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UNIQUE SOLVABILITY OF INITIAL BOUNDARY-VALUE PROBLEMS FOR HYPERBOLIC SYSTEMS IN CYLINDERS WHOSE BASE IS A CUSP DOMAIN

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Abstract. We study initial boundary-value problems for hyperbolic systems of divergence form of arbitrary order in cylinders whose base is a cusp domain. Our main results are to prove the existence, uniqueness and the smoothness with respect to time variable of generalized solutions of these problems by using the method which we will denote as "approximating boundary method".

1. Introduction

The boundary problems for hyperbolic systems in smooth cylinders have been well studied. Fichera [6] established the existence and the uniqueness of generalized solution, and he had proved that if the right-hand parts, the coefficients and the boundary are infinitely differentiable, so is the solution. In the case, when the non-smooth cylinders, the indicated methods can not be applied since it is impossible to straighten the boundary by a smooth transform.

In this paper, We consider the initial boundary-value problems for hyperbolic systems in cylinders $Q_T = \Omega \times (0,T), 0 < T < +\infty$, with base Ω is a cusp domain. In section 2, it is shown that the existence of a sequence of smooth domains $\{\Omega^\varepsilon\}$ such that $\Omega^\varepsilon \subset \Omega$ and $\lim_{\varepsilon\to 0} \Omega^\varepsilon = \Omega$; moreover, if Ω has a cusp point on boundary $\partial\Omega$, then $\Gamma = \partial\Omega^\varepsilon \cap \partial\Omega$ is a smooth $(n-1)$-dimensional manifold of the class C^∞. In section 3 we set up notation and state the initial boundary-value problems. Section 4 establishes the existence, uniqueness and the smoothness with respect to time variable of generalized solutions of these problems by the approximating boundary method and results in Fichera [6].

The main idea is to apply Theorem 2.1 from Section 2 to establish the unique solvability of the mentioned problem in $V^{m,1}(Q_T)$, where $V^{m,1}(Q_T)$ is a closed subspace of $H^{m,1}(Q_T)$, and Gårding's inequality holds in $V^{m,1}(Q_T)$.

2. Approximating boundary theorem

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Then there exists a sequence of smooth domains $\{\Omega^\varepsilon\}$ such that $\Omega^\varepsilon \subset \Omega$ and $\lim_{\varepsilon\to 0} \Omega^\varepsilon = \Omega$.*

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Proof. For $\varepsilon > 0$ arbitrary, set $S^\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \le \varepsilon\}, \Omega^\varepsilon = \Omega \setminus S^\varepsilon$ and $\partial\Omega^\varepsilon$ is the boundary of Ω^ε. Denote by $J(x)$ the characteristic function of Ω^ε and by $J_h(x)$ the mollification of $J(x)$, i.e,

$$J_h(x) = \int_{\mathbb{R}^n} \theta_h(x-y)J(y)dy,$$

where θ_h is a mollifier. If $h < \frac{\varepsilon}{2}$, then $J_h(x)$ has following properties:

(1) $J_h(x) = 0$ if $x \notin \Omega^{\varepsilon/2}$;
(2) $0 \le J_h(x) \le 1$;
(3) $J_h(x) = 1$ in $\Omega^{2\varepsilon}$;
(4) $J_h \in C^\infty(\mathbb{R}^n)$.

We now fix a constant $c \in (0,1)$, and set $\Omega^\varepsilon_c = \{x \in \Omega : J_h(x) > c\}$. It is obvious that $\Omega^{\varepsilon/2} \supset \Omega^\varepsilon_c \supset \Omega^{2\varepsilon}$. Therefore, $\Omega^\varepsilon_c \subset \Omega$ and $\lim_{\varepsilon\to 0} \Omega^\varepsilon_c = \Omega$, $\partial\Omega^\varepsilon_c = \{x \in \Omega : J_h(x) = c\}$.

Assume that K is a critical set of J_h. By Sard's theorem $\mu(J_h(K)) = 0$, it implies that there exists a constant $c_0 \in (0,1)$ such that $\{x \in \Omega : J_h(x) = c_0\}$ is not a critical set of J_h.

Denote $\Omega^\varepsilon_{c_0} = \{x \in \Omega : J_h(x) > c_0\}$ and $F(x) = J_h(x) - c_0$. If $x^0 \in \partial\Omega^\varepsilon_{c_0}$, then $F(x^0) = J_h(x^0) - c_0 = 0$ and $\operatorname{grad} J_h(x^0) \ne 0$. This implies there exists a $\frac{\partial J_h}{\partial x_i}(x^0) \ne 0$, without loss generality we can suppose that $\frac{\partial J_h}{\partial x_n}(x^0) \ne 0$. Using the implicit function theorem, there exists a neighbourhood W of $(x^0_1, \dots, x^0_{n-1})$ in $\mathbb{R}^{n-1}$ a neighbourhood V of x^0_n in $\mathbb{R}$ and an infinitely differentiable function $z : W \longrightarrow \mathbb{R}$ such that $x \in U_{x^0} \cap \partial\Omega^\varepsilon_{c_0}$, (where $U_{x^0} = W \times V$) if and only if $x = (x_1, \dots, x_n) \in U_{x^0}$, $x_n = z(x_1, \dots, x_{n-1})$. Hence, $\Omega^\varepsilon_{c_0}$ is smooth and $\lim_{\varepsilon\to 0} \Omega^\varepsilon_{c_0} = \Omega$. The theorem is proved. □

Suppose domain Ω is not smooth at one isolated point. The definition is a formal description of domains with a cusp point.

Definition 2.2. We call a bounded domain $\Omega \subset \mathbb{R}^n$ a exterior cusp domain if

(1) $O \in \partial\Omega, \partial\Omega \setminus \{O\}$ is a smooth $(n-1)$-dimensional manifold of the class C^∞.
(2) Denote $x' = (x_1, x_2, \dots, x_{n-1})$, then
$$\{x \in \Omega : 0 < x_n < 1\} \equiv \{x = (x', x_n) \in \mathbb{R}^n : |x'| < x_n^k\}, \quad k \ge 1.$$

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(1) *$O \in \partial\Omega, \partial\Omega \setminus \{O\}$ is a smooth $(n-1)$-dimensional manifold of the class C^∞.*
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$$\{x \in \Omega : 0 < x_n < 1\} \equiv \{x = (x', x_n) \in \mathbb{R}^n : |x'| > x_n^k\}, \quad k \ge 1.$$

Let $\varphi_\varepsilon \in C^\infty(\mathbb{R}_+)$, $0 < \varepsilon < 1/4$, satisfying $0 \le \varphi_\varepsilon \le 1$ and

$$\varphi_\varepsilon(t) = 1,\ \forall t < \varepsilon, \quad \varphi_\varepsilon(t) = 0,\ \forall t \ge 2\varepsilon.$$

When Ω is a exterior cusp domain, we set

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^n : x_n^k = (1 - \varphi_\varepsilon(|x'|))|x'| + 2\varepsilon\varphi_\varepsilon(|x'|),\ 0 < x_n < 1\}.$$

When Ω is a interior cusp domain, we set

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^n : x_n = (1 + \varphi_\varepsilon(|x'|))|x'|^{1/k} - (2\varepsilon)^{1/k}\varphi_\varepsilon(|x'|), x_n < 1\}$$

Denote $\partial_0\Omega = \{x \in \partial\Omega : x_n^k = |x'|, 0 < x_n < 1\}$. If $|x'| \geq 2\varepsilon$ then $\Gamma_\varepsilon \subset \partial_0\Omega$ else $\Gamma_\varepsilon \subset \Omega$. We will denote by $\Omega^\varepsilon \subset \Omega$ a domain with boundary $\partial\Omega^\varepsilon = \Gamma_\varepsilon \cup (\partial\Omega \setminus \partial_0\Omega)$ then $\{\Omega^\varepsilon\}$ is a smooth domain sequence, and $\lim_{\varepsilon\to 0} \Omega^\varepsilon = \Omega$.

3. Statement of the problem

Set $Q_T = \Omega \times (0,T)$, $Q_T^\varepsilon = \Omega^\varepsilon \times (0,T)$, $0 < T < +\infty$, $\Omega_\tau = \Omega \times \{t = \tau\}$. For each multi index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, set $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$. Let us consider the partial differential operator of order $2m$

$$L(x,t) = \sum_{|\alpha|,|\beta|=0}^{m} D^\alpha \left(a_{\alpha\beta}(x,t)D^\beta\right), \tag{3.1}$$

where $a_{\alpha\beta}$ are $s \times s$ matrices whose entries are complex valued functions, and $a_{\alpha\beta} = (-1)^{|\alpha|+|\beta|}{a^*}_{\alpha\beta}$. ${a^*}_{\alpha\beta}$ denotes the transposed conjugate matrix of $a_{\alpha\beta}$, and $a_{\alpha\beta}$ are infinity differentiable in $\overline{Q}_T$. We assume that there exist a constant $c_0 > 0$ such that

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x,t)\xi^\alpha\xi^\beta\eta\overline{\eta} \geq c_0|\xi|^{2m}|\eta|^2, \tag{3.2}$$

for all $\xi \in \mathbb{R}^n\backslash\{0\}, \eta \in \mathbb{C}^s\backslash\{0\}$ and all $(x,t) \in \overline{Q}_T$.

In this paper, we use the usual functional spaces:

$$\overset{\circ}{C}{}^{\infty}(Q_T), C^\infty(\overline{Q}_T), L_2(Q_T), L_2(\Omega), H^m(\Omega), \overset{\circ}{H}{}^m(\Omega)$$

(see [3, 4, 6, 7] for the precise definitions). We introduce some functional spaces which will be used in this paper.

$H^{m,1}(Q_T)$ is the space consisting of all functions $u = (u_1, \dots, u_s)$ from $L_2(Q_T)$ which have generalized derivatives up to order m by x and up to order 1 by t and belonging to $L_2(Q_T)$. The norm in this space is defined as follows:

$$\|u\|_{m,1} = \Big(\sum_{|\alpha|=0}^{m} \int_{Q_T} \Big(|D^\alpha u|^2 + |u_t|^2\Big)\,dx\,dt\Big)^{1/2},$$

where $|D^\alpha u|^2 = \sum_{i=1}^s |D^\alpha u_i|^2$, $|u_t|^2 = \sum_{i=1}^s \left|\partial u_i/\partial t\right|^2$.

$\overset{\circ}{H}{}^{m,1}(Q_T)$ is the closure in $H^{m,1}(Q_T)$ of the set consisting of all functions in $C^\infty(Q_T)$, vanish near $S_T = \partial\Omega \times (0,T)$.

$V^{m,1}(Q_T)$ is a closed subspace of $H^{m,1}(Q_T)$ having the following properties:

(i) $V^{m,1}(Q_T) \supset \overset{\circ}{H}{}^{m,1}(Q_T)$;

(ii) Denote by

$$B_T(u,v) = \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{|\alpha|} \int_{Q_T} a_{\alpha\beta}(x,t)D^\beta u\overline{D^\alpha v}\,dx\,dt$$

and for $t \in [0,T)$,

$$B(u,v)(t) = \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{|\alpha|} \int_{\Omega} a_{\alpha\beta}(x,t)D^\beta u\overline{D^\alpha v}dx, \quad \forall u,v \in V^{m,1}(Q_T),$$

then for all $u \in V^{m,1}(Q_T)$ satisfy

$$(-1)^m B(u,u)(t) \geq \gamma_0\|u(.,t)\|^2_{H^m(\Omega)}, \gamma_0 > 0, \quad \forall t \text{ in } [0,T). \tag{3.3}$$

(iii) Assume that $x_0 \in \partial\Omega, U$ is a neighbourhood of x_0 in $\mathbb{R}^n$ and denote by $I = (\Omega \cap U) \times (0,T)$. Let $\xi \in \overset{\circ}{C}{}^{\infty}(I)$ and $v \in V^{m,1}(Q_T)$, then $\xi v \in V^{m,1}(I)$.

In the case, $V^{m,1}(Q_T) = \overset{\circ}{H}{}^{m,1}(Q_T)$ or $V^{m,1}(Q_T) = H^{m,1}(Q_T)$, condition (iii) is obvious.

Suppose that $\{\Omega^\varepsilon\}$ is a sequence of smooth domains as above, we set

$$V_0^{m,1}(Q_T^\varepsilon) = \{u \in V^{m,1}(Q_T^\varepsilon) : \frac{\partial^j u}{\partial \nu^j}\Big|_{\Gamma_\varepsilon^1} = 0, j = 0, 1, \dots, m-1\}$$

where $\Gamma_\varepsilon^1 = \{x \in \Gamma_\varepsilon : |x'| \le 2\varepsilon\}$. Then $V_0^{m,1}(Q_T^\varepsilon)$ is closed subspace of $V^{m,1}(Q_T^\varepsilon)$.

By zero extension of $u \in V_0^{m,1}(Q_T^\varepsilon)$ out of Q_T^ε, we regard that $u \in V^{m,1}(Q_T)$; therefore, from (3.3) we get the inequality

$$(-1)^m B(u,u)(t) \ge \gamma_0 \|u(.,t)\|^2_{H^m(\Omega^\varepsilon)}, \gamma_0 > 0, \text{ for all } t \in [0,T), \tag{3.4}$$

holds for all $u \in V_0^{m,1}(Q_T^\varepsilon)$.

We have the following results obtained in [6].

Theorem 3.1 ([6]). *If $f \in C^\infty(\overline{Q}_T^\varepsilon)$ and $\frac{\partial^k f}{\partial t^k}\Big|_{t=0} = 0$, for $k = 0, 1, \dots,$ then there is the unique function $u \in C^\infty(\overline{Q}_T^\varepsilon)$ such that*

$$(-1)^{m-1} L(x,t)u - u_{tt} = f(x,t) \quad \text{in } Q_T^\varepsilon \tag{3.5}$$

satisfies initial conditions $u(x,0) = u_t(x,0) = 0$; moreover, $u \in V_0^{m,1}(Q_T^\varepsilon)$ and boundary conditions

$$\langle L(x,t)u, v\rangle_{L_2(\Omega^\varepsilon)} = B(u,v)(t) \tag{3.6}$$

holds for all $v \in V_0^{m,1}(Q_T^\varepsilon)$ and all $t \in [0,T]$, where the scalar product is

$$\langle u, v\rangle_{L_2(\Omega^\varepsilon)} = \int_{\Omega^\varepsilon} u\overline{v}dx.$$

In the cylinder Q_T, we consider systems

$$(-1)^{m-1} L(x,t)u - u_{tt} = f(x,t), \tag{3.7}$$

where $f \in L_2(Q_T)$.

Definition 3.2. A function $u \in V^{m,1}(Q_T)$ is a generalized solution of *initial boundary-value problems* for systems (3.7) if it satisfies following equalities:

$$(-1)^{m-1} B_T(u,\eta) + \langle u_t, \eta_t\rangle_{L_2(Q_T)} = \langle f, \eta\rangle_{L_2(Q_T)} \tag{3.8}$$

for all test function $\eta \in V^{m,1}(Q_T), \eta(x,T) = 0$, and initial conditions holds

$$u(x,0) = u_t(x,0) = 0. \tag{3.9}$$

In particular, $V^{m,1}(Q_T) = \overset{\circ}{H}{}^{m,1}(Q_T)$ or $V^{m,1}(Q_T) = H^{m,1}(Q_T)$, then we have definition of generalized solutions of *the fist initial boundary-value problem* or *second initial boundary-value problems* for system (3.7).

4. The uniqueness and existence theorems

In this section, we investigate the unique solvability of initial boundary-value problems for the systems (3.7). We start with studying the uniqueness theorem.

Theorem 4.1. *Assume that for a positive constant μ,*

$$\sup\Big\{\Big|\frac{\partial a_{\alpha\beta}}{\partial t}\Big|, |a_{\alpha\beta}| : (x,t) \in Q_T, 0 \le |\alpha|, |\beta| \le m\Big\} \le \mu.$$

Then the initial boundary-value problems (3.8), (3.9) for systems (3.7) has no more than one generalized solution in $V^{m,1}(Q_T)$.

Proof. Suppose that problems (3.8), (3.9) has two solutions u_1, u_2 in $V^{m,1}(Q_T)$. Put $u = u_1 - u_2$, (3.8) implies

$$(-1)^{m-1} B_T(u,\eta) + \langle u_t, \eta_t\rangle_{L_2(Q_T)} = 0 \tag{4.1}$$

for all $\eta \in V^{m,1}(Q_T), \eta(x,T) = 0$. For $b \in (0,T)$, we set

$$\eta(x,t) = \begin{cases} 0 & \text{if } t \in (b,T] \\ \int_b^t u(x,\tau)d\tau & \text{if } t \in [0,b]. \end{cases}$$

It is easy to check that $\eta(x,t) \in V^{m,1}(Q_T)$, $\eta_t(x,t) = u(x,t), \eta(x,T) = 0$. Put $\eta(x,t)$ in (4.1), we have

$$(-1)^{m-1} \int_0^b B(\eta_t,\eta)(t)dt + \int_{Q_b} \eta_{tt}\overline{\eta}_t \,dx\,dt = 0.$$

Adding this equality with its complex conjugate, we obtain

$$2\operatorname{Re}(-1)^{m-1} \int_0^b B(\eta_t,\eta)(t)dt + \int_{Q_b} \frac{\partial}{\partial t}|\eta_t|^2 \,dx\,dt = 0. \tag{4.2}$$

We have

$$\begin{aligned}
&2\operatorname{Re}\int_0^b (-1)^{m-1} B(\eta_t,\eta)(t)dt \\
&= \int_0^\tau (-1)^{m-1}\frac{\partial}{\partial t}\Big(B(\eta,\eta)(t)\Big)dt + \operatorname{Re}\sum_{|\alpha|,|\beta|=0}^m \int_{Q_b} (-1)^{|\alpha|+m}\frac{\partial a_{\alpha\beta}}{\partial t} D^\beta\eta\overline{D^\alpha\eta}\,dx\,dt \\
&= (-1)^{m-1}\big[B(\eta,\eta)(b) - B(\eta,\eta)(0)\big] \\
&\quad + \operatorname{Re}\sum_{|\alpha|,|\beta|=0}^m \int_{Q_b} (-1)^{|\alpha|+m}\frac{\partial a_{\alpha\beta}}{\partial t} D^\beta\eta\overline{D^\alpha\eta}\,dx\,dt.
\end{aligned}$$

Since $B(\eta,\eta)(b) = 0$, it implies

$$\begin{aligned}
&2\operatorname{Re}\int_0^b (-1)^{m-1} B(\eta_t,\eta)(t)dt \\
&= (-1)^m B(\eta,\eta)(0) + \operatorname{Re}\sum_{|\alpha|,|\beta|=0}^m \int_{Q_b} (-1)^{|\alpha|+m}\frac{\partial a_{\alpha\beta}}{\partial t} D^\beta\eta\overline{D^\alpha\eta}\,dx\,dt.
\end{aligned} \tag{4.3}$$

and since $\eta_t(x,0) = u(x,0) = 0$,

$$\int_\Omega \int_0^b \frac{\partial}{\partial t}|\eta_t|^2 dt dx = \int_\Omega |\eta_t(x,b)|^2 dx. \tag{4.4}$$

Put (4.3) and(4.4) in (4.2), we obtain

$$\begin{aligned} &(-1)^m B(\eta,\eta)(0) + \int_\Omega |\eta_t(x,b)|^2 dx \\ &= \operatorname{Re} \sum_{|\alpha|,|\beta|=0}^{m} \int_{Q_b} (-1)^{|\alpha|+m-1} \frac{\partial a_{\alpha\beta}}{\partial t} D^\beta \eta \overline{D^\alpha \eta}\, dx\, dt. \end{aligned} \tag{4.5}$$

Set

$$v^\alpha(x,t) = \int_t^0 D^\alpha u(x,\tau) d\tau, \quad t \in [0,b]$$

then

$$v^\alpha(x,b) = \int_b^0 D^\alpha u(x,\tau) d\tau = \int_b^0 D^\alpha \eta_t(x,\tau) d\tau = D^\alpha \eta(x,0).$$

Using (3.3) we have

$$(-1)^m B(\eta,\eta)(0) \ge \gamma_0 \int_\Omega \sum_{|\alpha|=0}^{m} |D^\alpha \eta(x,0)|^2 dx = \gamma_0 \int_\Omega \sum_{|\alpha|=0}^{m} |v^\alpha(x,b)|^2 dx. \tag{4.6}$$

From equality (4.5), we use (4.6) and Cauchy inequality, we will obtain

$$\gamma_0 \int_\Omega \sum_{|\alpha|=0}^{m} |v^\alpha(x,b)|^2 dx + \int_\Omega |v(x,b)|^2 dx \le K_1 \int_{Q_b} \sum_{|\alpha|=0}^{m} |v^\alpha(x,t)|^2\, dx\, dt$$

set $K = K_1/\gamma_0$ is a constant independent of b, then

$$\int_\Omega \sum_{|\alpha|=0}^{m} |v^\alpha(x,b)|^2 dx \le K \int_{Q_b} \sum_{|\alpha|=0}^{m} |v^\alpha(x,t)|^2 x dt$$

By setting

$$y(b) = \int_\Omega \sum_{|\alpha|=0}^{m} |v^\alpha(x,b)|^2 dx$$

we have

$$y(b) \le K \int_0^b y(t) dt,$$

The Gronwall-Bellmann inequality implies $y(b) = u(x,b) = 0$, for all $b \in (0,T)$ and all $x \in \Omega$; hence, $u \equiv 0$ in Q_T. $\square$

Now, we establish the existence of generalized solutions of mentioned problems by *the approximating boundary method*. Firstly, we will prove some following needed propositions.

Proposition 4.2. *If $f \in C^\infty(\overline{Q}_T)$, $\frac{\partial^k f}{\partial t^k}\Big|_{t=0} = 0$, for $k = 0, 1, \dots$ and*

$$\sup \big\{ \big|\frac{\partial a_{\alpha\beta}}{\partial t}\big|, |a_{\alpha\beta}| : (x,t) \in Q_T, 0 \le |\alpha|, |\beta| \le m \big\} \le \mu,$$

then the generalized solutions $u^\varepsilon \in V_0^{m,1}(Q_T^\varepsilon)$ of problem (3.8), (3.9), in smooth cylinders Q_T^ε, satisfies the estimate

$$\|u^\varepsilon\|_{m,1}^2 \le C \|f\|_{L_2(Q_T^\varepsilon)}^2$$

where C is a constant independent of ε.

Proof. By Theorem 3.1, we have $u^\varepsilon \in C^\infty(\overline{Q}^\varepsilon_T) \cap V_0^{m,1}(Q^\varepsilon_T)$, satisfying systems (3.5), boundary conditions (3.6) and initial conditions $u^\varepsilon(x,0) = u^\varepsilon_t(x,0) = 0$, it is clear that u^ε is the generalized solution of problem (3.8), (3.9) in Q^ε_T.

After multiplying (3.5) by $\overline{u}^\varepsilon_t$, integrating on $Q^\varepsilon_\tau, (\tau < T)$, we obtain

$$(-1)^{m-1}\int_0^\tau \int_{\Omega^\varepsilon} Lu^k\overline{u^\varepsilon_t}\,dx\,dt - \int_0^\tau \int_{\Omega^k} u^\varepsilon_{tt}\overline{u^\varepsilon_t}\,dx\,dt = \int_{Q^\varepsilon_\tau} f\overline{u^\varepsilon_t}\,dx\,dt.$$

From that, and using equality (3.6), we get

$$(-1)^m\int_0^\tau B(u^\varepsilon, u^\varepsilon_t)(t)dt + \int_0^\tau \int_{\Omega^\varepsilon} u^k_{tt}\overline{u^\varepsilon_t}\,dx\,dt = -\int_{Q^\varepsilon_\tau} f\overline{u^\varepsilon_t}\,dx\,dt$$

Adding this equality with its complex conjugate we obtain

$$(-1)^m 2\operatorname{Re}\int_0^\tau B(u^\varepsilon, u^\varepsilon_t)(t)dt + \int_{\Omega^\varepsilon}\int_0^\tau \frac{\partial}{\partial t}|u^\varepsilon_t|^2 dtdx = -2\operatorname{Re}\int_{Q^\varepsilon_\tau} f\overline{u^\varepsilon_t}\,dx\,dt. \tag{4.7}$$

We now change the left terms of (4.7)

$$\begin{aligned}
&2(-1)^m\operatorname{Re}\int_0^\tau B(u^\varepsilon, u^\varepsilon_t)(t)dt\\
&= (-1)^m\operatorname{Re}\int_0^\tau \frac{\partial}{\partial t}\Big(B(u^\varepsilon, u^\varepsilon)(t)\Big)dt\\
&\quad - \operatorname{Re}\int_{Q^\varepsilon_\tau}\sum_{\alpha,\beta=0}^m (-1)^{|\alpha|+m}\frac{\partial a_{\alpha\beta}}{\partial t}D^\beta u^\varepsilon\overline{D^\alpha u^\varepsilon}\,dx\,dt\\
&= (-1)^m\operatorname{Re}B(u^\varepsilon, u^\varepsilon)(\tau) - \operatorname{Re}\int_{Q^\varepsilon_\tau}\sum_{\alpha,\beta=0}^m (-1)^{|\alpha|+m}\frac{\partial a_{\alpha\beta}}{\partial t}D^\beta u^\varepsilon\overline{D^\alpha u^\varepsilon}\,dx\,dt.
\end{aligned}$$

Using the initial conditions, we get

$$\int_{\Omega^\varepsilon}\int_0^\tau \frac{\partial}{\partial t}|u^\varepsilon_t|^2 dtdx = \int_{\Omega^\varepsilon_\tau}|u^\varepsilon_t|^2 dx = \|u^\varepsilon_t(x,t)\|_{L_2(\Omega^\varepsilon_\tau)}$$

Therefore, basing on what has been discussed above, equality (4.7) can be rewritten as the form

$$\begin{aligned}
&(-1)^m B(u^\varepsilon, u^\varepsilon)(\tau) + \|u_t(x,t)\|_{L_2(\Omega^\varepsilon_\tau)}\\
&= \operatorname{Re}\int_{Q^\varepsilon_\tau}(-1)^{|\alpha|+m}\sum_{|\alpha|,|\beta|=0}^m \frac{\partial a_{\alpha\beta}}{\partial t}D^\beta u^\varepsilon\overline{D^\alpha u^\varepsilon}\,dx\,dt - 2\operatorname{Re}\int_{Q^\varepsilon_\tau} f\overline{u^\varepsilon_t}\,dx\,dt.
\end{aligned} \tag{4.8}$$

From (4.8), by using inequality (3.4), and Cauchy inequality, we obtain

$$\begin{aligned}
&\gamma_0\|u^\varepsilon(x,t)\|^2_{H^m(\Omega^\varepsilon_\tau)} + \|u^\varepsilon_t(x,t)\|_{L_2(\Omega^\varepsilon_\tau)}\\
&\le C_1\Big(\sum_{|\alpha|=0}^m\int_{Q^\varepsilon_\tau}\big(|D^\alpha u^\varepsilon|^2 + |u^\varepsilon_t|^2\big)\,dx\,dt + \|f\|^2_{L_2(Q^\varepsilon_T)}\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|u^\varepsilon(x,t)\|^2_{H^m(\Omega^\varepsilon_\tau)} + \|u^\varepsilon_t(x,t)\|_{L_2(\Omega^\varepsilon_\tau)}\\
&\le C_2\Big(\int_0^\tau\Big(\|u^\varepsilon(.,t)\|^2_{H^m(\Omega^\varepsilon_t)} + \|u^\varepsilon_t(.,t)\|_{L_2(\Omega^\varepsilon_t)}\Big)dt + \|f\|^2_{L_2(Q^\varepsilon_T)}\Big)
\end{aligned}$$

where $C_2 = C_1/\min\{\gamma_0; 1\}$. Denote

$$Z(\tau) = \|u^\varepsilon(x,t)\|^2_{H^m(\Omega^\varepsilon_\tau)} + \|u^\varepsilon_t(x,t)\|_{L_2(\Omega^\varepsilon_\tau)}$$

We get

$$Z(\tau) \le C_2\Big(\int_0^\tau Z(t)dt + \|f\|^2_{L_2(Q^\varepsilon_T)}\Big).$$

The Gronwall - Bellmann inequality implies

$$Z(\tau) \le C_2 e^{C_2\tau}\|f\|^2_{L_2(Q^\varepsilon_T)}, \quad \forall\tau \in (0,T)$$

By integrating with respect to τ from 0 to T this inequality, we obtain

$$\|u^\varepsilon\|^2_{m,1} \le C\|f\|^2_{L_2(Q^\varepsilon_T)}, \quad C = C_2(e^{C_2T} - 1),$$

where C is a absolute constant. □

In the next proposition, we prove result of proposition 4.2 without conditions $\frac{\partial^k f}{\partial t^k}\Big|_{t=0} = 0$, for $k = 0, 1, \dots$.

Proposition 4.3. *If $f \in C^\infty(\overline{Q}_T)$ and*

$$\sup\big\{\big|\frac{\partial a_{\alpha\beta}}{\partial t}\big|, |a_{\alpha\beta}| : (x,t) \in Q_T, 0 \le |\alpha|, |\beta| \le m\big\} \le \mu,$$

then generalized solution $u^\varepsilon \in V^{m,1}_0(Q^\varepsilon_T)$ of problem (3.8), (3.9) in smooth cylinders Q^ε_T satisfies the following estimates

$$\|u^\varepsilon\|^2_{m,1} \le C\|f\|^2_{L_2(Q^\varepsilon_T)}$$

where C is a constant independent of ε.

Proof. Denote

$$f_h(x,t) = \begin{cases} 0 & \text{if } (x,t) \ne Q^\varepsilon_T \\ f(x,t) & \text{if } (x,t) \in Q^\varepsilon_T, t > h \\ 0, & \text{if } (x,t) \in Q^\varepsilon_T, t \le h \end{cases}$$

for all $h > 0$. We will denote by $g_{\frac{h}{2}}$ the mollification of f_h; i.e.,

$$g_{\frac{h}{2}}(x,t) = \int_{\mathbb{R}^{n+1}} \theta_{\frac{h}{2}}(x-y, t-\tau) f_h(y,\tau)dyd\tau,$$

where θ_h is a mollifier. Then $g_{\frac{h}{2}} \in C^\infty(\overline{Q^\varepsilon_T}), g_{\frac{h}{2}} \equiv 0, t < \frac{h}{2}$ and $g_{\frac{h}{2}} \to f$ in $L_2(Q^\varepsilon_T)$. Applying proposition (4.2) to replace f by $g_{\frac{h}{2}}$, we get u^ε_h as a generalized solution of the problem (3.8), (3.9) in Q^ε_T and the following estimate holds

$$\|u^\varepsilon_h\|^2_{m,1} \le C\|g_{\frac{h}{2}}\|^2_{L_2(Q^\varepsilon_T)}$$

where C is a absolute constant. As $g_{\frac{h}{2}} \to f$ in $L_2(Q^\varepsilon_T)$, $\{u^\varepsilon_h\}$ is a Cauchy sequence in $V^{m,1}_0(Q^\varepsilon_T)$. Therefore, $u^\varepsilon_h \to u^\varepsilon, (h \to 0)$. It is easy to see that u^ε is a generalized solution of the problem and satisfies the following estimate

$$\|u^\varepsilon\|^2_{m,1} \le C\|f\|^2_{L_2(Q^\varepsilon_T)}.$$

□

We now prove the existence of a generalized solution to problem (3.8), (3.9) in Q_T when $f \in C^\infty(\overline{Q}_T)$.

Proposition 4.4. *Assume that $f \in C^\infty(\overline{Q}_T)$ and*

$$\sup \big\{ \big|\frac{\partial a_{\alpha\beta}}{\partial t}\big|, |a_{\alpha\beta}| : (x,t) \in Q_T, 0 \le |\alpha|, |\beta| \le m \big\} \le \mu.$$

Then problem (3.8), (3.9) in cylinder Q_T has the generalized solution $u \in V^{m,1}(Q_T)$ and

$$\|u\|^2_{m,1} \le C\|f\|^2_{L_2(Q_T)} \tag{4.9}$$

where C is a constant independent of u and f.

Proof. By proposition 4.3, the generalized solution $u^\varepsilon \in V_0^{m,1}(Q_T^\varepsilon)$ of problem (3.8), (3.9) in the smooth cylinder Q_T^ε satisfies the following inequality

$$\|u^\varepsilon\|^2_{m,1} \le C\|f\|^2_{L_2(Q_T^\varepsilon)} \tag{4.10}$$

where C is a constant independent of ε.

Since $\|f\|^2_{L_2(Q_T^\varepsilon)} \le \|f\|^2_{L_2(Q_T)}$, we have

$$\|u_\varepsilon\|^2_{m,1} \le C\|f\|^2_{L_2(Q_T)}.$$

Set

$$\widetilde{u}_\varepsilon(x,t) = \begin{cases} u^\varepsilon(x,t) & \text{if } (x,t) \in Q_T^\varepsilon \\ 0 & \text{if } (x,t) \in Q_T \setminus Q_T^\varepsilon \end{cases}$$

then

$$\|\widetilde{u}_\varepsilon\|^2_{m,1} = \|u^\varepsilon\|^2_{m,1} \le C\|f\|^2_{L_2(Q_T)}. \tag{4.11}$$

This means that the set $\{\widetilde{u}_\varepsilon\}_{\varepsilon>0}$ is uniform bounded in the space $V^{m,1}(Q_T)$. So we can take a subsequence, denote also by $\widetilde{u}_\varepsilon$ for convenience, which converges weakly to a function $\widetilde{u}_0 \in V^{m,1}(Q_T)$. We will show that $\widetilde{u}_0$ is a generalized solution of problem (3.8), (3.9) in cylinder Q_T. In fact, for all $\eta \in V^{m,1}(Q_T), \eta(x,T) = 0$ there exists $\eta_\delta \in C^\infty(\overline{Q_T})$ such that $\eta_\delta \equiv 0$ in $Q_T \setminus Q_T^\varepsilon, \eta_\delta(x,T) = 0$, and $\|\eta_\delta - \eta\|_{m,1} \longrightarrow 0$ when $\delta \to 0$. Since u^ε is a generalized solution of problem (3.8), (3.9) in smooth cylinder Q_T^ε, we have

$$(-1)^{m-1}B_T(u^\varepsilon, \eta_\delta) + \langle u_t^\varepsilon, \eta_{\delta t}\rangle_{L_2(Q_T^\varepsilon)} = \langle f, \eta_\delta\rangle_{L_2(Q_T^\varepsilon)}$$

Rewrite this equality in the form

$$(-1)^{m-1}B_T(\widetilde{u}_\varepsilon, \eta_\delta) + \langle \widetilde{u}_{\varepsilon t}, \eta_{\delta t}\rangle_{L_2(Q_T)} = \langle f, \eta_\delta\rangle_{L_2(Q_T^\varepsilon)}$$

Passing to the limit when $\varepsilon \to 0, \delta \to 0$ for the weakly convergent sequence, we get

$$(-1)^{m-1}B_T(\widetilde{u}_0, \eta) + \langle \widetilde{u}_{0t}, \eta_t\rangle_{L_2(Q_T)} = \langle f, \eta\rangle_{L_2(Q_T)}$$

Since $V^{m,1}(Q_T)$ is embedded continuously into $L_2(\Omega)$, the trace sequence $\{\widetilde{u}_\varepsilon(x,0)\}$ of $\{\widetilde{u}_\varepsilon(x,t)\}$ converges weakly to the trace $\widetilde{u}_0(x,0)$ of $\widetilde{u}_0(x,t)$ in $L_2(\Omega)$. On the other hand, $\widetilde{u}_\varepsilon(x,0) = 0$, so that $\widetilde{u}_0(x,0) = 0$. Hence, $\widetilde{u}_0(x,t)$ is a generalized solution of problem (3.8), (3.9). Moreover, from (4.11) we have

$$\|\widetilde{u}_0\|^2_{m,1} \le \varliminf_{\varepsilon\to 0} \|\widetilde{u}_\varepsilon\|^2_{m,1} \le C\|f\|^2_{L_2(Q_T)}.$$

□

Proposition 4.4 states the existence of generalized solutions of problem (3.8), (3.9) in $V^{m,1}(Q_T)$ when $f \in C^\infty(\overline{Q}_T)$. Using this proposition and properties of mollification of $f \in L_2(Q_T)$, we obtain the following theorem.

Theorem 4.5. *If $f \in L_2(Q_T)$, and*

$$\sup\big\{\big|\frac{\partial a_{\alpha\beta}}{\partial t}\big|, |a_{\alpha\beta}| : (x,t) \in Q_T, 0 \le |\alpha|, |\beta| \le m\big\} \le \mu,$$

then problem (3.8), (3.9) in the cylinder Q_T has generalized solutions $u \in V^{m,1}(Q_T)$ and

$$\|u\|_{m,1}^2 \le C\|f\|_{L_2(Q_T)}^2,$$

where C is a constant independent of u and f.

The following theorem shows that the generalized solution $u \in V^{m,1}(Q_T)$ of problem (3.8), (3.9) is smooth with respect to time variable t, if the right-hand side f and coefficients of the operator (3.1) are smooth enough with respect to t.

Theorem 4.6. *Let*

(i) $\frac{\partial^k f}{\partial t^k} \in L_2(Q_T), k \le h$;
(ii) $\frac{\partial^k f}{\partial t^k}\Big|_{t=0} = 0$, $x \in \Omega$, $k \le h-1$;
(iii) $\sup\big\{\big|\frac{\partial^k a_{\alpha\beta}}{\partial t^k}\big|, k < h : (x,t) \in Q_T, 0 \le |\alpha|, |\beta| \le m\big\} \le \mu$.

Then the generalized solution $u \in V^{m,1}(Q_T)$ of problem (3.8), (3.9) has generalized derivatives with respect to t up to order h in $V^{m,1}(Q_T)$ and

$$\big\|\frac{\partial^h u}{\partial t^h}\big\|_{m,1}^2 \le C\sum_{k=0}^{h}\big\|\frac{\partial^k f}{\partial t^k}\big\|_{L_2(Q_T)}^2$$

where C is a constant independent of u and f.

This theorem is proved by arguments analogous to those in the proof of propositions 4.2, 4.3, 4.4 and by induction on h.

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