initial displacement field u_0 belongs to V and satisfies

$$(F\varepsilon(u_0), \varepsilon(v-u_0))_Q + j(u_0, v-u_0) + r(\beta_0, u_0, v-u_0) \ge (f(0), v-u_0)_V \tag{2.18}$$

for all $v \in V$, where the initial bonding field β_0 satisfies

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \le \beta_0 \le 1 \quad \text{a.e. on } \Gamma_3. \tag{2.19}$$

As in [5], using Green's formula, we obtain the following variational formulation to Problem P_1.

Problem P_2**.** Find a displacement field $u \in W^{1,\infty}(0,T;V)$ and a bonding field $\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap B$ such that $u(0) = u_0$, $\beta(0) = \beta_0$ and for almost all $t \in [0,T]$:

$$\begin{aligned}&\Big(F\varepsilon(u(t)), \varepsilon(v-\dot u(t))\Big)_Q + j(u(t),v) - j(u(t),\dot u(t)) + r(\beta(t), u(t), v-\dot u(t)) \\ &\ge (f(t), v-\dot u(t))_V \quad \forall v \in V,\end{aligned} \tag{2.20}$$

$$\dot\beta(t) = -[\beta(t)(c_\nu(R_\nu(u_\nu(t)))^2 + c_\tau(|R_\tau(u_\tau(t))|)^2) - \varepsilon_a]_+ \quad \text{a.e. on } \Gamma_3, \tag{2.21}$$

where

$$r = r_\nu + r_\tau, \quad r_\nu(\beta,u,v) = -\int_{\Gamma_3} c_\nu \beta^2 R_\nu(u_\nu)v_\nu da, \tag{2.22}$$

$$r_\tau(\beta,u,v) = \int_{\Gamma_3} c_\tau \beta^2 R_\tau(u_\tau).v_\tau da. \tag{2.23}$$

Our main result of this section, which will be established in the next is the following theorem.

Theorem 2.1. *Let $T > 0$ and assume (2.12), (2.14), (2.15), (2.16), (2.17), (2.18), and (2.19). Then there exists a constant $\mu_* > 0$ such that for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_*$, Problem P_2 has at least one solution.*

3. A time-discretization

As in [5], we adopt the following time-discretization. For all $n \in \mathbf{N}^*$, we set $t_i = i\Delta t$, $0 \le i \le n$, and $\Delta t = T/n$. We denote respectively by u^i and β^i the approximation of u adn β at time t_i and $\Delta u^i = u^{i+1} - u^i$, $\Delta\beta^i = \beta^{i+1} - \beta^i$. For a continuous function $w(t)$, we use the notation $w^i = w(t_i)$. Then we obtain a sequence of time-discretized problems P_n^i defined for $u^0 = u_0$ and $\beta^{)} = \beta_0$ by:

Problem P_n^i. For $u^i \in V$, $\beta^i \in L^\infty(\Gamma_3)$, find $u^{i+1} \in V$, $\beta^{i+1} \in L^\infty(\Gamma_3)$ such that

$$\begin{aligned}
&(F\varepsilon(u^{i+1}), \varepsilon(w - u^{i+1}))_Q + j(u^{i+1}, w - u^i) \\
&- j(u^{i+1}, \Delta u^i) + r(\beta^{i+1}, u^{i+1}, w - u^{i+1}) \\
&\ge (f^{i+1}, w - u^{i+1})_V \quad \forall w \in V, \\
&\frac{\beta^{i+1} - \beta^i}{\Delta t} = -[\beta^{i+1}(c_\nu(R_\nu(u_\nu^{i+1}))^2 + c_\tau(|R_\tau(u_\tau^{i+1})|)^2) - \varepsilon_a]_+ \quad \text{a.e. on } \Gamma_3.
\end{aligned} \tag{3.1}$$

We have the following result.

Proposition 3.1. *There exists $\mu_c > 0$ such that for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_c$, Problem P_n^i has a unique solution.*

To show this proposition we introduce an intermediate problem. For $\eta \in V$, we introduce the following problem

Problem $P_{\eta n}^i$. For $u^i \in V$, $\beta^i \in L^\infty(\Gamma_3)$, find $u_\eta \in V$, $\beta_\eta \in L^\infty(\Gamma_3)$ such that

$$\begin{aligned}
&(F\varepsilon(u_\eta), \varepsilon(w - u_\eta))_Q + j(\eta, w - u^i) - j(\eta, u_\eta - u^i) + r(\beta_\eta, u_\eta, v - u_\eta) \\
&\ge (f^{i+1}, w - u_\eta)_V \quad \forall w \in V, \\
&\frac{\beta_\eta - \beta^i}{\Delta t} = -[\beta_\eta(c_\nu(R_\nu(u_{\eta\nu}))^2 + c_\tau(|R_\tau(u_{\eta\tau})|)^2) - \varepsilon_a]_+ \quad \text{a.e. on } \Gamma_3.
\end{aligned} \tag{3.2}$$

As in [5] we have the following lemma.

Lemma 3.2. *For any $\eta \in V$, Problem $P_{\eta n}^i$ has a unique solution (u_η, β_η), if Δt is small enough.*

To prove this lemma we introduce the following auxiliary problem.

Problem $P_{1\beta}$. For $u^i \in V$, $\beta \in L^\infty(\Gamma_3)$, find $u_\beta \in V$ such that

$$\begin{aligned}
&(F\varepsilon(u_\beta), \varepsilon(v - u_\beta))_Q + j(\eta, v - u^i) - j(\eta, u_\beta - u^i) + r(\beta, u_\beta, v - u_\beta) \\
&\ge (f^{i+1}, v - u_\beta)_V \quad \forall\, v \in V.
\end{aligned} \tag{3.3}$$

We have the following lemma.

Lemma 3.3. *Problem $P_{1\beta}$ has a unique solution.*

Proof. Let $A : V \to V$ be the operator given by

$$(Au, v)_V = (F\varepsilon(u), \varepsilon(v))_Q + \int_{\Gamma_3} (-c_\nu \beta^2 R_\nu(u_\nu) v_\nu + c_\tau \beta^2 R_\tau(u_\tau).v_\tau) da.$$

Using (2.14)(a), (2.11), (2.16), the properties of the operators R_ν and R_τ (see [18]) such that

$$|R_\nu(a) - R_\nu(b)| \le |a - b|, \forall a, b \in \mathbb{R};\ |R_\tau(a) - R_\tau(b)| \le |a - b|, \forall a, b \in \mathbb{R}^d, \tag{3.4}$$

it follows that A satisfies

$$|(Au - Av, w)_V| \le [M + (\|c_\nu\|_{L^\infty(\Gamma_3)} + \|c_\tau\|_{L^\infty(\Gamma_3)})d_\Omega^2]\|u - v\|_V \|w\|_V.$$

Also, we use (2.14)(b) to see that

$$\begin{aligned}(Au - Av, u - v)_V \ge m\|u - v\|_V^2 - \int_{\Gamma_3} \beta^2 c_\nu (R_\nu(u_\nu) - R_\nu(v_\nu))(u_\nu - v_\nu)da \\ + \int_{\Gamma_3} \beta^2 c_\tau (R_\tau(u_\tau) - R_\tau(v_\tau)).(u_\tau - v_\tau)da.\end{aligned}$$

As

$$\begin{gathered}(R_\nu(u_\nu) - R_\nu(v_\nu))(u_\nu - v_\nu) \le 0 \quad \text{a.e. on } \Gamma_3, \\ (R_\tau(u_\tau) - R_\tau(v_\tau)).(u_\tau - v_\tau) \ge 0 \quad \text{a.e. on } \Gamma_3,\end{gathered} \tag{3.5}$$

we get

$$(Au - Av, u - v)_V \ge m\|u - v\|_V^2,$$

which implies that A is strongly monotone. Therefore, A is an operator strongly monotone and Lipschitz continuous. On the other hand the functional $j_\eta : V \to \mathbb{R}$ defined by

$$j_\eta(v) = j(\eta, v - u^i) \quad \forall v \in V,$$

is convex, proper and lowly semicontinuous, then by a classical argument of elliptic variational inequalities [1], we deduce that the problem $P_{1\beta}$ has a unique solution u_β. □

We also consider the following problem.

Problem $P_{2\beta}$. For $\beta^i \in L^\infty(\Gamma_3)$, $u \in V$, find $\beta \in L^\infty(\Gamma_3)$ such that

$$\frac{\beta - \beta^i}{\Delta t} = -\big[\beta(c_\nu(R_\nu(u_{\beta\nu}))^2 + c_\tau(|R_\tau(u_{\beta\tau})|)^2) - \varepsilon_a\big]_+ \text{a.e. on } \Gamma_3.$$

Obviously, Problem $P_{2\beta}$ has a unique solution which is given by

$$\beta = \begin{cases} \beta^i, \quad \text{if } (c_\nu(R_\nu(u_{\beta\nu}))^2 + c_\tau(|R_\tau(u_{\beta\tau})|)^2)\beta^i - \varepsilon_a < 0, \\ \dfrac{\beta^i + \varepsilon_a \Delta t}{1 + \Delta t(c_\nu(R_\nu(u_{\beta\nu}))^2 + c_\tau(|R_\tau(u_{\beta\tau})|)^2)}, \\ \text{if } (c_\nu(R_\nu(u_{\beta\nu}))^2 + c_\tau(|R_\tau(u_{\beta\tau})|)^2)\beta^i - \varepsilon_a > 0, \end{cases}$$

and it satisfies $\beta \in [0, 1]$. To complete the proof of Lemma 3.2, let $v \in V$ and $\beta(v)$ be the corresponding solution of Problem $P_{2\beta}$. Let $u_{\beta(v)}$ be the corresponding solution of Problem $P_{1\beta}$, and define the mapping $\Psi : V \to V$ as

$$v \to \Psi(v) = u_{\beta(v)}\,.$$

Take $v = u_i$, $i = 1, 2$. As in [5, Lemma 2.3], there exists a positive constant C such that

$$\|\Psi(u_2) - \Psi(u_1)\|_V \le C\Delta t\|u_2 - u_1\|, \quad \forall u_1, u_2 \in V\,.$$

Then we conclude by a contraction argument that for Δt sufficiently small, Problem $P_{\eta n}^i$ has a unique solution (u_η, β_η). Next, we shall establish the proof of Proposition

3.1. Indeed, write the inequality (3.2) for $\eta = \eta_i$ and take $v = u_{\eta_j}$, $i, j = 1, 2$. Adding the two inequalities we have

$$\begin{aligned}&(F\varepsilon(u_{\eta_1}) - F\varepsilon(u_{\eta_2}), \varepsilon(u_{\eta_1} - u_{\eta_2}))_Q\\&\leq r(\beta_{\eta_1}, u_{\eta_1}, u_{\eta_2} - u_{\eta_1}) + r(\beta_{\eta_2}, u_{\eta_2}, u_{\eta_1} - u_{\eta_2}) + j(\eta_1, u_{\eta_2} - u^i)\\&\quad - j(\eta_1, u_{\eta_1} - u^i) + j(\eta_2, u_{\eta_1} - u^i) - j(\eta_2, u_{\eta_2} - u^i).\end{aligned}$$

Then

$$\begin{aligned}&r(\beta_{g_1}, u_{g_1}, u_{g_2} - u_{g_1}) + r(\beta_{g_2}, u_{g_2}, u_{g_1} - u_{g_2})\\&= \int_{\Gamma_3} [c_\tau(\beta_{\eta_1} - \beta_{\eta_2})(\beta_{\eta_1} + \beta_{\eta_2}) R(u_{\eta_{1\tau}})(u_{\eta_{2\tau}} - u_{\eta_{1\tau}})]da\\&\quad - \int_{\Gamma_3} [c_\nu(\beta_{\eta_1} - \beta_{\eta_2})(\beta_{\eta_1} + \beta_{\eta_2}) R(u_{\eta_{1\nu}})(u_{\eta_{1\nu}} - u_{\eta_{2\nu}})]da\\&\quad + \int_{\Gamma_3} [c_\nu \beta^2_{\eta_2} (R(u_{\eta_{1\nu}}) - R(u_{\eta_{2\nu}}))(u_{\eta_{1\nu}} - u_{\eta_{2\nu}})]da\\&\quad + \int_{\Gamma_3} [c_\tau \beta^2_{\eta_2} (R(u_{\eta_{1\tau}}) - R(u_{\eta_{2\tau}}))(u_{\eta_{2\tau}} - u_{\eta_{1\tau}})]da.\end{aligned}$$

Using the properties (3.5), we deduce

$$\begin{aligned}&(F\varepsilon(u_{\eta_1}) - F\varepsilon(u_{\eta_2}), \varepsilon(u_{\eta_1} - u_{\eta_2}))_Q\\&\leq \int_{\Gamma_3} [c_\tau(\beta_{\eta_1} - \beta_{\eta_2})(\beta_{\eta_1} + \beta_{\eta_2}) R(u_{\eta_{1\tau}})(u_{\eta_{2\tau}} - u_{\eta_{1\tau}})]da\\&\quad - \int_{\Gamma_3} [c_\nu(\beta_{\eta_1} - \beta_{\eta_2})(\beta_{u_{\eta_2 1}} + \beta_{\eta_2}) R(u_{\eta_{1\nu}})(u_{\eta_{1\nu}} - u_{\eta_{2\nu}})]da\\&\quad + j(\eta_1, u_{\eta_2} - u^i) - j(\eta_1, u_{\eta_1} - u^i) + j(\eta_2, u_{\eta_1} - u^i) - j(\eta_2, u_{\eta_2} - u^i).\end{aligned}$$

Now, using (2.11), (2.14)(b), (2.15)(b), (2.15)(c), the properties (3.4), $|R_\nu(u_\nu)| \leq L$, and $|R_\tau(u_\tau)| \leq L$, it follows that

$$\begin{aligned}m\|u_{\eta_1} - u_{\eta_2}\|_V &\leq L d_\Omega(\|c_\nu\|_{L^\infty(\Gamma_3)} + \|c_\tau\|_{L^\infty(\Gamma_3)})\|\beta_{\eta_1} - \beta_{\eta_2}\|_{L^2(\Gamma_3)}\\&\quad + L_p d^2_\Omega \|\mu\|_{L^\infty(\Gamma_3)} \|\eta_1 - \eta_2\|_V.\end{aligned} \tag{3.6}$$

On the other hand using that for $a, b \in \mathbb{R}$, $|a_+ - b_+| \leq |a - b|$, we deduce from the equality to relation (3.2) that

$$\begin{aligned}\|\frac{\beta_{\eta_1} - \beta_{\eta_2}}{\Delta t}\|_{L^2(\Gamma_3)} &\leq \|(\beta_{\eta_1} - \beta_{\eta_2})\Big(c_\nu(R_\nu(u_{\beta_{\eta_1}\nu}))^2 + c_\tau(|R_\tau(u_{\beta_{\eta_1}\tau})|)^2\Big)\|_{L^2(\Gamma_3)}\\&\quad + \|\beta_{\eta_2}\Big[\Big(c_\nu(R_\nu(u_{\beta_{\eta_1}\nu}))^2 + c_\tau(|R_\tau(u_{\beta_{\eta_1}\tau})|)^2\Big)\\&\quad - \Big(c_\nu(R_\nu(u_{\beta_{\eta_2}\nu}))^2 + c_\tau(|R_\tau(u_{\beta_{\eta_2}\tau})|)^2\Big)\Big]\|_{L^2(\Gamma_3)}.\end{aligned}$$

The above inequality implies

$$\begin{aligned}\|\frac{\beta_{\eta_1} - \beta_{\eta_2}}{\Delta t}\|_{L^2(\Gamma_3)} &\leq L^2(\|c_\nu\|_{L^\infty(\Gamma_3)} + \|c_\tau\|_{L^\infty(\Gamma_3)})\|\beta_{\eta_1} - \beta_{\eta_2}\|_{L^2(\Gamma_3)}\\&\quad + 2L(\|c_\nu\|_{L^\infty(\Gamma_3)} + \|c_\tau\|_{L^\infty(\Gamma_3)}) d_\Omega \|u_{\eta_1} - u_{\eta_2}\|_V.\end{aligned}$$

Therefore,

$$\begin{aligned}&[1-\Delta t L^2(\|c_\nu\|_{L^\infty(\Gamma_3)}+\|c_\tau\|_{L^\infty(\Gamma_3)})]\|\beta_{\eta_1}-\beta_{\eta_2}\|_{L^2(\Gamma_3)}\\&\le 2L(\|c_\nu\|_{L^\infty(\Gamma_3)}+\|c_\tau\|_{L^\infty(\Gamma_3)})d_\Omega\Delta t\|u_{\eta_1}-u_{\eta_2}\|_V\,.\end{aligned}$$

If

$$\Delta t<\frac{1}{L^2(\|c_\nu\|_{L^\infty(\Gamma_3)}+\|c_\tau\|_{L^\infty(\Gamma_3)})}\,,$$

there exists a constant $C_1>0$ such that

$$\|\beta_{\eta_1}-\beta_{\eta_2}\|_{L^2(\Gamma_3)}\le C_1\Delta t\|u_{\eta_1}-u_{\eta_2}\|_V\,.$$

Then from (3.6) we get

$$\begin{aligned}&[m-C_1Ld_\Omega(\|c_\nu\|_{L^\infty(\Gamma_3)}+\|c_\tau\|_{L^\infty(\Gamma_3)})\Delta t]\|u_{\eta_1}-u_{\eta_2}\|_V\\&\le L_pd_\Omega^2\|\mu\|_{L^\infty(\Gamma_3)}\|\eta_1-\eta_2\|_V,\end{aligned}$$

and thus for

$$\Delta t<\min\Big(\frac{m}{C_1Ld_\Omega(\|c_\nu\|_{L^\infty(\Gamma_3)}+\|c_\tau\|_{L^\infty(\Gamma_3)})}\,,\ \frac{1}{L^2(\|c_\nu\|_{L^\infty(\Gamma_3)}+\|c_\tau\|_{L^\infty(\Gamma_3)})}\Big),$$

there exists a constant $C_2>0$ such that

$$\|u_{\eta_1}-u_{\eta_2}\|_V\le C_2L_pd_\Omega^2\|\mu\|_{L^\infty(\Gamma_3)}\|\eta_1-\eta_2\|_V\,. \tag{3.7}$$

To complete the proof let us define the mapping $\Phi:V\to V$ as $\Phi(\eta)=u_\eta$. Then from (3.7) it follows

$$\|\Phi(\eta_1)-\Phi(\eta_2)\|_V\le C_2L_pd_\Omega^2\|\mu\|_{L^\infty(\Gamma_3)}\|\eta_1-\eta_2\|_V\,,\quad\forall\eta_1,\eta_2\in V\,.$$

Then when $\mu_c=\frac{1}{L_pd_\Omega^2C_2}$, the mapping Φ is a contraction for $\|\mu\|_{L^\infty(\Gamma_3)}<\mu_c$, thus it admits a unique fixed point η_c and $(u_{\eta_c},\beta_{\eta_c})$ is a unique solution to Problem P_n^i.

Now, to prove Theorem 2.1 it is necessary to establish the following estimates.

Lemma 3.4. *There exist two constants $C_3>0$, $C_4>0$ such that*

$$\|u^{i+1}\|_V\le C_3\|f^{i+1}\|_V,\quad\|\Delta u^i\|_V\le C_4(\|\Delta f^i\|_V+\Delta t). \tag{3.8}$$

Proof. We take $v=0$ in (3.1) to deduce

$$\begin{aligned}&\Big(F\varepsilon(u^{i+1}),\varepsilon(u^{i+1})\Big)_Q\\&\le j(u^{i+1},-u^i)-j(u^{i+1},\Delta u^i)+r(\beta^{i+1},u^{i+1},-u^{i+1})+(f^{i+1},u^{i+1})_V.\end{aligned}$$

Using the properties of j we have

$$j(u^{i+1},-u^i)-j(u^{i+1},\Delta u^i)\le d_\Omega^2L_p\|\mu\|_{L^\infty(\Gamma_3)}\|u^{i+1}\|_V^2.$$

On the other hand using $|R(u_\nu)|\le L$, $|R(u_\tau)|\le L$, and the relation (2.11), we have

$$|r(\beta^{i+1},u^{i+1},-u^{i+1})|\le d_\Omega L\sqrt{\operatorname{meas}\Gamma_3}\big(\|c_\nu\|_{L^\infty(\Gamma_3)}+\|c_\tau\|_{L^\infty(\Gamma_3)}\big)\|u^{i+1}\|_V$$

Using (2.14)(b), we get

$$\begin{aligned}m\|u^{i+1}\|_V^2&\le d_\Omega^2L_p\|\mu\|_{L^\infty(\Gamma_3)}\|u^{i+1}\|_V^2.+\|f^{i+1}\|_V\|u^{i+1}\|_V\\&\quad+d_\Omega L\sqrt{\operatorname{meas}\Gamma_3}(\|c_\nu\|_{L^\infty(\Gamma_3)}+\|c_\tau\|_{L^\infty(\Gamma_3)})\|u^{i+1}\|_V.\end{aligned}$$

Therefore, if we take

$$\mu_*=\min(\mu_c,\frac{m}{L_pd_\Omega^2}),$$

we deduce that for

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_*, \tag{3.9}$$

there exists a constant $C_3 > 0$ such that the first inequality holds. To show the second inequality (3.8) we consider the translated inequality to relation (3.1) at time t_i; that is,

$$\begin{aligned}(F\varepsilon(u^i), \varepsilon(w - u^i))_Q + j(u^i, w - u^{i-1}) - j(u^i, u^i - u^{i-1}) + r(\beta^i, u^i, w - u^i) \\ \geq (f^i, w - u^i)_V \quad \forall w \in V.\end{aligned} \tag{3.10}$$

Taking $w = u^i$ in the inequality to relation (3.1) and $w = u^{i+1}$ in the inequality (3.10) and adding the two inequalities, we obtain the inequality

$$\begin{aligned}&-(F\varepsilon(u^{i+1}) - F\varepsilon(u^i), \varepsilon(\Delta u^i))_Q - j(u^{i+1}, \Delta u^i) + j(u^i, u^{i+1} - u^{i-1}) \\ &- j(u^i, u^i - u^{i-1}) + r(\beta^{i+1}, u^{i+1}, u^i - u^{i+1}) + r(\beta^i, u^i, u^{i+1} - u^i) \\ &\geq (-\Delta f^i, \Delta u^i)_V.\end{aligned}$$

Then using the inequality

$$\big||u_\tau^{t_{i+1}} - u_\tau^{t_{i-1}}| - |u_\tau^{t_i} - u_\tau^{t_{i-1}}|\big| \leq |u_\tau^{t_{i+1}} - u_\tau^{t_i}|,$$

we have

$$j(u^i, u^{i+1} - u^{i-1}) - j(u^i, u^i - u^{i-1}) \leq \ j(u^i, \Delta u^i).$$

Therefore,

$$\begin{aligned}(F\varepsilon(u^{i+1}) - F\varepsilon(u^i), \varepsilon(\Delta u^i))_Q &\leq j(u^i, \Delta u^i) - j(u^{i+1}, \Delta u^i) + r(\beta^{i+1}, u^{i+1}, -\Delta u^i) \\ &\quad + r(\beta^i, u^i, \Delta u^i) + (\Delta f^i, \Delta u^i)_V.\end{aligned} \tag{3.11}$$

Using (2.11), (2.15)(b) and (2.15)(c), it follows that

$$j(u^{i+1}, \Delta u^i) - j(u^i, \Delta u^i) \leq \|\mu\|_{L^\infty(\Gamma_3)} L_p d_\Omega^2 \|\Delta u^i\|_V^2.$$

Moreover, using (2.11), $|R_\nu(u^j)| \leq L$, $|R_\tau(u^j)| \leq L$, $j = i, i+1$, and (3.5), we have

$$\begin{aligned}&r(\beta^{i+1}, u^{i+1}, -\Delta u^i) + r(\beta^i, u^i, \Delta u^i) \\ &\leq L d_\Omega(\|c_\nu\|_{L^\infty(\Gamma_3)} + \|c_\tau\|_{L^\infty(\Gamma_3)})\|\Delta u^i\|_V \|\Delta\beta^i\|_{L^2(\Gamma_3)}.\end{aligned}$$

On the other hand,

$$\|\Delta\beta^i\|_{L^2(\Gamma_3)} \leq \Delta t d_1,$$

where $d_1 > 0$. Combining the previous relations, we obtain from inequality (3.11) that for the same condition (3.9), there exists a constant $C_4 > 0$ such that

$$\|\Delta u^i\|_V \leq C_4(\|\Delta f^i\|_V + \Delta t).$$

□

4. Existence

In this section we prove our main result, Theorem 2.1. We consider the sequences of functions (u^n), (β^n) defined on $[0, T]$ by

$$u^n(t) = u^i + \frac{(t - t_i)}{\Delta t}\Delta u^i, \quad \beta^n(t) = \beta^i + \frac{(t - t_i)}{\Delta t}\Delta\beta^i$$

for $t \in [t_i, t_{i+1}]$, $i = 0, \dots, n-1$. As in [21, Proposition 4.2] we have the following lemma.

Lemma 4.1. *There exists $u \in W^{1,\infty}(0,T;V)$ and a subsequence (u^n), still denoted (u^n), such that*
$$u^n \to u \quad \text{weak} * \text{ in } W^{1,\infty}(0,T;V).$$

Proof. From (3.5) it follows that there exists a constant $C_5 > 0$ such that
$$\|u^n\|_{W^{1,\infty}(0,T;V)} \le C_5(\|f\|_{W^{1,\infty}(0,T;V)} + 1)$$
Consequently the sequence (u^n) is bounded in $W^{1,\infty}(0,T;V)$. Therefore, there exists a function $u \in W^{1,\infty}(0,T;V)$ and a subsequence, still denoted by (u^n), such that
$$u^n \to u \quad \text{weak} * \text{ in } W^{1,\infty}(0,T;V) \quad \text{as } n \to \infty.$$
□

Remark 4.2. As $W^{1,\infty}(0,T;V) \hookrightarrow C([0,T];V)$ we have $u^n(t) \to u(t)$ weakly in V for all $t \in [0,T]$.

Now let us introduce the sequences of functions $(\widetilde{u}^n)$, $(\tilde{f}^n)$, $(\tilde{\beta}^n)$ defined on $[0,T]$ by
$$\widetilde{u}^n(t) = u^{i+1}, \quad \widetilde{f}^n(t) = f(t_{i+1}), \quad \tilde{\beta}^n(t) = \beta^{i+1}$$
for $t \in (t_i, t_{i+1}]$, $i = 0, \dots, n-1$ and $\widetilde{u}^n(0) = u_0$, $\widetilde{f}^n(0) = f(0)$, $\tilde{\beta}^n(0) = \beta_0$. As in [21] we have the following result.

Lemma 4.3. *Passing to a subsequence again denoted $(\tilde{u}^n)$ we have*
- (i) $\widetilde{u}^n \to u$ *weak* $*$ *in* $L^\infty(0,T;V)$,
- (ii) $\widetilde{u}^n(t) \to u(t)$ *weakly in* V *a.e.* $t \in [0,T]$,
- (iii) $\widetilde{u}^n \to u$ *strongly in* $L^2(0,T;V)$.

Proof. For (i) and (ii) we refer the reader to [21, lemma 4.3]. For (iii) it suffices to give only some partial proof. Indeed, from the inequality of the relation (3.1) we deduce the inequality
$$\begin{aligned}
&(F\varepsilon(u^{i+1}), \varepsilon(w - u^{i+1}))_Q + j(u^{i+1}, w - u^{i+1}) + r(\beta^{i+1}, u^{i+1}, w - u^{i+1}) \\
&\ge (f^{i+1}, w - u^{i+1})_V \quad \forall w \in V,
\end{aligned}$$
which implies the inequality
$$\begin{aligned}
&(F\varepsilon(\tilde{u}^n(t)), \varepsilon(w - \tilde{u}^n(t)))_Q + j(\tilde{u}^n(t), w - \tilde{u}^n(t)) \\
&\quad + r\Big(\tilde{\beta}^n(t), \tilde{u}^n(t), w - \tilde{u}^n(t)\Big) \\
&\ge (\tilde{f}^n(t), w - \tilde{u}^n(t))_V \quad \forall w \in V, \quad \text{a.e. } t \in [0,T].
\end{aligned} \tag{4.1}$$
To show the strong convergence, we take $w = \tilde{u}^{n+m}(t)$ in (4.1) and $v = \tilde{u}^n(t)$ in the same inequality satisfied by $\tilde{u}^{n+m}(t)$, and adding the two inequalities, we obtain by using (2.15)(c)
$$\begin{aligned}
&(F\varepsilon(\tilde{u}^{n+m}(t)) - F\varepsilon(\tilde{u}^n(t)), \varepsilon(\tilde{u}^{n+m}(t) - \tilde{u}^n(t)))_Q \\
&\le \int_{\Gamma_3} \mu(p(\tilde{u}^{n+m}_\nu(t)) + p(\tilde{u}^n_\nu(t)))|\tilde{u}^{n+m}(t) - \tilde{u}^n(t)|da \\
&\quad + r\Big(\tilde{\beta}^n(t), \tilde{u}^n(t), \tilde{u}^{n+m}(t) - \tilde{u}^n(t)\Big) \\
&\quad + r\Big(\tilde{\beta}^{n+m}(t), \tilde{u}^{n+m}(t), \tilde{u}^n(t) - \tilde{u}^{n+m}(t)\Big), \quad \text{a.e. } t \in [0,T].
\end{aligned}$$

Using (2.15)(b) and (2.11) we deduce that there exists a constant $C_6 > 0$ such that

$$\begin{aligned}&\int_{\Gamma_3} \mu(p(\tilde{u}_\nu^{n+m}(t)) + p(\tilde{u}_\nu^n(t)))|\tilde{u}_\tau^{n+m}(t) - \tilde{u}_\tau^n(t)|da \\ &\le C_6\|\tilde{u}_\tau^{n+m}(t) - \tilde{u}_\tau^n(t)\|_{(L^2(\Gamma_3))^d}.\end{aligned}$$

In the same away there exists a constant $C_7 > 0$ such that

$$\begin{aligned}&r\Big(\tilde{\beta}^n(t), \tilde{u}^n(t), \tilde{u}^{n+m}(t) - \tilde{u}^n(t)\Big) \\ &+ r\Big(\tilde{\beta}^{n+m}(t), \tilde{u}^{n+m}(t), \tilde{u}^n(t) - \tilde{u}^{n+m}(t)\Big) \\ &\le C_7\Big(\|\tilde{u}_\tau^{n+m}(t) - \tilde{u}_\tau^n(t)\|_{(L^2(\Gamma_3))^d} + \|\tilde{u}_\nu^{n+m}(t) - \tilde{u}_\nu^n(t)\|_{L^2(\Gamma_3)}\Big).\end{aligned}$$

Using (2.14)(b) it follows that there exists a constant $C_8 > 0$ such that

$$\begin{aligned}&m\|\tilde{u}^{n+m}(t) - \tilde{u}^n(t)\|_V^2 \\ &\le C_8\Big(\|\tilde{u}_\tau^{n+m}(t) - \tilde{u}_\tau^n(t)\|_{(L^2(\Gamma_3))^d} + \|\tilde{u}_\nu^{n+m}(t) - \tilde{u}_\nu^n(t)\|_{L^2(\Gamma_3)}\Big), \quad \text{a.e. } t \in [0,T].\end{aligned}$$

Now, to complete the proof we refer the reader to [21, Proposition 4.5]. □

Next, we consider the problem.

Problem P_a. Find a bonding field $\beta : [0,T] \to L^\infty(\Gamma_3)$ such that

$$\dot{\beta}(t) = -[\beta(t)(c_\nu(R_\nu(u_\nu(t)))^2 + c_\tau(|R_\tau(u_\tau(t))|)^2) - \varepsilon_a]_+ \quad \text{a.e. } t \in (0,T),$$
$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3,$$

where u is a weak limit founded in Lemma 4.1. We have the following result.

Proposition 4.4. *There exists a unique solution to Problem P_a and it satisfies*

$$\beta \in W^{1,\infty}(0,T; L^2(\Gamma_3)) \cap B.$$

Proof. As in [5] let $k > 0$ and

$$X = \big\{\beta \in C([0,T]; L^2(\Gamma_3)); \quad \sup_{t\in[0,T]}[\exp(-kt)\|\beta(t)\|_{L^2(\Gamma_3)}] < +\infty\big\}.$$

X is a Banach space with the norm

$$\|\beta\|_X = \sup_{t\in[0,T]}[\exp(-kt)\|\beta(t)\|_{L^2(\Gamma_3)}],$$

and consider the mapping $T : X \to X$ given by

$$T\beta(t) = \beta_0 - \int_0^t [\beta(s)(c_\nu(R_\nu(u_\nu(s)))^2 + c_\tau(|R_\tau(u_\tau(s))|)^2) - \varepsilon_a]_+ ds.$$

Then there exists a constant $c_1' > 0$ such that

$$\begin{aligned}&|T\beta_1(t) - T\beta_2(t)|^2 \\ &\le c_1'\int_0^t (c_\nu(R_\nu(u_\nu(s)))^2 + c_\tau|R_\tau(u_\tau(s))|^2)(\beta_1(s) - \beta_2(s))^2 ds.\end{aligned}$$

Using $|R_\nu(u_\nu(s))| \le L$, $|R_\tau(u_\tau(s))| \le L$, it follows that

$$\begin{aligned}\|T\beta_1(t) - T\beta_2(t)\|^2_{L^2(\Gamma_3)} &\le c_2' \int_0^t \|\beta_1(s) - \beta_2(s)\|^2_{L^2(\Gamma_3)} ds \\ &\le c_2' \|\beta_1 - \beta_2\|_X^2 \frac{\exp(2kt)}{2k},\end{aligned}$$

for some constant $c_2' > 0$. So we obtain

$$\|T\beta_1 - T\beta_2\|_X \le \sqrt{\frac{c_2'}{2k}} \|\beta_1 - \beta_2\|_X,$$

and then for k sufficiently large T has a unique fixed point β. To show that $\beta \in [0,1]$ for all $t \in [0,T]$, a.e. on Γ_3, we refer the reader to [19, Remark 3.1]. □

Next, we prove a convergence result.

Lemma 4.5. *Let β be the unique solution to Problem P_a. Then we have:*

(i) *$\beta^n \to \beta$ strongly in $L^\infty(0,T; L^2(\Gamma_3))$,*

(ii) *$\tilde\beta^n \to \beta$ strongly in $L^\infty(0,T; L^2(\Gamma_3))$.*

Proof. (i), Since $\dot\beta^n(t) = \frac{\Delta\beta^i}{\Delta t}$, for all $t \in]t_i, t_{i+1}[$, we have

$$\beta^n(t) = \beta^i - \int_{t_i}^t [\tilde\beta^n(s)(c_\nu(R_\nu(\tilde u_\nu^n(s)))^2 + c_\tau(|R_\tau(\tilde u_\tau^n(s))|)^2) - \varepsilon_a]_+ ds,$$

$$\beta(t) = \beta(t_i) - \int_{t_i}^t [\beta(s)(c_\nu(R_\nu(u_\nu(s)))^2 + c_\tau(|R_\tau(u_\tau(s))|)^2) - \varepsilon_a]_+ ds.$$

Then

$$\begin{aligned}\beta^n(t) - \beta(t) = \beta^i - \beta(t_i) &- \int_{t_i}^t [\tilde\beta^n(s)(c_\nu(R_\nu(\tilde u_\nu^n(s)))^2 + c_\tau(|R_\tau(\tilde u_\tau^n(s))|)^2) - \varepsilon_a]_+ ds \\ &+ \int_{t_i}^t [\beta(s)(c_\nu(R_\nu(u_\nu(s)))^2 + c_\tau(|R_\tau(u_\tau(s))|)^2) - \varepsilon_a]_+ ds.\end{aligned}$$

Thus,

$$\begin{aligned}\|\beta^n(t) - \beta(t)\|_{L^2(\Gamma_3)} &\le \|\beta^i - \beta(t_i)\|_{L^2(\Gamma_3)} \\ &\quad + \int_0^t \|\tilde\beta^n(s)\Big(c_\nu(R_\nu(\tilde u_\nu^n(s)))^2 + c_\tau(|R_\tau(\tilde u_\tau^n(s))|)^2\Big) \\ &\quad - \beta(s)\Big(c_\nu(R_\nu(u_\nu(s)))^2 + c_\tau(|R_\tau(u_\tau(s))|)^2\Big)\|_{L^2(\Gamma_3)} ds.\end{aligned}$$

Using the properties of R_l, $l = \nu, \tau$ (see [18]) such that $|R_l(u_l)| \le L$ and (3.4), we have

$$\begin{aligned}&\|c_\nu\tilde\beta^n(s)(R_\nu(\tilde u_\nu^n(s)))^2 - c_\nu\beta(s)(R_\nu(u_\nu(s)))^2\|_{L^2(\Gamma_3)} \\ &\le \Big\|\tilde\beta^n(s)c_\nu\Big((R_\nu(\tilde u_\nu^n(s)))^2 - (R_\nu(u_\nu(s)))^2\Big) \\ &\quad + \Big(\tilde\beta^n(s) - \beta^n(s)\Big)c_\nu(R_\nu(u_\nu(s)))^2\Big\|_{L^2(\Gamma_3)} \\ &\quad + \|(\beta^n(s) - \beta(s))c_\nu(R_\nu(u_\nu(s)))^2\|_{L^2(\Gamma_3)} \\ &\le 2L\|c_\nu\|_{L^\infty(\Gamma_3)}\|\tilde u_\nu^n(s) - u_\nu(s)\|_{L^2(\Gamma_3)} \\ &\quad + L^2\|c_\nu\|_{L^\infty(\Gamma_3)}\Delta t c_1' + L^2\|c_\nu\|_{L^\infty(\Gamma_3)}\|\beta^n(s) - \beta(s)\|_{L^2(\Gamma_3)}.\end{aligned}$$

Also we have

$$
\begin{aligned}
&\|\tilde\beta^n(s)c_\tau(|R_\tau(\tilde u^n_\nu(s))|)^2 - c_\tau\beta(t)(|R_\tau(u_\tau(s))|)^2\|_{L^2(\Gamma_3)}\\
&\le \|\tilde\beta^n(s)c_\tau\Big((|R_\tau(\tilde u^n_\tau(s))|)^2 - (|R_\tau(u_\tau(s))|)^2\Big)\\
&\quad + \Big(\tilde\beta^n(s) - \beta^n(s)\Big)c_\tau(|R_\tau(u_\tau(s))|)^2\|_{L^2(\Gamma_3)}\\
&\quad + \|(\beta^n(s) - \beta(s))c_\tau(|R_\tau(u_\tau(s))|)^2\|_{L^2(\Gamma_3)}\\
&\le 2L\|c_\tau\|_{L^\infty(\Gamma_3)}\|\tilde u^n_\tau(s) - u_\tau(s)\|_{(L^2(\Gamma_3))^d}\\
&\quad + L^2\|c_\tau\|_{L^\infty(\Gamma_3)}\Delta t c_1' + L^2\|c_\tau\|_{L^\infty(\Gamma_3)}\|\beta^n(s) - \beta(s)\|_{L^2(\Gamma_3)}.
\end{aligned}
$$

From the above inequalities, we deduce

$$
\begin{aligned}
&\|\beta^n(t) - \beta(t)\|_{L^2(\Gamma_3)}\\
&\le \|\beta^i - \beta(t_i)\|_{L^2(\Gamma_3)} + 2L\Big(\|c_\nu\|_{L^\infty(\Gamma_3)}\int_0^t \|\tilde u^n_\nu(s) - u_\nu(s)\|_{L^2(\Gamma_3)}ds\\
&\quad + \|c_\tau\|_{L^\infty(\Gamma_3)}\int_0^t \|\tilde u^n_\tau(s) - u_\tau(s)\|_{(L^2(\Gamma_3))^d}ds\Big)\\
&\quad + L^2\Big(\|c_\nu\|_{L^\infty(\Gamma_3)} + \|c_\tau\|_{L^\infty(\Gamma_3)}\Big)\int_0^t \|\beta^n(s) - \beta(s)\|_{L^2(\Gamma_3)}ds\\
&\quad + \Big(\|c_\nu\|_{L^\infty(\Gamma_3)} + \|c_\tau\|_{L^\infty(\Gamma_3)}\Big)TL^2\Delta t c_1'.
\end{aligned}
$$

Now using a Gronwall-type argument it follows that there exists a constant $C_9 > 0$ such that

$$
\begin{aligned}
\|\beta^n(t) - \beta(t)\|_{L^2(\Gamma_3)} \le C_9\Big(&\|\beta^i - \beta(t_i)\|_{L^2(\Gamma_3)} + \int_0^t \Big(\|\tilde u^n_\nu(s) - u_\nu(s)\|_{L^2(\Gamma_3)}\\
&+ \|\tilde u^n_\tau(s) - u_\tau(s)\|_{(L^2(\Gamma_3))^d}\Big)ds + \Delta t\Big).
\end{aligned}
$$

Using (2.11), the above inequality implies

$$
\begin{aligned}
&\max_{t\in[0,T]}\|\beta^n(t) - \beta(t)\|_{L^2(\Gamma_3)}\\
&\le C_9\Big(\max_{i=0,\dots,n}\|\beta^i - \beta(t_i)\|_{L^2(\Gamma_3)} + 2d_\Omega\int_0^T \|\tilde u^n(s) - u(s)\|_V ds + \Delta t\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\max_{t\in[0,T]}\|\beta^n(t) - \beta(t)\|_{L^2(\Gamma_3)}\\
&\le C_9\Big(\max_{i=0,\dots,n}\|\beta^i - \beta(t_i)\|_{L^2(\Gamma_3)} + 2d_\Omega\sqrt{T}\|\tilde u^n - u\|_{L^2(0,T;V)} + \Delta t\Big).
\end{aligned}
$$

As in [5, Lemma 3.5], we still have

$$
\lim_{n\to+\infty}\max_{i=0,\dots,n}\|\beta^i - \beta(t_i)\|_{L^2(\Gamma_3)} = 0\,.
$$

Using (iii) of Lemma 4.3, one obtains

$$
\lim_{n\to+\infty}\max_{t\in[0,T]}\|\beta^n(t) - \beta(t)\|_{L^2(\Gamma_3)} = 0\,.
$$

So (i) is proved. To prove (ii) it suffices to remark that there exists a constant $C_{10} > 0$ such that

$$\begin{aligned}\|\tilde{\beta}^n(t) - \beta(t)\|_{L^2(\Gamma_3)} &\le \|\tilde{\beta}^n(t) - \beta^n(t)\|_{L^2(\Gamma_3)} + \|\beta^n(t) - \beta(t)\|_{L^2(\Gamma_3)} \\ &\le C_{10}\Delta t + \|\beta^n(t) - \beta(t)\|_{L^2(\Gamma_3)}.\end{aligned}$$

□

Now we have all the ingredients to prove the following proposition.

Proposition 4.6. *(u, β) is a solution to Problem P_2.*

Proof. In the inequality (3.1), for $v \in V$ set $w = u^i + v\Delta t$ and divide by Δt; we obtain the inequality

$$\begin{aligned}&(F\varepsilon(u^{i+1}), \varepsilon(v - \frac{\Delta u^i}{\Delta t}))_Q + j(u^{i+1}, v) - j(u^{i+1}, \frac{\Delta u^i}{\Delta t}) + r(\beta^{i+1}, u^{i+1}, v - \frac{\Delta u^i}{\Delta t}) \\ &\ge (f^{i+1}, v - \frac{\Delta u^i}{\Delta t})_V\end{aligned}$$

Whence for any $v \in L^2(0, T; V)$, we have

$$\begin{aligned}&(F\varepsilon(\widetilde{u}^n(t)), \varepsilon(v(t) - \dot{u}^n(t)))_Q + j(\widetilde{u}^n(t), v(t)) - j(\widetilde{u}^n(t), \dot{u}^n(t)) \\ &+ r(\tilde{\beta}^n(t), \widetilde{u}^n(t), v(t) - \dot{u}^n(t)) \\ &\ge (\widetilde{f}^n(t), v(t) - \dot{u}^n(t))_V \quad \text{a.e. } t \in [0, T].\end{aligned}$$

Integrating both sides of the above inequality on $(0, T)$, we obtain

$$\begin{aligned}&\int_0^T (F\varepsilon(\widetilde{u}^n(t)), \varepsilon(v(t) - \dot{u}^n(t)))_Q dt + \int_0^T j(\widetilde{u}^n(t), v(t)) dt \\ &- \int_0^T j(\widetilde{u}^n(t), \dot{u}^n(t)) dt + \int_0^T r\Big(\tilde{\beta}^n(t), \widetilde{u}^n(t), v(t) - \dot{u}^n(t)\Big) dt \\ &\ge \int_0^T (\widetilde{f}^n(t), v(t) - \dot{u}^n(t))_V dt.\end{aligned} \tag{4.2}$$

To pass in the limit in this inequality we need to establish the following properties. □

Lemma 4.7. *We have the following properties for $v \in L^2(0, T; V)$:*

$$\lim_{n\to\infty} \int_0^T (F\varepsilon(\widetilde{u}^n(t)), \varepsilon(v(t) - \dot{u}^n))_Q dt = \int_0^T (F\varepsilon(u(t)), \varepsilon(v(t) - \dot{u}(t)))_Q dt, \tag{4.3}$$

$$\liminf_{n\to\infty} \int_0^T j(\widetilde{u}^n(t), \dot{u}^n(t)) dt \ge \int_0^T j(u(t), \dot{u}(t)) dt, \tag{4.4}$$

$$\lim_{n\to\infty} \int_0^T j(\widetilde{u}^n(t), v(t)) dt = \int_0^T j(u(t), v(t)) dt, \tag{4.5}$$

$$\lim_{n\to\infty} \int_0^T (\widetilde{f}^n(t), v(t) - \dot{u}^n(t))_V dt = \int_0^T (f(t), v(t) - \dot{u}(t))_V dt, \tag{4.6}$$

$$\lim_{n\to\infty} \int_0^T r(\tilde{\beta}^n(t), u^n(t), v(t) - \dot{u}^n(t)) dt = \int_0^T r(\beta(t), u(t), v(t) - \dot{u}(t)) dt. \tag{4.7}$$

Proof. For the proof of (4.3) and (4.6) we refer the reader to [21, Lemma 4.6]. To prove (4.4), it suffices to see [12, Lemma 3.5]. To prove (4.5), it suffices to use (iii) of Lemma 4.3. Finally for the proof of (4.7) we refer the reader to [5, Lemma 3.8] and use the properties (3.4).

Now using lemma 4.5 (ii) and Lemma 4.7 we pass to the limit as $n \to +\infty$ in the inequality (4.2) to obtain

$$\begin{aligned}&\int_0^T (F\varepsilon(u(t)), \varepsilon(v(t) - \dot{u}(t)))_Q dt + \int_0^T j(u(t), v(t))dt\\&- \int_0^T j(u(t), \dot{u}(t))dt + \int_0^T r(\beta(t), u(t), v(t) - \dot{u}(t))dt\\&\geq \int_0^T (f(t), v(t) - \dot{u}(t))_V dt,\end{aligned}$$

from which we deduce the inequality (2.20) and also that β is the unique solution of the differential equation (2.21). □

Remark 4.8. We can consider another quasistatic frictional contact problem with adhesion. In Problem P_1 the contact conditions on Γ_3 (2.5) and (2.6) are modified as follows.

$$-\sigma_\nu = p(u_\nu) - c_\nu \beta^2 R_\nu(u_\nu) \quad \text{on } \Gamma_3 \times (0,T),$$

$$\begin{cases} |\sigma_\tau + c_\tau \beta^2 R_\tau(u_\tau)| \leq \mu |p(u_\nu) - c_\nu \beta^2 R_\nu(u_\nu)|, \\ |\sigma_\tau + c_\tau \beta^2 R_\tau(u_\tau)| < \mu |p(u_\nu) - c_\nu \beta^2 R_\nu(u_\nu)| \Longrightarrow \dot{u}_\tau = 0, \\ |\sigma_\tau + c_\tau \beta^2 R_\tau(u_\tau)| = \mu |p(u_\nu) - c_\nu \beta^2 R_\nu(u_\nu)| \Longrightarrow \\ \exists \lambda \geq 0 \text{ such that } \dot{u}_\tau = -\lambda(\sigma_\tau + c_\tau \beta^2 R_\tau(u_\tau)), \end{cases} \quad \text{on } \Gamma_3 \times (0,T),$$

Using the new contact conditions, as in Problem P_2 the corresponding variational problem is written with the functional $j : V \times V \to \mathbb{R}$ defined by

$$j(u,v) = \int_{\Gamma_3} (p(u_\nu)v_\nu + \mu |p(u_\nu) - c_\nu \beta^2 R_\nu(u_\nu)||v_\tau|)da \quad \forall u, v \in V.$$

In the same away we show that if there exists a constant $\mu_* > 0$, this problem admits at least one solution for

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_*.$$

Conclusion. In this paper we have studied a mathematical model which describes a quasistatic frictional contact problem with adhesion for nonlinear elastic materials. The adhesive and frictional contact is modelled with a normal compliance condition and the associated version of Coulomb's law of dry friction. An existence result of a weak solution was proved under a smallness assumption of the friction coefficient. Finally, we note that the important question of uniqueness of the solution is not resolved here, and remains still open.

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References

[1] H. Brezis; *Equations et inéquations non linéaires dans les espaces vectoriels en dualité.* Annales Inst. Fourier, 18, 115-175, 1968.

[2] L. Cangémi; *Frottement et adhérence: modèle, traitement numérique et application à l'interface fibre/matrice,* Ph. D. Thesis, Univ. Méditerranée, Aix Marseille I, 1997.

[3] O. Chau, J. R. Fernandez, M. Shillor and M. Sofonea; Variational and numerical analysis of a quasistatic viscoelastic contact problem with adhesion, *Journal of Computational and Applied Mathematics* 159 (2003), 431-465.

[4] O. Chau, M. Shillor and M. Sofonea; Dynamic frictionless contact with adhesion, *J. Appl. Math. Phys.* (ZAMP) 55 (2004), 32-47.

[5] M. Cocou and R. Rocca; Existence results for unilateral quasistatic contact problems with friction and adhesion, *Math. Model. Num. Anal.* 34 (2000), 981-1001.

[6] M. Cocu, E. Pratt, M. Raous; Formulation and approximation of quasistatic frictional contact, Int. J. Engng Sc.,34, 7, 783-798, 1996.

[7] G. Duvaut, J-L Lions; *Les inéquations en mécanique et en physique*, Dunod, Paris, 1972.

[8] J. R. Fernandez, M. Shillor and M. Sofonea; Analysis and numerical simulations of a dynamic contact problem with adhesion, *Math. Comput. Modelling*, 37 (2003) 1317-1333.

[9] M. Frémond; Adhérence des solides; *J. Mécanique Théorique et Appliquée*, 6, 383-407 (1987).

[10] M. Frémond; Equilibre des structures qui adhèrent à leur support, *C. R. Acad. Sci. Paris*, 295, série II, 913-916 (1982).

[11] Frémond; *Non smooth Thermomechanics*, Springer, Berlin 2002.

[12] M. Motreanu and M. Sofonea; Evolutionary variational inequalities arising in quasistatic frictional contact problems for elastic materials, Abstract and Analysis, vol. 4, No. 4, 255-279, 1999.

[13] S.A. Nassar, T. Andrews, S. Kruk, and M. Shillor; *Modelling and simulations of a bonded rod*, Math. Comput. Modelling, 42 (2005), 553-572.

[14] M. Raous, L. Cangémi and M. Cocu; A consistent model coupling adhesion, friction, and unilateral contact, *Comput. Meth. Appl. Mech. Engng.* 177 (1999), 383-399.

[15] M. Raous, Jean M., and J. J. Moreau (eds.); *Contact Mechanics,* Plenum Press, New York, 1995.

[16] J. Rojek and J. J. Telega; Contact problems with friction, adhesion and wear in orthopeadic biomechanics. I: General developements, *J. Theor. Appl. Mech.*, 39 (2001), 655-677.

[17] M. Shillor, M. Sofonea, and J. J. Telega; *Models and Variational Analysis of Quasistatic Contact,* Lecture Notes Physics, vol.655, Springer, Berlin, 2004.

[18] M. Sofonea, W. Han, and M. Shillor; *Analysis and Approximations of Contact Problems with Adhesion or Damage*, Pure and Applied Mathematics 276, Chapman & Hall / CRC Press, Boca Raton, Florida, 2006.

[19] M. Sofonea and T. V. Hoarau-Mantel; *Elastic frictionless contact problems with adhesion*, *Adv. Math. Sci. Appl.*, 15 (2005), No. 1, 49-68.

[20] M. Sofonea, R. Arhab, and R. Tarraf; Analysis of electroelastic frictionless contact problems with adhesion, *Journal of Applied Mathematics*, vol. 2006, ID 64217, pp.1-25.

[21] A. Touzaline, A. Mignot; Existence of solutions for quasistatic problems of unilateral contact with nonlocal friction for nonlinear elastic materials, Electronic Journal of Differential Equations, 2005 (2005), No. 99, pp. 1-13.

Addendum posted on January 8, 2009.

The author wants to correct some misprints:

- Page 7: In the first displayed inequality, the norm of β has been included:

$$|(Au - Av, w)_V| \leq \big[M + \big(\|c_\nu\|_{L^\infty(\Gamma_3)} + \|c_\tau\|_{L^\infty(\Gamma_3)}\big)\|\beta\|^2_{L^\infty(\Gamma_3)} d_\Omega^2\big]\|u - v\|_V \|w\|_V$$

- Page 7, Problem $P_{2\beta}$: In the second part of the definition of β, > 0 has been replaced by ≥ 0:

$$\beta = \begin{cases} \beta^i, & \text{if } (c_\nu(R_\nu(u_{\beta\nu}))^2 + c_\tau(|R_\tau(u_{\beta\tau})|)^2)\beta^i - \varepsilon_a < 0, \\ \dfrac{\beta^i + \varepsilon_a \Delta t}{1 + \Delta t(c_\nu(R_\nu(u_{\beta\nu}))^2 + c_\tau(|R_\tau(u_{\beta\tau})|)^2)}, \\ \text{if } (c_\nu(R_\nu(u_{\beta\nu}))^2 + c_\tau(|R_\tau(u_{\beta\tau})|)^2)\beta^i - \varepsilon_a \geq 0, \end{cases}$$

- Page 9, Lemma 3.4: +1 has been attached to the right-hand side of the first inequality:

$$\|u^{i+1}\|_V \leq C_3\big(\|f^{i+1}\|_V + 1\big), \quad \|\Delta u^i\|_V \leq C_4(\|\Delta f^i\|_V + \Delta t). \tag{3.8}$$

- Page 11, in the last displayed inequality: The second and third terms on the right-hand side have been modified as follows:

$$\begin{aligned} &(F\varepsilon(\tilde{u}^{n+m}(t)) - F\varepsilon(\tilde{u}^n(t)), \varepsilon(\tilde{u}^{n+m}(t) - \tilde{u}^n(t)))_Q \\ &\leq \int_{\Gamma_3} \mu(p(\tilde{u}_\nu^{n+m}(t)) + p(\tilde{u}_\nu^n(t)))|\tilde{u}^{n+m}(t) - \tilde{u}^n(t)|da \\ &\quad + r\Big(\tilde{\beta}^{n+m}(t), \tilde{u}^{n+m}(t), \tilde{u}^n(t) - \tilde{u}^{n+m}(t)\Big) \\ &\quad + \Big(\tilde{f}^{n+m}(t) - \tilde{f}^n(t), \tilde{u}^{n+m}(t) - \tilde{u}^n(t)\Big)_V, \quad \text{a.e. } t \in [0,T]. \end{aligned}$$

- Page 12, lines 9 and 10, after "Using (2.14)(b) it follows that there exists a constant $C_8 > 0$ such that": Replace the displayed inequality by

$$\begin{aligned} \|\tilde{u}^{n+m}(t) - \tilde{u}^n(t)\|_V^2 \leq C_8\Big(&\|\tilde{u}_\tau^{n+m}(t) - \tilde{u}_\tau^n(t)\|_{(L^2(\Gamma_3))^d} + \|\tilde{u}_\nu^{n+m}(t) - \tilde{u}_\nu^n(t)\|_{L^2\Gamma_3)} \\ &+ \|\tilde{f}^{n+m}(t) - \tilde{f}^n(t)\|_V^2\Big), \quad \text{a.e. } t \in [0,T]. \end{aligned}$$

End of addendum.

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