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LARGE TIME BEHAVIOR FOR SOLUTIONS OF NONLINEAR PARABOLIC PROBLEMS WITH SIGN-CHANGING MEASURE DATA

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Abstract. Let $\Omega \subseteq \mathbb{R}^N$ a bounded open set, $N \geq 2$, and let $p > 1$; in this paper we study the asymptotic behavior with respect to the time variable t of the entropy solution of nonlinear parabolic problems whose model is

$$\begin{aligned} u_t(x,t) - \Delta_p u(x,t) &= \mu \quad \text{in } \Omega \times (0,\infty), \\ u(x,0) &= u_0(x) \quad \text{in } \Omega, \end{aligned}$$

where $u_0 \in L^1(\Omega)$, and $\mu \in \mathcal{M}_0(Q)$ is a measure with bounded variation over $Q = \Omega \times (0,\infty)$ which does not charge the sets of zero p-capacity; moreover we consider μ that does not depend on time. In particular, we prove that solutions of such problems converge to stationary solutions.

1. Introduction

A large number of papers was devoted to the study of asymptotic behavior for solution of parabolic problems under various assumptions and in different contexts: for a review on classical results see [10, 1, 21], and references therein. More recently in [11] the same problem was studied for bounded data and a class of operators rather different to the one we will discuss.

Moreover, in [13] and [16] it was used an approach similar to our one, to face, respectively, the quasilinear case with natural growth terms of the type $g(u)|\nabla u|^2$ and the linear case with general measure data. While the same problem was studied in [17] for nonnegative data. Here we want to generalize this result to changing sign measure data.

Let $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function (i.e. $a(\cdot,\xi)$ is measurable on Ω, for all $\xi \in \mathbb{R}^N$, and $a(x,\cdot)$ is continuous on $\mathbb{R}^N$ for a.e. $x \in \Omega$) such that the following holds:

$$a(x,\xi)\cdot\xi \geq \alpha|\xi|^p, \tag{1.1}$$

$$|a(x,\xi)| \leq \beta[b(x) + |\xi|^{p-1}], \tag{1.2}$$

$$(a(x,\xi) - a(x,\eta))\cdot(\xi - \eta) > 0, \tag{1.3}$$

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for almost every $x \in \Omega$, for all $\xi,\ \eta \in \mathbb{R}^N$ with $\xi \neq \eta$, where $p > 1$ and $\alpha,\ \beta$ are positive constants and b is a nonnegative function in $L^{p'}(\Omega)$. For every $u \in W_0^{1,p}(\Omega)$, let us define the differential operator

$$A(u) = -\operatorname{div}(a(x, \nabla u)),$$

that, thanks to the assumptions on a, turns out to be a coercive monotone operator acting from the space $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$. We shall deal with the solutions of the initial boundary-value problem

$$\begin{gathered} u_t + A(u) = \mu \quad \text{in } \Omega \times (0, \infty), \\ u(x, 0) = u_0(x) \quad \text{in } \Omega, \\ u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, \infty), \end{gathered} \tag{1.4}$$

where μ is a measure with bounded variation over $Q = \Omega \times (0, \infty)$ that does not depend on time, and $u_0 \in L^1(\Omega)$.

Let us fix $T > 0$. If $\mu \in L^{p'}(0, T; W^{-1,p'}(\Omega))$, it is well known that problem (1.4) has a unique variational solution in $Q_T = \Omega \times (0, T)$ such that $u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap C([0, T]; L^2(\Omega))$ and $u_t \in L^{p'}(0, T; W^{-1,p'}(\Omega))$, that is

$$\begin{aligned} &\int_0^T \langle u_t, \varphi \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)}\, dt + \int_{Q_T} a(x, \nabla u) \cdot \nabla\varphi \, dx\, dt \\ &= \int_0^T \langle \mu, \varphi \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)}\, dt, \end{aligned}$$

for all $\varphi \in L^p(0, T; W_0^{1,p}(\Omega))$ (see [14] for the case $p \geq 2$ and [12] for $1 < p < 2$).

With the symbol $\mathcal{M}_0(Q)$ we mean a measure with bounded variation over Q which does not charge the sets of zero p-capacity. We refer the reader to [8] for further specifications about parabolic p-capacity. Let us only mention that a measure in $\mathcal{M}_0(Q)$ which does not depend on time is in some sense a measure in $\mathcal{M}_0(\Omega)$, the set of all Radon bounded measures absolutely continuous with respect to the elliptic p-capacity. In fact, if μ does not depend on the time variable t, then there exists a bounded Radon measure ν on Ω such that, for any Borel set $B \subseteq \Omega$, and $0 < t_0 < t_1 < \infty$, we have $\mu(B \times (t_0, t_1)) = (t_1 - t_0)\nu(B)$. In [17] it was proved that actually ν is absolutely continuous with respect to the elliptic p-capacity, and so, thanks to a result of [6], we deduce that ν can be decomposed as $\nu = f - \operatorname{div}(g)$, where $f \in L^1(\Omega)$ and $g \in (L^{p'}(\Omega))^N$.

In [3] (for more details see also [6]) the concept of entropy solution of the elliptic boundary-value problem associated to (1.4) was introduced: let $\mu \in \mathcal{M}_0(\Omega)$ be a measure with bounded variation over Ω which does not charge the sets of zero elliptic p-capacity; we call v an entropy solution for the boundary-value problem

$$\begin{gathered} A(v) = \mu \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.5}$$

if v is finite a.e. and its truncated function $T_k(v) \in W_0^{1,p}(\Omega)$ (recall that $T_k(s) = \max(-k, \min(k, s))$, for all $k > 0$, and it holds

$$\int_\Omega a(x, \nabla v) \cdot \nabla T_k(v - \varphi)\, dx \leq \int_\Omega T_k(v - \varphi)\, d\mu, \tag{1.6}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, for all $k > 0$

An analogous definition will be given in the parabolic case following [20]. To our aim, it suffices to give the definition in the the case of measures which do not depend on time.

Definition 1.1. Let $k > 0$ and define

$$\Theta_k(z) = \int_0^z T_k(s)\,ds,$$

as the primitive function of the truncation function; let $\mu \in \mathcal{M}_0(Q)$ be independent of t, and $u_0 \in L^1(\Omega)$. We say that $u(x,t) \in C([0,\infty); L^1(\Omega))$ is an *entropy solution* of the problem

$$\begin{gathered} u_t + A(u) = \mu \quad \text{in } \Omega \times (0,\infty), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \\ u(x,t) = 0, \quad \text{on } \partial\Omega \times (0,\infty), \end{gathered} \tag{1.7}$$

if, for all $k, T > 0$, we have that $T_k(u) \in L^p(0,T; W_0^{1,p}(\Omega))$, and it holds

$$\begin{aligned} &\int_\Omega \Theta_k(u-\varphi)(T)\,dx - \int_\Omega \Theta_k(u_0 - \varphi(0))\,dx \\ &+ \int_0^T \langle \varphi_t, T_k(u-\varphi)\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)}\,dt + \int_{Q_T} a(x,\nabla u)\cdot \nabla T_k(u-\varphi)\,dx\,dt \\ &\le \int_{Q_T} T_k(u-\varphi)\,d\mu, \end{aligned} \tag{1.8}$$

for any $\varphi \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q_T) \cap C([0,T]; L^1(\Omega))$ with φ_t in the space $L^{p'}(0,T; W^{-1,p'}(\Omega))$.

Remark 1.2. The entropy solution u of the problem (1.7) exists and is unique as shown in [20] for L^1 data; this result was improved in many papers for more general measure data. In [19] it was proved via the notion of *renormalized solution* which turns out to be equivalent to the one of entropy solution with this kind of data (see [9]). Moreover, the solution u is such that $|a(x,\nabla u)| \in L^q(Q_T)$ for all $q < 1 + \frac{1}{(N+1)(p-1)}$, $T > 0$, even if its approximated gradient may not belong to any Lebesgue space.

Let us finally remark that the continuity of the entropy solution with values in $L^1(\Omega)$, which is false in general for measure data (see [18]), turns out to hold true in our framework since the measure μ is supposed to be independent of t (see [17]).

Our main result reads as follows.

Theorem 1.3. *Let $\mu \in \mathcal{M}_0(Q)$ be independent of the variable t, $p > \frac{2N+1}{N+1}$, $u_0 \in L^1(\Omega)$ be a function; let $u(x,t)$ be the entropy solution of problem* (1.4)*, and v the entropy solution of the corresponding elliptic problem* (1.5)*. Then*

$$\lim_{t\to+\infty} u(x,t) = v(x),$$

in $L^1(\Omega)$.

2. Proof of main result

Before passing to the proof of our main result let us state some interesting results about the entropy solution v of the elliptic problem (1.5).

According to [3] (see also [6]) we have that v is in the Marcinkiewicz space $M^{\frac{N(p-1)}{N-p}}(\Omega)$ that implies $v \in L^q(\Omega)$ fon any $q < \frac{N(p-1)}{N-p}$; hence, if $p > \frac{2N}{N+1}$, we have $v \in C(0,\infty; L^1(\Omega))$. So let us suppose $p > \frac{2N}{N+1}$ and let us observe that such a solution actually turns out to be an entropy solution of the initial boundary-value problem (1.7) with initial datum $u_0(x) = v(x)$, since, for all $T > 0$, we have

$$
\begin{aligned}
&\int_\Omega \Theta_k(v-\varphi)(T)\,dx - \int_\Omega \Theta_k(v-\varphi)(0)\,dx \\
&= \int_{Q_T} \frac{d}{dt}\Theta_k(v-\varphi)\,dx\,dt = \int_0^T \langle (v-\varphi)_t, T_k(v-\varphi)\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,dt \\
&= -\int_0^T \langle \varphi_t, T_k(v-\varphi)\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,dt
\end{aligned}
$$

that can be cancelled out with the analogous term in (1.8) getting the right formulation (1.6) for v.

For technical reasons we shall use the stronger assumption

$$
p > \frac{2N+1}{N+1} \tag{2.1}
$$

throughout this note; notice that, according to [5] (see also [8]), in this case a solution u of problem (1.4) belongs to $L^r(0,T;W_0^{1,r})$ for any $r < p - \frac{N}{N+1}$, $T > 0$. Observe that $p - \frac{N}{N+1} > 1$ if and only if (2.1) holds true; hence, in this case, the gradient of the entropy solution u (that coincides with the distributional one) actually belong $L^1(Q_T)$, for any $T > 0$. Moreover, this is the same assumption used in [20] since it allows, for instance, to get continuity of the solution with values in $L^1(\Omega)$ directly by using the trace result of [19].

Most part of our work will rely on comparison between suitable entropy subsolutions and supersolutions of problem (1.4). The notion of entropy subsolution and supersolution for the parabolic problem has been given as a natural extension of the one for the elliptic case (see for instance [15]) in [17]. Let us recall it.

Definition 2.1. A function $\underline{u}(x,t) \in C([0,\infty); L^1(\Omega))$ is an *entropy subsolution* of problem (1.4) if, for all $k, T > 0$, we have that $T_k(\underline{u}) \in L^p(0,T;W_0^{1,p}(\Omega))$, and holds

$$
\begin{aligned}
&\int_\Omega \Theta_k((\underline{u}-\varphi)^+)(T)\,dx - \int_\Omega \Theta_k((\underline{u}_0-\varphi(0))^+)\,dx \\
&+ \int_0^T \langle \varphi_t, T_k(\underline{u}-\varphi)^+\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,dt + \int_{Q_T} a(x,\nabla \underline{u})\cdot \nabla T_k(\underline{u}-\varphi)^+\,dx\,dt \\
&\le \int_{Q_T} T_k(\underline{u}-\varphi)^+\,d\mu,
\end{aligned}
$$

for all $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T) \cap C([0,T];L^1(\Omega))$ with φ_t in the space $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $\underline{u}(x,0) \equiv \underline{u}_0(x) \le u_0(x)$ almost everywhere on Ω with $\underline{u}_0 \in L^1(\Omega)$.

On the other hand, $\overline{u}(x,t) \in C([0,\infty); L^1(\Omega))$ is an *entropy supersolution* of problem (1.4) if, for all $k, T > 0$, we have that $T_k(\overline{u}) \in L^p(0,T;W_0^{1,p}(\Omega))$, and holds

$$\begin{aligned}
&\int_\Omega \Theta_k((\overline{u}-\varphi)^-)(T)\,dx - \int_\Omega \Theta_k((\overline{u}_0-\varphi(0))^-)\,dx \\
&+ \int_0^T \langle \varphi_t, T_k(\overline{u}-\varphi)^-\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,dt + \int_{Q_T} a(x,\nabla\overline{u})\cdot\nabla T_k(\overline{u}-\varphi)^-\,dx\,dt \\
&\geq \int_{Q_T} T_k(\overline{u}-\varphi)^-\,d\mu,
\end{aligned}$$

for all $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T) \cap C([0,T];L^1(\Omega))$ with φ_t in the space $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $\overline{u}(x,0) \equiv \overline{u}_0(x) \geq u_0(x)$ almost everywhere on Ω with $\overline{u}_0 \in L^1(\Omega)$.

In [17] the author proved the following result.

Lemma 2.2. *Let $\mu \in \mathcal{M}_0(\Omega)$, and let $\underline{u}$ and $\overline{u}$ be, respectively, an entropy subsolution and an entropy supersolution of problem* (1.4)*, and let u be the unique entropy solution of the same problem. Then, for any $t > 0$, $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$, a.e. in Ω.*

Thanks to this result we are able to prove Theorem 1.3. For the sake of simplicity, in what follows, the convergences are all understood to be taken up to a suitable subsequence extraction, even if no explicitly claimed.

Proof of Theorem 1.3. We will prove it in a few steps. As usual, the symbol C will indicate any positive constant whose value may change from line to line. Let us consider $v^\oplus$ and $v^\ominus$ as the entropy solutions of, respectively,

$$\begin{gathered}
A(v) = \mu^+ \quad \text{in } \Omega, \\
v = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{2.2}$$

and

$$\begin{gathered}
A(v) = -\mu^- \quad \text{in } \Omega, \\
v = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{2.3}$$

By comparison [15], we have both $v^\ominus \leq 0 \leq v^\oplus$ and

$$v^\ominus \leq v \leq v^\oplus\,. \tag{2.4}$$

Moreover, it is easy to see that both $v^\oplus$ and $v^\ominus$ are stationary solution of the associated parabolic problem with themselves as initial data.

Step 1. $u_0 = v^\oplus$. Some *a Priori Estimates.* To simplify the notation, during this proof, we will indicate by Q the parabolic cylinder of height one $\Omega \times (0,1)$, instead of Q_1 as usual; let $n \in \mathbb{N} \cup \{0\}$, and define $u^n(x,t)$ as the entropy solution of the initial boundary-value problem

$$\begin{gathered}
u_t^n + A(u^n) = \mu \quad \text{in } \Omega \times (0,1), \\
u^n(x,0) = u(x,n) \quad \text{in } \Omega, \\
u^n(x,t) = 0 \quad \text{on } \partial\Omega \times (0,1),
\end{gathered} \tag{2.5}$$

with $u(x,0) = v^\oplus$. Notice that, since μ does not depend on t, u^n turns out to be nothing but the time-translation (of length n) of the solution u with initial datum $v^\oplus$.

Thanks to Lemma 2.2 we readily have $u(x,t) \leq v^\oplus$, for any $t > 0$. So, using again the fact that the datum μ does not depend on time, we can apply the comparison result also between $u(x,t+s)$ solution with $u_0 = u(x,s)$, with s a positive parameter, and $u(x,t)$, the solution with $u_0 = v^\oplus$ as initial datum; so we obtain $u(x,t+s) \leq u(x,t)$ for all $t, s \geq 0$, a.e. in Ω.

Recall that, since $u \in C([0,\infty); L^1(\Omega))$, then $u(x,n) \in L^1(\Omega)$ is well defined. Now, let us look for some *a priori estimates* concerning the sequence u^n.

Following the same outline of [17], we can perform the same calculations to prove first

$$\int_Q |\nabla T_k(u^n)|^p \, dx \, dt \leq Ck; \tag{2.6}$$

moreover, from (2.6), we deduce that the sequence u^n is uniformly bounded in the *Marcinkiewicz* space $M^{p-1+\frac{p}{N}}(Q)$; this fact implies, since in particular $p > \frac{2N}{N+1}$, that u^n is uniformly bounded in $L^m(Q)$ for all $1 \leq m < p - 1 + \frac{p}{N}$ (for further properties of Marcinkiewicz spaces see for instance [22]). Finally, for every $n \geq 0$, $|\nabla u^n|$ is equi-bounded in $M^\gamma(Q)$, with $\gamma = p - \frac{N}{N+1}$, and so, since $p > \frac{2N+1}{N+1}$, $|\nabla u^n|$ is uniformly bounded in $L^s(Q)$ with $1 \leq s < p - \frac{N}{N+1}$.

Now, we shall use the above estimates to prove some *compactness* results that will be useful to pass to the limit in the entropy formulation for u^n. Indeed, thanks to these estimates, we can say that there exists a function $\overline{u} \in L^q(0,1; W_0^{1,q}(\Omega))$, for all $q < p - \frac{N}{N+1}$, such that u^n converges to $\overline{u}$ weakly in $L^q(0,1; W_0^{1,q}(\Omega))$. On the other hand from the equation we deduce that u_t^n is uniformly bounded, with respect to n, in the space $L^1(Q) + L^{s'}(0,1; W^{-1,s'}(\Omega))$, where $s' = \frac{q}{p-1}$, for all $q < p - \frac{N}{N+1}$. So that, thanks to the *Aubin-Simon* type result proved in [7] we have that u^n actually converges to $\overline{u}$ in $L^1(Q)$. Moreover, using the estimate (2.6) on the truncations of u^n, we deduce, from the boundedness and continuity of $T_k(s)$, that, for every $k > 0$,

$$\begin{gathered}
T_k(u^n) \rightharpoonup T_k(\overline{u}), \quad \text{weakly in} \quad L^p(0,1; W_0^{1,p}(\Omega)), \\
T_k(u^n) \to T_k(\overline{u}), \quad \text{strongly in } L^p(Q).
\end{gathered}$$

Finally, the sequence u^n satisfies the hypotheses of [5, Theorem 3.3], and so we get

$$\nabla u^n \to \nabla \overline{u} \quad \text{a.e. in } \Omega.$$

All these results allow us to pass to the limit in the entropy formulation of u^n; indeed, for all $k > 0$, u^n satisfies

$$\int_\Omega \Theta_k(u^n - \varphi)(1) \, dx \tag{2.7}$$

$$- \int_\Omega \Theta_k(u^n(x,0) - \varphi(0)) \, dx \tag{2.8}$$

$$+ \int_0^1 \langle \varphi_t, T_k(u^n - \varphi) \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} \, dt \tag{2.9}$$

$$+ \int_Q a(x, \nabla u^n) \cdot \nabla T_k(u^n - \varphi) \, dx \, dt \tag{2.10}$$

$$\leq \int_Q T_k(u^n - \varphi) \, d\mu, \tag{2.11}$$

for $\varphi \in L^p(0,1; W_0^{1,p}(\Omega)) \cap L^\infty(Q) \cap C([0,1]; L^1(\Omega))$ with $\varphi_t \in L^{p'}(0,1; W^{-1,p'}(\Omega))$. Let us analyze this inequality term by term: recalling that μ can be decomposed as $\mu = f - \operatorname{div}(g)$, where $f \in L^1(\Omega)$ and $g \in (L^{p'}(\Omega))^N$, then, since $T_k(u^n - \varphi)$

converges to $T_k(\overline{u}-\varphi)$ $*$-weakly in $L^\infty(Q)$, and $T_k(u^n-\varphi)$ converges to $T_k(\overline{u}-\varphi)$ also weakly in $L^p(0,1;W_0^{1,p}(\Omega))$, we have

$$\int_Q T_k(u^n-\varphi)\,d\mu \xrightarrow{n} \int_Q T_k(\overline{u}-\varphi)\,d\mu;$$

moreover, we can write

$$\begin{aligned} &\int_Q a(x,\nabla u^n)\cdot\nabla T_k(u^n-\varphi)\,dx\,dt \\ &= \int_Q (a(x,\nabla u^n)-a(x,\nabla\varphi))\cdot\nabla T_k(u^n-\varphi)\,dx\,dt \\ &\quad+\int_Q a(x,\nabla\varphi)\cdot\nabla T_k(u^n-\varphi)\,dx\,dt, \end{aligned} \tag{2.12}$$

and the second term on the right-hand side of (2.12) converges, as n tends to infinity, to

$$\int_Q a(x,\nabla\varphi)\cdot\nabla T_k(\overline{u}-\varphi)\,dx\,dt,$$

while to deal with the nonnegative first term of the right hand side of (2.12), we must use the a.e. convergence of the gradients; then, applying *Fatou's lemma*, we get

$$\begin{aligned} &\int_Q (a(x,\nabla\overline{u})-a(x,\nabla\varphi))\cdot\nabla T_k(\overline{u}-\varphi)\,dx\,dt \\ &\le \liminf_n \int_Q (a(x,\nabla u^n)-a(x,\nabla\varphi))\cdot\nabla T_k(u^n-\varphi)\,dx\,dt. \end{aligned}$$

On the other hand, since $u(x,t)$, is monotone nonincreasing in t and recalling (2.4), we have that there exists a function w such that

$$v(x)\le w(x)\le u(x,t)\le v^{\oplus}(x)$$

and $u(x,t)$ converges to w a.e. in Ω as t tends to infinity. Clearly w does not depend on t and, thanks to dominated convergence theorem, $u(x,t)$ converges to w in $L^1(\Omega)$.

Our goal is to prove that $\overline{u}=v$ almost everywhere in Ω; to do that, it is enough to observe that $\overline{u}$ does not depend on time (in fact, $\overline{u}(x,t)=w(x)$, since $u^n(x,t)=u(x,t+n)$), and that (2.7)+(2.8)+(2.9) converges to zero as n tends to infinity. Indeed, if that holds true, we obtain that $\overline{u}$ satisfies the entropy formulation for the elliptic problem (1.5), and so, since the entropy solution is unique, we get that $\overline{u}=v$ a.e. in Ω.

Let us check that (2.7)+(2.8)+(2.9) approaches zero as n goes to infinity. Using the *monotone convergence theorem*, we get

$$\begin{aligned} \lim_n[(2.7)+(2.8)] &= \int_\Omega \Theta_k(w(x)-\varphi(1))\,dx - \int_\Omega \Theta_k(w(x)-\varphi(0))\,dx \\ &= \int_\Omega\int_0^1 \frac{d}{dt}\Theta(w(x)-\varphi)\,dtdx \\ &= \int_0^1 \langle (w(x)-\varphi)_t, T_k(w(x)-\varphi)\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,dt, \end{aligned}$$

while, since $T_k(u^n-\varphi)$ converges to $T_k(w-\varphi)$ weakly in $L^p(0,1;W_0^{1,p}(\Omega))$, we have

$$\begin{aligned}&\int_0^1 \langle \varphi_t, T_k(u^n-\varphi)\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,dt\\ &\xrightarrow{n} \int_0^1 \langle \varphi_t, T_k(w-\varphi)\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,dt.\end{aligned}$$

Finally we can sum all these terms and, since w does not depend on time, we find

$$\lim_n[(2.7)+(2.8)+(2.9)] = \int_0^1 \langle w_t, T_k(w-\varphi)\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,dt = 0;$$

and, as we mentioned above, this is enough to prove that $w(x)=v(x)$. The same argument can be developed to prove that the solution of (1.4) with $v^\ominus$ as initial data converges in $L^1(\Omega)$ to v.

Using again Lemma 2.2, we easily deduce that the result holds true for any solution of problem (1.4) with u_0 such that $v^\ominus \le u_0 \le v^\oplus$.

Step 2. $v^{\ominus,\tau} \le u_0 \le v^{\oplus,\tau}$. Let us fix $\tau>1$. Then, we can easily readapt the idea of [17] to show that the same result holds true even for initial data data $v^{\ominus,\tau} \le u_0 \le v^{\oplus,\tau}$, where $v^{\oplus,\tau}$ and $v^{\ominus,\tau}$ solve (1.5) with, respectively,

$$\mu_{\oplus,\tau} = \begin{cases} \tau\mu^+ & \text{if } f^+=0,\\ \tau f^+ - \operatorname{div}(g^+) & \text{if } f^+\neq 0,\end{cases}$$

and

$$\mu_{\ominus,\tau} = \begin{cases} -\tau\mu^- & \text{if } f^-=0,\\ -\tau f^- + \operatorname{div}(g^-) & \text{if } f^-\neq 0.\end{cases}$$

as data. Here, thanks to the decomposition result of [6], $\mu^\pm = f^\pm - \operatorname{div}(g^\pm)$ ($f^\pm \ge 0$ in $L^1(\Omega)$, $g^\pm \in (L^{p'}(\Omega))^N$).

Step 3. $u_0\in L^1(\Omega)$ and $\mu\neq 0$. Let us consider the general case of a solution $u(x,t)$ with initial datum $u_0\in L^1(\Omega)$ and let suppose that $\mu\neq 0$ since, if $\mu=0$, then the result it is well known; let us define the family of functions

$$u_{0,\tau} = \begin{cases} \min(u_0, v^{\oplus,\tau}) & \text{if } u_0\ge 0\\ \max(u_0, v^{\ominus,\tau}) & \text{if } u_0<0.\end{cases}.$$

As we have shown in Step 2, for every fixed $\tau>1$, $u_\tau(x,t)$, the entropy solution of problem (1.4) with $u_{0,\tau}$ as initial datum, converges to v a.e. in Ω, as t tends to infinity. Moreover, we have also that $T_k(u_\tau(x,t))$ converges to $T_k(v)$ weakly in $W_0^{1,p}(\Omega)$ as t diverges, for every fixed $k>0$. So, using Lemma 3.4 of [17], we can easily check that $u_{0,\tau}$ converges to u_0 in $L^1(\Omega)$ as τ tends to infinity. Therefore, using a stability result of entropy solution (see for instance [19]) we obtain that $T_k(u_\tau(x,t))$ converges to $T_k(u(x,t))$ strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ as τ tends to infinity.

Now, making the same calculations used in [20] to prove the uniqueness of entropy solutions applied to u and u_τ, where u_τ is considered as the solution obtained as limit of approximating solutions with smooth data, we can easily find, for any fixed $\tau>1$, the following estimate

$$\int_\Omega \Theta_k(u-u_\tau)(t)\,dx \le \int_\Omega \Theta_k(u_0-u_{0,\tau})\,dx,$$

for every $k, t > 0$. Then, let us divide the above inequality by k, and let us pass to the limit as k tends to 0; we obtain

$$\|u(x,t) - u_\tau(x,t)\|_{L^1(\Omega)} \le \|u_0(x) - u_{0,\tau}(x)\|_{L^1(\Omega)}, \tag{2.13}$$

for every $t > 0$. Hence, we have

$$\|u(x,t) - v(x)\|_{L^1(\Omega)} \le \|u(x,t) - u_\tau(x,t)\|_{L^1(\Omega)} + \|u_\tau(x,t) - v(x)\|_{L^1(\Omega)};$$

then, thanks to the fact that the estimate in (2.13) is uniform in t, for every fixed ε, we can choose $\bar\tau$ large enough such that

$$\|u(x,t) - u_{\bar\tau}(x,t)\|_{L^1(\Omega)} \le \frac{\varepsilon}{2},$$

for every $t > 0$; on the other hand, according to Step 2, there exists $\bar t$ such that

$$\|u_{\bar\tau}(x,t) - v(x)\|_{L^1(\Omega)} \le \frac{\varepsilon}{2},$$

for every $t > \bar t$, and this proves our result. □

Remark 2.3. As we said before, in many cases, the convergence in norm to the stationary solution can be improved depending on the regularity of the limit solution (or equivalently to the regularity of the datum); for instance, according to Lemma 2.2, we have that, if $0 \le u_0 \le v$,

$$0 \le u(x,t) \le v(x), \quad \text{for all } t \in (0,\infty), \text{a.e. in } \Omega;$$

so, if $\mu \in L^q(\Omega)$ with $q > \frac{N}{p}$, then Stampacchia's type estimates ensure that the solution v of the stationary problem

$$\begin{gathered} A(v) = \mu \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

is in $L^\infty(\Omega)$ and so the convergence of $u(x,t)$ to v of Theorem 1.3 is at least $*$-weak in $L^\infty(\Omega)$ and almost everywhere. Reasoning similarly one can refine, depending on the data, the asymptotic result of Theorem 1.3.

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